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LOGICAL OPENS OF EXPONENTIAL OBJECTS

by Oscar P. BRUNO

RÉSUMÉ. Soit $X = \bar{A}$ et $Y = \bar{B}$, avec $A = C^\infty(\mathbf{R}^p)/J$, $B = C^\infty(\mathbf{R}^n)/I$, deux objets représentables dans le topos de Dubuc. L'ensemble des sections globales de l'exponentielle Y^X est identifié à l'ensemble $Z(I, A^n) \subset A^n$ de zéros de l'idéal I dans A^n et est ainsi muni d'une topologie C^∞-compact-ouvert. Dans cet article, on étudie les ouverts de Penon de l'objet Y^X. On montre qu'ils coïncident avec les ouverts C^∞-CO de $Z(I, A^n)$ dans le cas où J a des extensions déterminées par des lignes (Définition 0.3) ou bien si $I = \{0\}$. On donne un exemple d'un ouvert de Penon qui n'est pas C^∞-CO en prenant l'idéal J de fonctions à germe nul.

INTRODUCTION.

Let $X = \overline{C^\infty(\mathbf{R}^p)/J}$, $Y = \overline{C^\infty(\mathbf{R}^n)/I}$ be two representable objects in the Dubuc topos D (see Section 0) where J has line determined extensions (0.3). The main result in this paper (Theorem 1.11) says that the global section functor Γ establishes a bijection between Penon open subobjects of Y^X and open subsets of $\Gamma(Y^X)$ in the C^∞-CO topology. We show also that when $I = \{0\}$, we can assume J arbitrary (1.12). However, the restriction on J (of having line determined extensions) is seen to be unavoidable in general.

We precede the article with a Section 0 where we recall all these notions and fix the notations.

SECTION 0.

Let D denote the Dubuc topos (see [3, 4]). We recall that D is the topos defined by the following site :

i) The category $\mathcal{B}$, dual to that of finitely generated C^∞-rings $C^\infty(\mathbf{R}^n)/I$ presented by an ideal of local nature (see [4] and Remark below).

ii) The open cover topology (see [3] and Remark below).

0.1. Remark. i) Let U be an open subset of $\mathbf{R}^n$ (in most of the cases $U = \mathbf{R}^n$) and $I \subset C^\infty(U)$ an ideal. Then I is of local nature (or of local character, or germ determined) iff for every $f \in C^\infty(U)$, $f \in I$ iff there exists an open covering $\{U_\alpha\}_\alpha$ of U such that

$$f|U_\alpha \in I|U_\alpha = \text{ideal generated in } C^\infty(U_\alpha) \text{ by } \{h|U_\alpha : h \in I\}.$$

We remark that if $I \subset C^\infty(\mathbf{R}^n)$ is an ideal of local character and U is an open subset of $\mathbf{R}^n$, $I|U$ may not be of local character. If $I \subset C^\infty(U)$ is any ideal, there exists a smallest local nature ideal $\hat{I}$ which contains I. In fact, $f \in \hat{I}$ iff there exists an open covering $\{U_\alpha\}$ of U such that $f|U_\alpha \in I|U_\alpha$. $\hat{I}$ is called the local nature closure of I.

ii) We recall that the generating covers of the open cover topology are families of the form

$$j_{U_\alpha} : \overline{C^\infty(U_\alpha)} \rightarrowtail \overline{C^\infty(\mathbf{R}^n)}$$

where $\{U_\alpha\}$ is an open covering of $\mathbf{R}^n$ and j_{U_α} are the maps corresponding to the restriction morphisms. The coverings of an arbitrary

$$T = \overline{C^\infty(\mathbf{R}^n)/I} \in B$$

are obtained by pulling-back these covers (see [3]). It can be seen then that they are families of the form

$$\overline{C^\infty(U_\alpha)/\hat{I}|U_\alpha} \rightarrowtail \overline{C^\infty(\mathbf{R}^n)/I}$$

where U_α is a covering of the set of zeroes of I, Z(I).

0.2. Remark. Let $X = \overline{C^\infty(\mathbf{R}^p)/J}$, $Y = \overline{C^\infty(\mathbf{R}^n)/I} \in B$. We recall that the (cartesian) product of X and Y in B is

$$X \times Y = \overline{C^\infty(\mathbf{R}^{n+p})/(J(\bar{x}, \bar{t}) \hat{+} J(\bar{t}, \bar{x})}$$

where this notation should be understood as follows : since we consider the elements of $C^\infty(\mathbf{R}^p)$ (resp. $C^\infty(\mathbf{R}^n)$, $C^\infty(\mathbf{R}^{p+n})$) functions of the variables

$$\bar{x} = (x_1, ..., x_p) \quad (\text{resp. } \bar{t} = (t_1, ..., t_n), \ (\bar{x}, \bar{t}) = (x_1, ..., x_p, t_1, ..., t_n)))$$

the ideal J is an ideal in the variable $\bar{x}$: $J = J(\bar{x})$. Now $J(\bar{x}, \bar{t})$ is the ideal generated in $C^\infty(\mathbf{R}^{p+n})$ by the functions of $J(x)$. On the other hand, the symbol $\hat{+}$ means the *local nature sum*, i.e., to sum and take local nature closure.

