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A CONVENIENT SETTING FOR HOLOMORPHY

*by A. KRIEGL and L.D. NEL**

RÉSUMÉ. On montre que les applications Fa-holomorphes entre espaces localement convexes convenables (= bornologiques, séparés et Mackey complets) donnent un cadre pour l'holomorphie qui a plusieurs propriétés intéressantes. (a) La classe de ces espaces est assez grande pour contenir à peu près tous les espaces particuliers importants, en particulier les espaces de Fréchet. Elle est aussi assez restreinte pour que tous les espaces soient tonnelés. (b) On vérifie qu'une fonction $f: E \to F$ entre ces espaces est Fa-holomorphe en composant avec les formes linéaires $\ell : F \to \mathbf{C}$, ou en composant avec les courbes holomorphes $h : \mathbf{D} \to E$, ou en composant avec les inclusions linéaires de Banach $E_B \to E$. (c) La Fa-holomorphie (notion introduite par L. Fantappié il y a plus de 50 ans et oubliée par la plupart des auteurs) se réduit dans des situations particulières importantes à d'autres notions moins bonnes mais plus étudiées. (d) La catégorie *CLC* formée des espaces précédents et des applications linéaires Fa-holomorphes (= applications linéaires continues) est (co-)complète et admet une loi exponentielle $[E \otimes F, G] \simeq [E, [F, G]]$ qui étend celle connue pour les espaces de Banach. (e) Les applications Fa-holomorphes entre ces *CLC*-espaces admet aussi une loi exponentielle. Mais de plus il y a un plongement plein dans une très bonne catégorie *Holo* (espaces holologiques) à structures initiales et finales et avec une loi exponentielle $(W \times X, Y) \simeq (W, (X, Y))$. L'espace de Fréchet classique $H(\Omega, \mathbf{C})$ apparaît comme espace d'applications holologique canonique. (f) On a une catégorie *CHV* (espaces vectoriels holologiques convenables) qui est isomorphe à *CLC* mais enrichie pour admettre des lois exponentielles externes pour des espaces holologiques non vectoriels. Ceci fournit de nouveaux outils pour l'étude de l'holomorphie. (g) La théorie de la Fa-holomorphie développée ici a une analogie avec la théorie de la différentiabilité. Nous réaffirmons que "holomorphie = différentiablité infinie + **C**-linéarité de la dérivée".

CONTENTS.

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INTRODUCTION.

Several concepts of holomorphy have been used in order to develop the theory in infinite dimensional setting. As will be noted explicitly at the end of Section 2, most of these concepts display pathological behaviour which renders them inadequate as general framework and at best suitable as ad hoc assumptions for specialized results. Although the special results obtained via these concepts constitute significant advances, an appropriate unifying framework still appears to be lacking in the theory. We hope to show that this can be remedied.

There are of course notable exceptions to our remark about the neglect of Fa-holomorphy. One such is the study of [Pizanelli 71] which brought to light several relations as well as contrasts between Fa-holomorphy and other notions. It should also be pointed out that in the book [Colombeau 82] the good behaviour of Fa-holomorphy is indirectly indicated through the study of Silva holomorphy in the extended sense, a concept which may seem quite different from Fa-holomorphy at first glance but reduces to it in our setting.

In Section 1 we assemble some facts about the useful ancillary concept of G-holomorphy to prepare for the further development of Fa-holomorphy in Section 2. Fantappié's concept is the weakest one for which the chain rule works and it emerges as the clear choice to achieve the features mentioned above. However, to realize its full potential, one must apply Fa-holomorphy to functions between suitably complete spaces. Among numerous possibilities, c^{∞}-completeness (equivalent to Mackey completeness and weaker than sequential completeness) turns out to be the appropriate one for our purposes. For Fa-holomorphic maps between *CLC* -spaces pleasing simplifications take place. One can now test $f : E \to F$ for holomorphy not only by composing with linear functionals $\ell : F \to \mathbf{C}$ (the celebrated technique of functional analysis) but alternatively by composing with holomorphic curves $h : \mathbf{D} \to E$ or by composing with linear Banach inclusions $E_B \to E$. These are powerful tools which frequently reduce study of a general map $f : E \to F$ effectively to the simpler situation where either $F = \mathbf{C}$ or E is a Banach space or a space of scalars. Sometimes the reduction is all the way to the scalar situation.