Recall that if $H \rightarrowtail F$ is a subobject of F in a topos, then H is said to be Penon open iff the following formula holds internally :

$$\forall h \in H \ \ \forall q \in F \ (\neg(h = q) \vee q \in H)$$

(see [6, 1]).

Let $\widetilde{Top}_2$ be the topos of sheaves over the site of Hausdorff topological spaces with open coverings, $\widetilde{Zar}$ be the topos of sheaves over the site given by the category *dual* to that of finitely presented k-algebras with coverings $\overline{B[a_i^{-1}]} \rightarrowtail \overline{B}$ where $\Sigma a_i = 1$ and k is an algebraically closed field ; and let D be the Dubuc topos already presented. It has been proved by J. Penon (see [5]) that if either $E = \widetilde{Top}_2$ or $E = \widetilde{Zar}$

or E = D and F is *representable,* then a subobject of F is Penon open iff it is representable and represented by : in the first case an open subset of F, in the second a Zariski open, and in the third, if $F = \overline{C^\infty(\mathbf{R}^n)/I}$ by a subobject of the form $\overline{C^\infty(U)/I|U}$ where U is an open subset of $\mathbf{R}^n$. We study here Penon opens of Y^X where

$$Y = \overline{C^\infty(\mathbf{R}^n)/I}, \quad X = \overline{C^\infty(\mathbf{R}^p)/J} \in D$$

are representables (I and J of local character). In some cases we will need to assume that the ideal J has line determined extensions :

0.3. Definition (see [2]). An ideal $J \subset C^\infty(\mathbf{R}^p)$ is said to have *line determined extensions* iff it satisfies the following condition: for every $n \in \mathbf{N}$ and $f \in C^\infty(\mathbf{R}^{p+n})$, $f \in J(\bar{x}, \bar{t})$ iff for every fixed $\bar{a} \in \mathbf{R}^n$, $f(\bar{x}, \bar{a}) \in J$.

We recall from [2] that a large class of finitely generated ideals (including those generated by a finite number of analytic functions) have line determined extensions and there are some examples of non-finitely generated ideals which also have line determined extensions. As a matter of fact, these ideals are characterized as *universally closed,* i.e., C^∞-CO closed ideals such that the extension $J(\bar{x}, \bar{t})$ to $C^\infty(\mathbf{R}^{p+n})$ for all n is C^∞-CO closed.

0.4. Definition. The C^∞-CO *topology* in $C^\infty(\mathbf{R}^n)$ is the topology for which a sequence f_k of elements of $C^\infty(\mathbf{R}^n)$ converges to $f \in C^\infty(\mathbf{R}^n)$ iff f_k and all its derivatives converge uniformely on compacts to f and its respective derivatives.

A result which is closely related to the notion of ideal with line determined extensions is the following :

0.5. ***Theorem*** *(Calderon-Reyes-Qué, see [7]). Let C, D be closed subsets of* $\mathbf{R}^p$ *and* $\mathbf{R}^n$ *respectively, and let*

$$J \subset C^\infty(\mathbf{R}^p), \quad I \subset C^\infty(\mathbf{R}^n) \quad \textit{and} \quad S \subset C^\infty(\mathbf{R}^{n+p})$$

be the ideals of all flat functions on C, D and CxD respectively. (Recall that a function $f \in C^\infty(\mathbf{R}^k)$ *is said to be flat on a closed subset K of* $\mathbf{R}^k$ *iff f and all its derivatives vanish on K). Then*

$$S = J(\bar{x}, \bar{t}) + I(\bar{t}, \bar{x}). \qquad \diamond$$

Finally we recall a well known lemma. By the way we remark that it is this lemma which implies that the congruence associated in the standard way to any ideal $I \subset C^\infty(\mathbf{R}^n)$ is a C^∞-ring congruence (see [4]).

0.6. Lemma. *a) For every n+p-variables C^∞-function $h : \mathbf{R}^{n+p} \to \mathbf{R}$ and for every integer $m \geq 0$ there exist C^∞-functions*

$$f_k \text{ of } n \text{ variables } \{k = (k_1, \ldots, k_p) : \Sigma k_i \leq m\},$$
$$\ell_k \text{ of } n+p \text{ variables } \{k = (k_1, \ldots, k_p) : \Sigma k_i = m+1\}$$

such that the equality

$$h(\bar{t}, \bar{x}) = \sum_k f_k(\bar{t})\bar{x}^k + \sum_k \ell_k(\bar{t}, \bar{x})\bar{x}^k$$

holds for every

$$(\bar{t}, \bar{x}) = (t_1, \ldots, t_n, x_1, \ldots, x_p) \in \mathbf{R}^{n+p}$$

where $\bar{x}^k = x_1^{k_1} \cdot \ldots \cdot x_p^{k_p}$. *Of course we have*

$$f_k(\bar{t}) = \frac{1}{k!}\frac{\partial^{|k|} h}{\partial \bar{x}^k}(\bar{t}, 0).$$

b) We will use this Lemma in the following particular case : If $h \in C^\infty(\mathbf{R}^n)$ then there exist functions $\ell_i \in C^\infty(\mathbf{R}^{2n})$ such that, for every $\bar{y}_1, \bar{y} \in \mathbf{R}$ we have

$$h(\bar{y}_1) - h(\bar{y}) = \sum_{i=1}^{n} (y_1^i - y^i).\ell_i(\bar{y}_1, \bar{y}).$$

SECTION 1.

We prove first some auxiliary results (1.1 to 1.5).

Let $B = C^\infty(\mathbf{R}^n)/I$, $A = C^\infty(\mathbf{R}^p)/J$ be any two C^∞-rings in B^{op} and

$$X = \bar{A}, \quad y = \bar{B} \in B \subset D.$$

Let $\Gamma : D \to$ Sets be the global section functor $\Gamma(F) = \mathrm{Hom}(1, F)$. We have

1.1. Proposition. $\Gamma(Y^X) = Z(I, A^n)$, *where*

$$Z(I, A^n) = \{(f_1, \ldots, f_n) \in A^n : \forall h \in I, h(f_1, \ldots, f_n) = 0\}$$

(Notice that the last definition makes sense since smooth functions may be evaluated in C^∞-rings.)

1.2. Definition. The C^∞-CO *topology on* A is the quotient topology determined by the C^∞-CO topology of $C^\infty(\mathbf{R}^p)$ (see 0.4).

The C^∞-CO topology of A^n is just the product topology, and we give the subspace topology to $Z(I, A^n)$.

Recall that the quotient map $C^\infty(\mathbf{R}^p) \twoheadrightarrow A$ is open, thus it follows :

1.3. Lemma. *If a sequence h_k of elements of A converges to $h \in A$ in the C^∞-CO topology, then there exist a sequence $\{f_k\} \subset C^\infty(\mathbf{R}^p)$*

and $f \in C^{\infty}(R^p)$ *such that*

i) $[f_k] = h_k$ *and* $[f] = h.$

(The brackets mean "equivalence class of".)

ii) f_k *converges to* f *in the* C^{∞}*-CO topology of* $C^{\infty}(\mathbf{R}^p)$.

1.4. Lemma. *Let* X, Y *be as above and* $i : H \rightarrowtail Y^X$ *be a subobject of* Y^X. *Then* H *is Penon open iff it satisfies the following conditions :*

a) For every representable sheaf $T = \overline{C^{\infty}(\mathbf{R}^k)/K} \in B$, *arrow* $q : T \to Y^X$ *and* $\bar{s}_0 \in Z(K) \subset \mathbf{R}^k$, $\bar{s}_0 : 1 \to T$, *if* $q \circ s_0$ *factors through* H, *then there exists a neighborhood* V *of* $\bar{s}_0$ *in* $\mathbf{R}^k$ *such that* $q \circ j_V$ *factors through* H *(where*

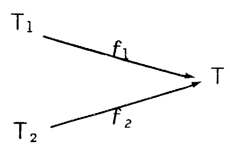
$$j_V : \overline{C^{\infty}(V)/K|V} \to \overline{C^{\infty}(\mathbf{R}^k)/K}$$

is the map corresponding to the restriction). In other words, if "$q(\bar{s}_0) \in H$*" then there exists a neighborhood* V *of* $\bar{s}_0 \in \mathbf{R}^k$ *such that*

" $q(V \cap Z(K)) \subset H$ ".

b) If $T = \overline{C^{\infty}(\mathbf{R}^k)/K}$ *is any representable sheaf and* q, h *are arrows* $q : T \to Y^X$, $h : T \to H$, *and there exists a sequence* $\bar{s}_r$ *of elements of* Z(K) *converging to* $\bar{s}_0 \in Z(K)$ *such that* $q \circ \bar{s}_r = i \circ h \circ \bar{s}_r$, *then* $q \circ \bar{s}_0$ *factors through* H. (Notice that this condition is vacuous if the ideal J is C^{∞}-CO closed since in this case we have $q \circ \bar{s}_0 = i \circ h \circ \bar{s}_0$.*)*

Proof. Kripke-Joyal semantics (see [1]) tells us that H is Penon open iff for every $T \in B$ and for every $q : T \to Y^X$, $h : T \to H$ there exists a covering of T



such that $(q \circ f_1, i \circ h \circ f_1)$ verifies the formula $\neg(h = q)$ and $q \circ f_2$ factors through H. We must prove that this K-J statement is equivalent to the statement of the Lemma.

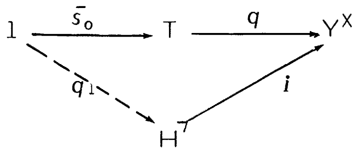
Statement of the Lemma implies K-J statement : Assume H verifies the statement of the Lemma. Because of the sheaf axiom on H it suffices to show that for every $\bar{s}_0 \in Z(K)$ either

i) there exists an open neighborhood V of $\bar{s}_0$ in $\mathbf{R}^k$ such that $q \circ j_V$ factors through H, or

ii) There exists an open neighborhood V of $\bar{s}_0$ in $\mathbf{R}^k$ such that for $\bar{s} \in V \cap Z(K)$ we have $i \circ h \circ \bar{s} \neq q \circ \bar{s}$.