As another important simplification we show that (also in the infinite dimensional situation) holomorphy is nothing but smoothness plus **C**-linearity of the derivative. Accordingly, the complex theory can in large measure be reduced to the real theory. This approach is the more valuable in view of the recent advances of smoothness theory in [Frölicher 82], [Kriegl 82, 83], [Frölicher, Gisin & Kriegl 84] and [Nel 84c]. In fact, the present study was prompted by these papers : we wondered whether a parallel theory can be developed in the complex case. Thus we began this study fore-armed with several conjectures suggested by the smooth case in conjunction with category theoretical considerations. These conjectures generally turned out to be true, although the proofs of certain crucial steps are quite different in the holomorphic case.

Reflexiveness with respect to the external duality

$$(\text{topology} \rightleftarrows \text{bornology})$$

used in the studies of [Hogbe-Nlend 77] and [Colombeau 82] plays an important role in holomorphy. In Section 3 we point out that in the present setting this external reflexiveness is equivalent to reflexiveness with respect to a canonical internal duality. One just has to choose the "right" category of ("convenient") bornological vector spaces : an isomorph of *CLC*.

In the last Section we study an intrinsic holomorphy structure ("holological spaces") : a very pleasant category in which all canonical formalisms and closure properties that one could reasonably wish for are in fact present. From the associated category of vector spaces we carefully construct the subcategory of *CHV* ("convenient holological vector spaces") so as to be a nice category in which all spaces are nice (Mackey complete and functionally separated). It is reassuring to find that this *CHV* is also isomorphic to *CLC*. Thus *CLC* emerges as precisely the Mackey complete linear part of holomorphy. Three different types of structures become unified and new tools for the study of holomorphy become available. All of this is quite analogous to the smooth case.

PRELIMINARIES.

Our standard reference for locally convex spaces will be [Jarchow 81] and the abbreviations we will use are :

lc	locally convex
lcs	locally convex separated topological vector space
nbh	neighborhood
E'	vector space of continuous linear functionals on an lcs E
B(E, F)	vector space of bounded linear maps from E to F supplied with the lc-topology of uniform convergence on bounded subsets
D	open unit disk $\{\lambda \in \mathbf{C} \mid |\lambda| < 1\}$ in **C**.

Next we want to mention the basic concepts developed in [Kriegl 82] for **R**-lcs and how they relate to **C**-lcs : For this one should be aware that a **C**-lcs can equivalently be described as an **R**-lcs with a continuous linear transformation J satisfying $J^2 = -1$ (that represents the multiplication with the imaginary unit *i*). Hence we can use all concepts developed for **R**-lcs equally for **C**-lcs by considering just the underlying **R**-lcs.

An important thing is the von Neumann bornology on an lcs. That is a the family of so called bounded sets, i.e., those subsets that are absorbed by every 0-nbh. For bornologies we refer to [Hogbe-Nlend 77]. All constructions and definitions developed in [Kriegl 82] in terms of the bornology depend only on a basis of the bornology, and most often the basis formed by all (closed) bounded disks was used. But if a set is disked in the underlying real vector space of a **C**-lcs, then it need not be disked in the **C**-lcs. Nevertheless the **C**-disked (closed) bounded sets form again a basis (since the **C**-disked closed 0-nbhs form a basis in every **C**-lcs). Hence in the following constructions for a **C**-lcs it does not matter whether one uses **R**-disks or **C**-disks.

One important construction associated with the bornology is the normed space E_B formed by the vector subspace of the lcs E generated by the bounded disk B and supplied with the Minkowsky-functional for B. This space is **C**-normed iff B is **C**-disked. And B is called a Banach disk iff E_B is complete (hence a Banach space).

In [Kriegl 82] the c^∞-topology on an lcs was defined to be the final topology generated by the smooth curves and was proved to be equal to the final topology with respect to the inclusions $E_B \to E$ for bounded disks B (i.e., the Mackey-closure topology, cf. [Hogbe-Nlend 77]).

Furthermore in [Kriegl 82] the concept of c^∞-completeness was developed and proved to be best-adapted to differentiation theory. A lcs was called c^∞-complete iff (for example) every smooth curve has an antiderivative and it was proved that this is equivalent to the existence of a basis for the bornology that consists of Banach disks B (this is Mackey-completeness, cf. [Hogbe-Nlend 77]; or locally completeness, cf. [Jarchow 81, p. 196]).

We should mention also the concept of b-compact subsets (cf. [Hogbe-Nlend 77]), that are those subsets of an lcs which are compact in one of the normed spaces E_B.