So, take $\bar{s}_0 \in Z(K)$ and assume that point ii is not verified. It follows that there exists a sequence $\bar{s}_r$ of points of Z(K) converging to $\bar{s}_0$ such that for every $r \in \mathbf{N}$, $q \circ \bar{s}_r = i \circ h \circ \bar{s}_r$. We remark that this does not imply $q \circ \bar{s}_0 = i \circ h \circ \bar{s}_0$, but in virtue of b it follows that $q \circ \bar{s}_0$ factors through H and so, by a, we have that $\bar{s}_0$ verifies point i.

K-J statement implies statement of the Lemma : a) Take $q : T \to Y^X$ and consider the following commutative diagram

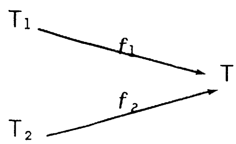


(1)

With this data we may consider the arrows

$$q : T \to Y^X \qquad \text{and} \qquad T \xrightarrow{a} 1 \xrightarrow{q_1} H.$$

By K-J statement, there exists a covering



such that $(q \circ f_1, i \circ q_1 \circ a \circ f_1)$ verifies the formula $\neg(q = h)$ and $q \circ f_2$ factors through H. Since T_1, T_2 is a covering $\bar{s}_0 : 1 \to T$ must factor either through T_1 or through T_2. But it cannot factor through T_1 since this would imply that

$$(q \circ \bar{s}_0, i \circ q_1 \circ a \circ \bar{s}_0) = (q \circ \bar{s}_0, i \circ q_1)$$

verifies the formula $\neg(q = h)$, which contradicts the commutativity of (1).

b) Immediate. ◊

1.5. *Lemma* (see [5]). *Let* F *be an object in the topos* D.

a) The correspondence $R \to \Gamma(R)$ *from the set of subobjects of* F *to the set of subsets of* $\Gamma(F)$ *has a right adjoint* E, *i.e., for every* $S \subset \Gamma(F)$, *there exists* $E(S) \leq F$ *such that for every* $R \leq F$ *we have*

$$\Gamma(R) \subset S \qquad \textit{iff} \qquad R \leq E(S).$$

Moreover $\Gamma(E(S)) = S$. *In fact, for* $S \subset \Gamma(F)$, $E(S)$ *is defined by the following rule : an arrow* $f : T \to F$ $(T \in B)$ *factors through* $E(S)$ *iff* $\Gamma(f) : \Gamma(T) \to \Gamma(F)$ *factors through* S.

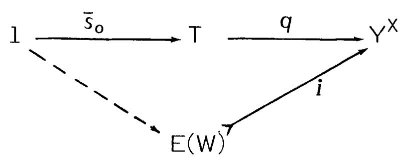
b) If $H \rightarrowtail F$ *is Penon open, then* $E(\Gamma(H)) = H$.

Proof. a) It must be seen that the sub-presheaf defined is actually a sheaf. This is easily done.

b) It must be seen that an arrow $f : T \to F$ factors through H iff it factors through $E(\Gamma(H))$. Now $\Rightarrow$ is immediate. To see $\Leftarrow$ assume $f : T \to F$ factors through $E(\Gamma(H))$. This means that for every global section $\bar{s}_0 : 1 \to T$, $f \circ \bar{s}_0$ factors through H. Now use 1.4 a and the sheaf axiom on H. ◊

1.6. Proposition. *Let* W *be a* C^∞-CO *open subset of* $Z(I, A^n)$. *Then* $E(W) \leqq Y^X$ *is Penon open.*

Proof. We use 1.4. Let us see that E(W) verifies 1.4.a. Take arrows as in the commutative diagram



$T = \overline{C^\infty(\mathbf{R}^k)/K(\bar{s})}$. It follows that $q \circ \bar{s}_0 \in W$. Now q is represented by an

$$[\bar{f}] \in (C^\infty(\mathbf{R}^{k+p})/(K(\bar{s}, \bar{x}) \hat{+} J(\bar{x}, \bar{s})))^n,$$

$[\bar{f}] = ([f_1], \ldots, [f_n])$, and so $q \circ \bar{s}_0$ is represented by $[\bar{f}(\bar{s}_0, \bar{x})] \in W$. So, since W is open, it follows that $[\bar{f}(\bar{s}, \bar{x})] \in W$ for every fixed $\bar{s}$ in a certain neighborhood V' of $\bar{s}_0$ in Z(K). Then, calling $V \subset \mathbf{R}^k$ an open set such that $V' = V \cap Z(K)$ we have that $q \circ j_V$ factors through E(W), where

$$j_V : \overline{C^\infty(V)/K\upharpoonright V} \rightarrowtail \overline{C^\infty(\mathbf{R}^k)/K}$$

is the arrow corresponding to the restriction morphism. Let us now see that E(W) verifies 1.4.b. To do this, take arrows $q : T \to Y^X$, $h : T \to E(W)$ and a sequence $\bar{s}_r$ of elements of Z(K) converging to

$$\bar{s}_0 \in Z(K) \quad \text{such that} \quad q \circ \bar{s}_r = i \circ h \circ \bar{s}_r .$$