Finally smoothness of a map between lcs was defined in [Kriegl 83] as having all compositions with smooth curves again smooth.

1. G-HOLOMORPHY.

For the definition of G-holomorphy we have to assume that all straight lines have an open intersection with U.

1.1. Definition. Let E and F be lcs, U a subset of E, such that the trace of U on all finite dimensional subspaces is open (i.e., U is finitely open), f a map from U to F. Then f is called **G-holomorphic** (after [Gâteaux 19]) $:\Leftrightarrow$ for all $z \in U$, $v \in E$ the Gâteaux derivatives

$$f'(z).v := \frac{d}{d\lambda}\Big|_{\lambda=0} f(z+\lambda v)$$

exist.

If E is one-dimensional this already gives the appropriate concept of holomorphy :

1.2. *Proposition* [*Bochnak & Siciak 71, p. 81*]. *Let* U *be an open subset of* **C**, F *a* c^∞*-complete lcs, f a map from* U *to* F, *then :*

(1) f is G-holomorphic

$\Leftrightarrow$ *(2) f is continuous and* $\int_{\partial T} f = 0$ *for all triangles* T *in* U

$\Leftrightarrow$ *(3) all complex derivatives* $f^{(n)}(z)$ *exist and*

$$f(z+v) = \Sigma_{n=0}^{\infty} f^{(n)}(z)\frac{v^n}{n!}$$

locally

$\Leftrightarrow$ *(4) f has an antiderivative on every simple connected subset*

$\Leftrightarrow$ *(5) f is smooth (or differentiable) as a map from* $U \subset \mathbf{R}^2$ *into* F, *and the derivative is* C*-linear at every point.*

Remark. This was proved in [Bochnak & Siciak 71, p. 82] under the stronger assumption that F is sequential complete. But sequential completeness was only used for a lemma saying that scalar holomorphy implies holomorphy. This we are going to prove under the relaxed assumption of c^∞-completeness.

1.3. *Lemma* *[Colombeau 82, p. 115]. Let* U *be an open subset of* **C**, F *a* c^∞*-complete lcs, and* $f : U \to F$ *a map, then :*

(1) f is G-holomorphic

$\Leftrightarrow$ *(2) for all* C*-linear functionals* ℓ *the composition* $\ell \circ f$ *is G-holomorphic.*

Proof. (We give a proof different from [Colombeau 82] to emphasize similarity with the corresponding statement for real differentiable curves) The non-trivial implication is ($1 \Leftarrow 2$). So, suppose $\ell \circ f$ is holomorphic for all ℓ. Since $\ell \circ f =: g$ is a map from U to **C** we can use the classical theory to deduce that g' is holomorphic, and

$$v \mapsto \int_0^1 g'(z + tv)\, dt = \frac{g(z+v) - g(z)}{v} =: \text{slope } g(z, v)$$

is holomorphic as well. Therefore

$$\frac{\text{slope } g(z, v_1) - \text{slope } g(z, v_2)}{v_1 - v_2}$$

is locally bounded. But since

$$g(z, v) = \ell\,(\text{slope } f(z, v))$$

we can conclude that slope $f(z, .)$ is locally Lipschitz. And therefore slope $f(z, .)$ is a *M*-Cauchy net, hence converges in F since F is c^∞-complete. This proves that f is G-holomorphic. ◊

($5 \Leftrightarrow 1$) For F = **C** this is the classical Cauchy-Riemann Theorem. So let us reduce the general case to this specific one :

f is holomorphic $\Leftrightarrow$ $\ell \circ f$ is holomorphic for all $\ell \Leftrightarrow \ell \circ f$ is smooth and $(\ell \circ f)'(z)$ is C-linear $\underset{[\text{Kriegl } 83]}{\Longleftrightarrow}$ f is smooth and $f'(z)$ is C-linear. ◊

This lemma is one reason for assuming all lcs to be c^∞-complete. But it should be mentioned that c^∞-completeness is not necessary for this equivalence.

1.4. Example. There exists a non c^∞-complete lcs where nevertheless every scalarly holomorphic curve is already holomorphic. Furthermore there exist different topologies and bornologies on certain vector spaces having the same holomorphic curves.

To see this consider any lcs with countable algebraic dimension, i.e., there are linearly independent points e_i for $i \in \mathbf{N}$ having as linear hull the whole space.