Let

$$[\bar{f}], [\bar{g}] \in (C^\infty(\mathbf{R}^{k+p})/(K(\bar{s}, \bar{x}) \hat{+} J(\bar{x}, \bar{s}))^n$$

represent h and q respectively. The equality $q \circ \bar{s}_r = i \circ h \circ \bar{s}_r$ means that

$$[\bar{f}(\bar{s}_r, \bar{x})] = [\bar{g}(\bar{s}_r, \bar{s})] \quad \text{in} \quad (C^\infty(\mathbf{R}^p)/J)^n$$

for every $r \in \mathbf{N}$, or, in the other words,

$$\bar{f}(\bar{s}_r, \bar{x}) - \bar{g}(\bar{s}_r, \bar{x}) \in J^n.$$

Now $\bar{g}(\bar{s}_0, \bar{x}) + (\bar{f}(\bar{s}_r, \bar{x}) - \bar{g}(\bar{s}_r, \bar{x}))$ C^∞-CO converges to $\bar{f}(\bar{s}_0, \bar{x})$ as $r \to \infty$. Then

$$[\bar{g}(\bar{s}_0, \bar{x})] = [\bar{g}(\bar{s}_0, \bar{x}) + (\bar{f}(\bar{s}_r, \bar{x}) - \bar{g}(\bar{s}_r, \bar{x}))]$$

converges to $[f(\bar{s}_o, \bar{x})]$. But we know that $[\bar{f}(\bar{s}_o, \bar{x})] = h \circ \bar{s}_o$ is in W. So, $[\bar{g}(\bar{s}_o, \bar{x})] \in W$ or, in other words, $q \circ \bar{s}_o$ factors through E(W). ◊

In order to prove the converse of 1.6 (in the case that J has line determined extensions) we need two lemmas.

1.7. Lemma *(Glueing Lemma). If a sequence f_ℓ of elements of $C^\infty(\mathbf{R}^p)$ and $f \in C^\infty(\mathbf{R}^p)$ are such that for every compact set $K \subset \mathbf{R}^p$ and every $d \in \mathbf{N}$ there exists $L_{K,d} \in \mathbf{R}$ such that*

$$|D^\alpha(f_l - f)| < L_{K,d} \cdot e^{-\ell}$$

in K for $|\alpha| < d$ and $\ell \geqq \ell_o$ for certain $\ell_o \in \mathbf{N}$ then there exists

$$F \in C^\infty(\mathbf{R}^{p+1})$$

such that

$$\begin{cases} F(\bar{x}, s) = f_\ell(\bar{x}) & \text{if } \quad 1/\ell - 1/4\ell(\ell+1) < s < 1/\ell + 1/\ell(\ell+1) \\ F(\bar{x}, s) = f(\bar{x}) & \text{if } \quad s \leqq 0 \end{cases}$$

and $F(\bar{x}, s_o)$ belongs to the ideal generated by $\{f_\ell : \ell \in \mathbf{N}\} \cup \{f\}$ for every fixed $s_o \in \mathbf{R}$.

Proof. We may assume $f = 0$. Take $\varphi \in C^\infty(\mathbf{R})$ such that

$$\text{supp}(\varphi) \subset (-1, 1) \quad \text{and} \quad \varphi([-\tfrac{1}{2}, \tfrac{1}{2}]) = 1.$$

Let us call

$$\varphi_\ell(s) = \varphi(2\ell(\ell+1)(s - 1/\ell)).$$

We have that

$$\text{supp}(\varphi_\ell) \cap \text{supp}(\varphi_k) = \emptyset \quad \text{if} \quad \ell \neq k.$$

It is easily seen that

$$F(\bar{x}, s) = \begin{cases} f_\ell(\bar{x})\,\varphi_\ell(s) & \text{if} \quad s \in \text{supp}(\varphi_\ell) \\ 0 & \text{otherwise} \end{cases}$$

is C^∞ and has the required properties. ◊

1.8. Lemma. *a) Assume J has line determined extensions. Let $\bar{h}_k$ be a sequence of elements of $Z(I, A^n)$ C^∞-CO converging to $\bar{h} \in Z(I, A^n)$. Let $N \subset C^\infty(\mathbf{R})$ be the ideal of all functions vanishing at $1/\ell$ and 0 ($\ell \in \mathbf{N}$), and let $S = C^\infty(\mathbf{R})/N$. (We call S the generic convergent sequence). Then there exists a subsequence $\bar{h}_{k_\ell}$ of $\bar{h}_k$ and an arrow*

$$F : S \to Y^X \quad \text{such that} \quad F \circ 1/\ell = \bar{h}_{k_\ell} \quad \text{and} \quad F \circ 0 = \bar{h},$$

where $1/\ell : 1 \to S$ and $0 : 1 \to S$ are the arrows corresponding to evaluation at $1/\ell$ and 0 respectively.