Let us first show that scalarly holomorphic curves c into such a space are holomorphic in some finite dimensional subspace : For this choose linear continuous $\ell_n : E \to \mathbf{C}$ with

$$\ell_n(e_n) = 1 \quad \text{and} \quad \ell_n(e_k) = 0 \quad \text{for} \quad k < n.$$

Then $c_n := \ell_n \circ c$ is holomorphic into $\mathbf{C}$. Now consider $A := \{n \in \mathbf{N} \mid c_n \neq 0\}$ and suppose that this set is infinite. For every $n \in A$ the set

$$Z_n := \{\lambda \in \tfrac{1}{2}\mathbf{D} \mid c_n(\lambda) = 0\}$$

is finite (by the identity Theorem for holomorphic maps). Hence $\bigcup_{n \in A} Z_n$ is countable; on the other hand it has to be equal to $\frac{1}{2}\mathbf{D}$ since every $c(\lambda)$ is a finite linear combination of e_i and therefore $\ell_n(c(\lambda)) = 0$ for n sufficiently large. So we can conclude that $\ell_n \circ c = c_n = 0$ for $n \geqq N$. From this it follows that the image of c lies in the subspace generated by $\{e_n \mid n < N\}$ (Otherwise

$$c(\lambda) = \Sigma_{n=1}^{M} x_n e_n \text{ with } M \geqq N \text{ and } x_M \neq 0,$$

then apply ℓ_M).

There are even infinitely many metrizable non-isomorphic lcs with countable algebraic dimension (take the space of finite sequences in any ℓ^p). And there is a most natural finest lc-topology on such a space induced by $\mathbf{C}^{\mathbf{N}}$, which is the only c^∞-complete bornological topology on a countable dimensional vector space.

This shows that unlike the case of real smooth functions neither the topology nor the bornology is determined by the holomorphic curves. And for two lc-structures on a vector space having the same holomorphic curves it might well be that one is c^∞-complete and the other not.

1.5. *Lemma.* *Let* E *be a* c^∞*-complete lcs,* $b_n \in E$*, then*

(1) $\{r^n b_n \mid n \in \mathbf{N}\}$ *is bounded for all* $0 < r < 1$

$\Longleftrightarrow$ *(2)* $\Sigma_{n=0} \lambda^n b_n$ *converges (and* $\lambda \mapsto \Sigma_{n=0} \lambda^n b_n$ *is holomorphic) on* $\mathbf{D}$.

Proof. ($1 \Rightarrow 2$) Let a_n be bounded, then $\Sigma_{n=0}^{k} \lambda^n a_n$ is a M-Cauchy sequence for $|\lambda| < 1$ and therefore converges in E (it even converges uniformly on compact subsets of $\mathbf{D}$, since

$$\Sigma_{n=k+1}^{\infty} \lambda^n a_n \in \frac{\lambda^{k+1}}{1-\lambda} B \ ;$$

and hence the same is true for the series formed by taking derivatives of the summands. From this it is easy to see that the power series represents a holomorphic curve). Let now $a_n := r^n b_n$ be bounded, then by argument above

$$\Sigma_{n=0}^{\infty} \tau^n b_n = \Sigma_{n=0}^{\infty} \lambda^n a_n$$

represents a holomorphic curve defined on $\lambda =: \tau/r < 1$. Since this is true for all $0 < r < 1$ this part is proved.

(2 $\Rightarrow$ 1) Suppose $\lambda \mapsto \Sigma_{n=0} \lambda^n b_n$ is holomorphic, then

$$\lambda \mapsto \ell(\Sigma_{n=0}^{\infty} \lambda^n b_n) = \Sigma_{n=0}^{\infty} \lambda^n \ell(b_n)$$

is holomorphic on **D** and by the classical theory it converges absolutely, hence the summands have to be bounded, i.e., $\{r^n \ell(b_n) \mid n \in \mathbf{N}\}$ is bounded for $0 < r < 1$. Consequently the same is true for $\{r^n b_n \mid n \in \mathbf{N}\}$. ◊

Another difference from the behaviour of smooth curves into **R**-lcs is :

1.6. *Lemma.* *Let* $c : D \to E$ *be a G-holomorphic map, then there exists a bounded disk* B *such that* c *is locally G-holomorphic from* **D** *into* E_B.