b) Let J be <u>any</u> ideal [of local character] and $I = \{0\}$. Let $\bar{h}_k$

be a sequence of elements of $Z(\{0\}, A^n) = A^n$ C^∞-CO *converging to* $\bar{h} \in A^n$. *Then, there exists a subsequence* $\bar{h}_{k_\ell}$ *of* $\bar{h}_k$ *and an arrow*

$$F : R \to \overline{C^\infty(\mathbf{R}^n)} \text{ such that } F \circ 1/\ell = \bar{h}_{k_\ell} \text{ and } F \circ 0 = \bar{h},$$

where $R = C^\infty(\mathbf{R})$ *is the line.*

Proof. We prove only point a. Point b follows similarly although more directly. By 1.3, there exists a sequence $\{\bar{f}_k\} \subset C^\infty(\mathbf{R}^p)^n$ and $\bar{f} \in C^\infty(\mathbf{R}^p)^n$ such that $\bar{f}_k$ C^∞-CO converges to $\bar{f}$ and $[\bar{f}_k] = \bar{h}_k$, $[\bar{f}] = \bar{h}$. Let us take a subsequence $\bar{f}_{k_\ell}$ of $\bar{f}_k$ such that

$$|D^\alpha(f^i_{k_\ell} - f^i)| < e^{-\ell} \qquad (1 \leq i \leq n)$$

in $[-\ell, \ell]$ for every α such that $|\alpha| \leq \ell$. Thus, by 1.7, there exists an $\bar{F} \in C^\infty(\mathbf{R}^{p+1})$ such that

$$\begin{cases} \bar{F}(\bar{x}, s) = \bar{f}_{k_\ell}(\bar{x}) & \text{for } s \in (\frac{1}{\ell} - \frac{1}{4\ell(\ell+1)}, \frac{1}{\ell} + \frac{1}{4\ell(\ell+1)}) \\ \bar{F}(\bar{x}, s) = \bar{f}(\bar{x}) & \text{for } s \leq 0 . \end{cases}$$

We will show that this $\bar{F}$ defines an arrow $F : S \to Y^X$. As it happens, such an arrow is a zero of I in

$$C^\infty(\mathbf{R}^{p+1})/(N(s, \bar{x}) \hat{+} J(\bar{x}, s))^n$$

(Recall that $\hat{+}$ means "local nature closure of the sum"). So, we must show that

$$[\bar{F}] \in (C^\infty(\mathbf{R}^{p+1})/(N(s, \bar{x}) \hat{+} J(\bar{x}, s))^n$$

is a zero of I. Take $g \in$ I. We have that

$$g([\bar{F}]) = [g(\bar{F})] \in C^\infty(\mathbf{R}^{p+1})/(N(s, \bar{x}) \hat{+} J(\bar{x}, s))$$

(this is the C^∞-ring structure in a quotient of this type, see [4]). And

$$g(\bar{F}(\bar{x}, 0)) = g(\bar{f}(\bar{x})) \in J$$

and for every

$$\ell \in \mathbf{N} \quad \text{and} \quad s \in (\frac{1}{\ell} - \frac{1}{4\ell(\ell+1)}, \frac{1}{\ell} + \frac{1}{4\ell(\ell+1)}),$$

$\bar{g}(\bar{F}(\bar{x}, s)) = g(\bar{f}_{k_\ell}(\bar{x})) \in J$. Now from 0.6.b, it follows that $g(\bar{f}_{k_\ell})$, $g(\bar{f})$ satisfy the hypothesis of 1.7 (because $\bar{f}_{k_\ell}$, $\bar{f}$ do). Call $G \in C^\infty(\mathbf{R}^{p+1})$ the function given by 1.7 :

$$\begin{cases} G(\bar{x}, s) = g(\bar{f}_{k_\ell}(\bar{x})) & \text{if } \frac{1}{\ell} - \frac{1}{4\ell(\ell+1)} < s < \frac{1}{\ell} + \frac{1}{4\ell(\ell+1)} \\ G(\bar{x}, s) = g(\bar{f}(\bar{x})) & \text{if } s \leq 0 \end{cases}$$

and for every fixed $s_0 \in \mathbf{R}$, $G(\bar{x}, s_0) \in J$. Since J has line determined extensions, it follows that $G(\bar{x}, s) \in J(\bar{x}, s)$. On the other hand,

$$g(\bar{F}) - G \in C^{\infty}(\mathbf{R}^{p+1})$$

is a function flat on $\mathbf{R}^p \times (\{1/k : k \in \mathbf{N}\} \cup \{0\})$. So, by 0.5,

$$(g(\bar{F}) - G) \in N(s, \bar{x}).$$

So

$$g(\bar{F}) = G + (g(\bar{F}) - G) \in N(s, \bar{x}) \hat{+} J(\bar{x}, s).$$

It is immediate to verify that the arrow F just defined verifies

$$F \circ 1/\ell = \bar{h}_{k_\ell} \qquad \text{and} \qquad F \circ 0 = \bar{h}\,.$$ ◊

1.9. *Proposition.* *Assume* J *has line determined extensions and let* U *be a Penon open subobject of* Y^X. *Then* $\Gamma(U)$ *is a* C^∞-CO *open subset of* $Z(I, A^n)$.

Proof. Suppose $\Gamma(U)$ is not C^∞-CO open in $Z(I, A^n)$. This means that there is a sequence $\bar{h}_k$ of elements of $Z(I, A^n) \backslash \Gamma(U)$ C^∞-CO converging to a certain $\bar{h} \in \Gamma(U)$. By 1.8.a, there exist a subsequence $\bar{h}_{k_\ell}$ of $\bar{h}_k$ and an arrow

$$F : S = \overline{C^\infty(R)/N} \to Y^X \quad \text{such that} \quad F \circ 1/\ell = \bar{h}_{k_\ell} \text{ and } F \circ 0 = \bar{h}.$$

Now, since U is Penon open, we have from 1.4 that there exists an open neighborhood V of $0 \in \mathbf{R}$ such that $F \circ j_V$ factors through U. This is a contradiction. ◊

1.10. *Proposition.* *Let* J *be any local character ideal and* $I = \{0\} \subset C^\infty(\mathbf{R}^n)$. *Let* U *be a Penon open subobject of* $Y^X = \overline{C^\infty(R^n)}^X$. *Then* $\Gamma(U)$ *is a* C^∞-CO *open subset of*

$$A^n = (C^\infty(\mathbf{R}^p)/J)^n = Z(\{0\}, A^n).$$

Proof. Similar to the proof of 1.9 (use 1.8.b instead of 1.8.a). ◊

From 1.5, 1.6 and 1.9 it follows :

1.11. *Theorem.* *Let*

$$X = \overline{C^\infty(\mathbf{R}^p)/J} = \bar{A}, \quad Y = \overline{C^\infty(\mathbf{R}^n)/I} = \bar{B}$$

and let us assume that J *has line determined extensions. Then the mapping* $U \mapsto \Gamma(U)$ *from the set of subobjects of* Y^X *to the set of subobjects of* $Z(I, A^n)$ *determines a bijection between the set of Penon open subobjects of* Y^X *and the set of* C^∞-CO *open subsets of* $Z(I, A^n)$.

Example. An easy instance of 1.11 is D^D ($D = \overline{C^\infty(\mathbf{R})/(X^2)}$). One may see

that its open subobjects "coincide" with usual open subsets of **R**.
As it was said in the Introduction the hypothesis on J of having line determined extensions is essential : it cannot be avoided in general (see Example 1.14 below). However Theorem 1.11 holds in some cases for ideals J not having line determined extensions. This is the case for instance, if the ideal I is $\{0\}$.

1.12. *Theorem.* *Let*

$$X = \overline{C^\infty(\mathbf{R}^p)/J} = \bar{A}, \quad Y = \overline{C^\infty(\mathbf{R}^n)} = \bar{B},$$

where J is any ideal of local character . Then the mapping $U \mapsto \Gamma(U)$ *from the set of subobjects of* Y^X *to the set of subsets of* A^n *determines a bijection between the set of Penon open subobjects of* Y^X *and the set of* C^∞*-CO open subsets of* A^n*.*

Proof. Follows from 1.5, 1.6 and 1.10.

1.13. Examples. i) Consider R^R in the Dubuc topos, where $R = \overline{C^\infty(\mathbf{R})}$ is the line. In this case, Theorem 1.12 just says that Γ establishes a bijection between the set of Penon open subobjects of R^R and the set of C^∞-CO open subsets of $C^\infty(\mathbf{R})$. This was conjectured by M. Bunge at the workshop which took place in Aarhus in June 1983 and answered independently by I. Moerdijk and the author.

ii) Let $\Delta = \overline{C^\infty(\mathbf{R})/J}$ where J is the ideal of all $f \in C^\infty(\mathbf{R})$ such that f vanishes in a neighborhood of $0 \in \mathbf{R}$. R^Δ is the *internal* ring of germs at 0 of smooth one-variable functions. By 1.12, the Penon topology of R^Δ "coincides" with the C^∞-CO topology on $C^\infty(\mathbf{R})/\Delta$ which is the set-theoretical ring of germs at 0 of smooth one variable functions.

1.14. Example. Let $w \in C^\infty(\mathbf{R}^2)$ be a function vanishing in

$$C = \{(x, y) \in \mathbf{R}^2 : |x| \leq |y|\} \cup \{(x, y) \mid y = 0\}$$

and different from zero everywhere else. Let $I \subset C^\infty(R^2)$ be the ideal generated by w, and $J \subset C^\infty(\mathbf{R})$ be the ideal of all smooth functions vanishing in a neighborhood of $0 \in R$. Let

$$X = \overline{C^\infty(R)/J} \quad \text{and} \quad Y = \overline{C^\infty(\mathbf{R}^2)/I}.$$

We have that

$$\Gamma(Y^X) = \{([f_1],[f_2]) \in (C^\infty(R)/J)^2 : w(f_1(x), f_2(x)) \text{ vanishes in a neighborhood of } 0 \in \mathbf{R}\}$$

Let $V \subset \Gamma(Y^X)$ be the set

$$V = \{([f_1],[0]) : \frac{df_1}{dx}(0) \neq 0\} \subset \Gamma(Y^X).$$

Our example is $E(V) \rightarrowtail Y$ (see 1.5) : $E(V)$ is Penon open in Y^X while it is easily seen that $\Gamma(E(V)) = V$ is not C^∞-CO open in $\Gamma(Y^X)$. In order to see that $E(V)$ is Penon open we need the following lemma, whose proof we omit.