Proof. (As for finite order real differentiability we can proceed as follows :) By the mean-value Theorem it is enough to use for B the closed disked hull of the values of c, c', c'' on some nbh. ◊

Remark. It would be very nice to know whether this E_B can be chosen to be complete even for non c^{∞}-complete lcs E, since this would show that the Banach disks and the holomorphic curves determine each other.

1.7. *Theorem* *[Taylor 39]. Let* U *be open in* **C**, $f : U \to B(E, F)$ *a map such that for all* $z \in E$ *the composition* $ev_z \circ f$ *is holomorphic, then* f *is holomorphic.*

Proof. With ev_z we denote the functional defined on any function space which maps a function f to $f(z)$. We will give a proof different from [Taylor 39], since we want to stress again the connection to the real theory. We have to prove that f is smooth and the R-derivative is C-linear. Since $ev_z \circ f : U \to F$ is holomorphic, it is smooth from $U \subset R^2$ into $F_{\mathbf{R}}$ and the derivative $(ev_z \circ f)'(\lambda)$ is C-linear. The real theory [Kriegl 83] gives us that $f: U_{\mathbf{R}} \to B(E_{\mathbf{R}}, F_{\mathbf{R}})$ is smooth. Since

$$B(E, F) = \{ u \in B(E_{\mathbf{R}}, F_{\mathbf{R}}) \mid u(iz) = iu(z) \}$$

is a closed subspace of $B(E_{\mathbf{R}}, F_{\mathbf{R}})$ and f takes values in this space it is smooth into this space. Finally $ev_z \circ f' = (ev_z \circ f)'$ is **C**-linear, and hence so is f' itself. ◊

The next less simple case is when E is of finite dimension and therefore isomorphic to $\mathbb{C}^n$.

1.8. *Proposition* *(Hartogs'Theorem, cf. [Bochnak & Siciak 71, pp. 84]). Let* U *be open in* $\mathbb{C}^n$, F *be a* c^∞*-complete lcs, and* $f : U \to F$ *a map, then*

(1) f is G-holomorphic

$\Leftrightarrow$ *(2)* $\ell \circ f$ *is G-holomorphic for every continuous* **C***-linear functional* ℓ

$\Leftrightarrow$ *(3) all higher directional derivatives* $f^{(n)}(z)(v_1, \ldots, v_n)$ *exist and*

$$f(z+v) = \Sigma_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z)(v, \ldots, v)$$

locally.

Proof. ($1 \Leftrightarrow 2$) f G-holomorphic $\Leftrightarrow \lambda \mapsto f(z+\lambda v)$ is holomorphic $\overset{}{\underset{(1.3)}{\Longleftrightarrow}} \lambda \mapsto \ell(f(z+\lambda z))$ is holomorphic for all $\ell \Leftrightarrow \ell \circ f$ is G-holomorphic.

($2 \Leftrightarrow 3$) Since both statements can be checked with the continuous linear functionals this is essentially Hartogs' Theorem. ◊

1.9. Definition. Consequently we will use the term **holomorphic** instead of G-holomophic in the case of a finite dimensional domain.

Let us turn now towards the case where E is an arbitrary lcs.

1.10. *Proposition.* *Let* E *and* F *lcs with* F c^∞*-complete,* U *be finitely open in* E *and* $f : U \to F$ *a G-holomorphic map, then :*

(1) f has all G-derivatives of arbitrary order and these derivatives can be calculated by the formula :

$$f^{(n)}(z).(v_1, \ldots, v_n) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} f(z + \Sigma_{k=0}^{n} \lambda_k v_k) \Pi_{k=0}^{n} \lambda_k^{-2} \, d\lambda_1 \ldots d\lambda_n$$

and

$$f^{(n)}(z).(v, \ldots, v) = \frac{n!}{2\pi i} \int_{\partial D} f(z + \lambda v) \lambda^{-n-1} \, d\lambda$$

[Bochnak & Siciak 71, p. 94].

(2) These G-derivatives are multilinear [Dineen 81, p. 55].

(3)
$$f(z+v) = \Sigma_{n=0}^{\infty} f^{(n)}(z).(v, \ldots, v) \frac{1}{n!}$$

for $v \in \{v \mid z + \lambda v \in U$ *for all* $|\lambda| < 1\}$ *[Dineen 81, p. 55].* ◊

Remark. The set described in (3) is open (resp. finitely open, resp. c^∞-open) if U is it.

But nevertheless G-holomorphy in this general setting is too weak a concept, since if we try to prove the chain rule, we recognize that the first function makes a holomorphic curve out of the straight line, with which we start, and we do not know what the behaviour of the second G-holomorphic function on this curve is like.

An example showing that the chain rule fails for G-holomorphic maps is obtained by every G-holomorphic map between Banach spaces which is not holomorphic (e.g., a linear non-continuous map), then (2.12) shows that it is not Fa-holomorphic, i.e., there is a holomorphic curve into E, such that the composition with it is not holomorphic.

2. Fa-HOLOMORPHY.

But as in the case of R-differentiability, we can take this as a motivation to define a stronger concept of holomorphy as follows :

2.1. Definition. Let E and F be c^∞-complete lcs, U a subset of E open in the c^∞-topology, and f a map from U to F, then f is called **Fa-holomorphic** (after [Fantappié 30], cf. [Pizanelli 72]) $:\Leftrightarrow$ $f \circ c$ is holomorphic for all holomorphic $c: \mathbf{D} \to U \subset E$.

One could equally define a map f to be Fa-holomorphic when $f \circ c$ is holomorphic on $c^{-1}(U)$ for all locally defined holomorphic curves c into E.

It should be mentioned here that for metrizable lcs among others the c^∞-topology is the original one.

Furthermore we will show later (2.5) that although the definition above makes sense for subsets whose inverse images under holomorphic curves are open it does not lead to a nice theory.

Obviously every Fa-holomorphic map is G-holomorphic, since straight lines are holomorphic. Although the converse is true for finite dimensional domains it is not true for general Banach spaces [Pizanelli 72, p. 184] : To see this take any linear, non-continuous map between Banach spaces. Then it is obviously G-holomorphic with $f'(z)v = f(v)$, but f is not Fa-holomorphic, since by (2.6) it would then be continuous.

Since the definition of Fa-holomorphy is quite similar to smoothness in the real case, one can translate many of the theorems on smooth functions between real lcs to such Fa-holomorphic functions between complex lcs and use essentially the analogous proofs to verify them. However the propositions on maps between finite dimensional spaces have often different proofs, like Boman's Theorem and the corresponding Hartogs' Theorem.

2.2. *Proposition.* *Let E and F be c^∞-complete lcs, U be a c^∞-open subset of E and $f: U \to F$ a map, then :*

(1) f is Fa-holomorphic

$\Leftrightarrow$ *(2) $\ell \circ f$ is Fa-holomorphic for all continuous linear ℓ.*

Proof. This is an immediate consequence of (1.3). ◊

2.3. Theorem *(Hartogs' Theorem). Let* E_k *and* F *be lcs with* F c^∞*-complete,* U_k c^∞*-open in* E_k *and* $f : U_1 \times U_2 \to F$ *a map, then :*

(1) f is Fa-holomorphic

$\Leftrightarrow$ *(2) f is separately Fa-holomorphic (i.e., the maps* $f(., y)$ *and* $f(x, .)$ *are Fa-holomorphic for all* $x \in U_1$ *and* $y \in U_2$*).*

Proof. This can be reduced immediately to the classical theorem by testing with continuous linear functionals and pairs of holomorphic curves. ◊

It should be mentioned that a c^∞-open subset of $E_1 \times E_2$ need not be open in $c^\infty E_1 \times c^\infty E_2$ except when one factor is finite dimensional!

It can be deduced easily from this theorem that a function on a product is Fa-holomorphic iff its associated function into the function space of Fa-holomorphic maps defined on the second factor and supplied with the lc-topology of pointwise convergence is Fa-holomorphic. But the problem with this topology is that the hom-space is not c^∞-complete, and hence not internal. So we have to find an lcs-topology on this function space which is c^∞-complete and for which the same exponential law is true. The existence of such a structure (proved in 2.14) will show that two lcs may have the same holomorphic structures, in the sense that the identity is Fa-holomorphic in both directions, but not a bornological isomorphism. See also the example (1.4).

Let us next inspect the G-derivative of Fa-holomorphic maps more closely.