1.15. *Lemma.* *Let* V, C, X, Y *be as above and*

$$\overline{F} = (F_1, F_2) \in C^\infty(R^{k+1})^2$$

be such that for certain $\bar{s}_0 \in R$, $[\overline{F}(\bar{s}_0, x)] \in V$, *but there exists a sequence* $\bar{s}_r$ *of points of* R , $\bar{s}_r \to \bar{s}_0$ *as* $r \to \infty$ *such that* $[\overline{F}(\bar{s}_r, x)] \in \Gamma(Y^X)\backslash V$. *Then there exist a sequence* x_r *of real numbers* $x_r \to 0$ *as* $r \to \infty$ *and* $r_0 \in \mathbf{N}$ *such that for* $r \geqq r_0$ *we have* $\overline{F}(\bar{s}_r, x_r) \notin C$. ◊

Let us now see that $E(V)$ is Penon open. We use 1.4. Let us see first that $E(V)$ verifies 1.4.b. Take $T = \overline{C^\infty(R^k)/K}$, a pair of arrows : $q : T \to Y^X$, $h : T \to E(V)$ and a sequence $\bar{s}_r$ of elements of $Z(K)$ converging to $\bar{s}_0 \in Z(K)$ such that

$$q \circ \bar{s}_r = i \circ h \circ \bar{s}_r.$$

Let us assume that q and $i \circ h$ are represented by

$$[\bar{f}], [\bar{g}] \in (C^\infty(R^{k+1})/(K(\bar{s}, x) \mathbin{\hat{+}} J(x, \bar{s})))^2$$

respectively. It follows that $q \circ \bar{s}_r$, $i \circ h \circ \bar{s}_r$ are represented by

$$[\bar{f}(\bar{s}_r, x)], [g(\bar{s}_r, x)] \in \Gamma(Y^X) \subset (C^\infty(R)/J)^2.$$

We have

$$\bar{f}(\bar{s}_r, x) - \bar{g}(\bar{s}_r, x) \quad \in J^2 \quad \text{for all} \quad r \in \mathbf{N},$$

then

$$\bar{f}(\bar{s}_0, x) - \bar{g}(\bar{s}_0, x) \in \text{closure } (J^2).$$

Since closure(J) is the ideal of all flat functions at $0 \in R$ and $[\bar{g}(s_0, x)] \in V$ it follows that $\bar{f}(\bar{s}_0, x) \in V$, as the reader may check (use that $w(\bar{f}(\bar{s}_0, x))$ must vanish in a neighborhood of $0 \in R$).

Now, let us see that $E(V)$ verifies 1.4.a. Take $T = \overline{C^\infty(R^k)/K}$, an arrow $q : T \to Y^X$ and $\bar{s}_0 : 1 \to T$, $\bar{s}_0 \in Z(K)$ such that $q \circ \bar{s}_0$ factors through $E(V)$, i.e., $q \circ \bar{s}_0 \in V$. We have that q is represented by an element

$$[\overline{F}] \in Z(I, (C^\infty(R^{k+1})/(K(\bar{s}, x) \mathbin{\hat{+}} J(x, \bar{s})))^2)$$

and so, $q \circ \bar{s}_0$ is represented by

$$[\overline{F}(\bar{s}_0, x)] \in \Gamma(Y^X).$$

We must show that there exists an open neighborhood W of $\bar{s}_0$ in **R** such that $q \circ j_W$ factors through $E(V)$. Now, by 1.5, the condition

" $t \circ j_W$ factors through $E(V)$ "

means

$$" q \circ \bar{s} \in V \text{ for every } \bar{s} \in W \cap Z(K) ".$$

Assume that such W does not exist. This means that there exists a sequence $\bar{s}_r$ of points of $Z(K) \subset \mathbf{R}^k$ converging to $\bar{s}_o$ and such that

$$\bar{F}(\bar{s}_r, x) \in \Gamma(Y^X) \setminus V.$$

By 1.15, it follows that there exists a sequence x_r of real numbers which tends to zero as $r \to \infty$ such that $\bar{F}(\bar{s}_r, x_r) \notin C$ (i.e., $w(\bar{F}(\bar{s}_r, \bar{x}_r)) \neq 0$) Now, we know that $w(\bar{F}) \in K(\bar{s}, x) \hat{+} J(x, \bar{s})$ and so, since the functions of J vanish in a neighborhood of 0, there exists a neighborhood W of $(s_o, 0) \in \mathbf{R}^{k+1}$ such that in W, $w(\bar{F})$ = an element of $K(\bar{s}, x)$. For some $\varepsilon > 0$

$$(s_o^1 - \varepsilon, s_o^1 + \varepsilon) \times \dots \times (s_o^k - \varepsilon, s_o^k + \varepsilon) \times (-\varepsilon, \varepsilon)$$

is contained in W, and so we should have $w(\bar{F}(\bar{s}_r, x)) = 0$ for $x \in (-\varepsilon, \varepsilon)$ and every $r \geq r_o$ for some $r_o \in \mathbf{N}$. This is a contradiction. ◊

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