2.4. Proposition. *Let* E *and* F *be lcs,* U c^∞*-open in* E *and* $f : U \to F$ *a Fa-holomorphic map, then* $f' : U \times E \to F$ *is Fa-holomorphic and* **C***-linear in the second variable.*

Proof. As in the smooth case we consider the c^∞-locally defined Fa-holomorphic map $(\lambda, z, v) \mapsto z + \lambda v$. By (2.3) the composition with f is holomorphic and hence so is

$$(z, v) \mapsto f'(z)v = \frac{d}{d\lambda}\Big|_{\lambda=0} f(z + \lambda v).$$

The **C**-linearity is already a consequence of G-holomorphy (1.2). ◊

Now let us show that this proposition breaks down if we assume U to have only open inverse images under holomorphic curves :

2.5. Example. There exists a Fa-holomorphic map defined on a (non c^∞-open) subset that has open inverse images under holomorphic curves and the derivative of f is not Fa-holomorphic any more. Let

$$U := \{(x, y) \mid |x| \neq 0 \Rightarrow |y| \neq e^{-1/|x|}\} \subset \mathbf{C}^2,$$
$$U_1 := \{(x, y) \mid |x| \neq 0 \Rightarrow |y| > e^{-1/|x|}\}$$
$$U_2 := \{(x, y) \mid |x| \neq 0 \ \wedge |y| < e^{-1/|x|}\}$$

and $f : U \to \mathbf{C}$ be defined by

$$f(x, y) := 0 \quad \text{for} \quad (x, y) \in U_1 \quad \text{and} \quad f(x, y) := \frac{y}{x} \quad \text{for } (x, y) \in U_2.$$

Let us show first that U is open in the sense mentioned above although it is not open in $\mathbf{C}^2$ and furthermore that f is Fa-holomorphic : For this let $c : \mathbf{D} \to \mathbf{C}^2$ be a holomorphic curve. If $c(0) \in U_1 \setminus \{0\}$ (resp. U_2) then it is obvious that $c \in U_1$ (resp. U_2) locally. This shows at the same time that in these cases $f \circ c$ is locally holomorphic.

So suppose now that $c(0) = 0$. If at least one coordinate of c is locally identical to 0, then c lies on one axis, hence is locally in U and $f \circ c = 0$ is locally holomorphic. So it remains to consider the case where

$$c(\lambda) = (\lambda^n x(\lambda), \lambda^m y(\lambda))$$

with x and y holomorphic and unequal to 0 at 0, say

$$a := 2|x(0)| \neq 0 \quad \text{and} \quad b := \tfrac{1}{2}|y(0)| \neq 0.$$

Since the zeros of a holomorphic curve are isolated we conclude that $x(\lambda) \neq 0$ locally for $\lambda \neq 0$. On the other hand

$$|pr_2(c(\lambda))| = |\lambda|^m |y(\lambda)| \geqq \lambda^m b > e^{-|\lambda|^{-n} a^{-1}} \geqq e^{-|pr_1(c(\lambda))|^{-1}}$$

since

$$|pr_1(c(\lambda))| = |\lambda|^n |x(\lambda)| .$$

Hence $c(\lambda) \in U_1$ locally for $\lambda \neq 0$ and $f \circ c = 0$ is holomorphic.

Now let us show that the derivative is not Fa-holomorphic any more : It is easy to see that

$$f'(x, y)(v, w) = 0 \quad \text{for} \quad (x, y) \in U_1$$

and

$$f'(x, y)(v, w) = f_x(x, y)v + f_y(x, y)w \quad \text{for} \quad (x, y) \in U_2,$$

where

$$f_x(x, y) = -y/x^2 \quad \text{and} \quad f_y(x, y) = 1/y.$$

Hence the derivative along the x-axis $\lambda \mapsto (\lambda, 0)$ in direction $(v, w) = (0, 1)$ is $\lambda \mapsto 1/\lambda$ for $\lambda \neq 0$ and 0 for $\lambda = 0$. This is obviously not holomorphic. ◊

A central result is the following :

2.6. Theorem. *Let f be a multilinear map between c^∞-complete lcs, then :*

(1) f is Fa-holomorphic

$\Leftrightarrow$ *(2) f is bounded on images of compact sets under holomorphic curves*

$\Leftrightarrow$ *(3) f is bounded.*

Proof. Let first f be linear, and restrict furthermore to the situation where E is a Banach space and $F = \mathbb{C}$. Then this proposition is due to [Sebastião e Silva 53]:

(1 ⇒ 3) Let f be a Fa-holomorphic non-bounded (i.e., non-continuous) map, then there exist $x_n \to x$ (= 0 without loss of generality) with $|f(x_n) - f(x)| \geq \varepsilon$ (= 1 without loss of generality). The sequence $n!(x_n - x_{n-1})$ is bounded (by using a subsequence and $x_{-1} := 0$), hence

$$c : \lambda \mapsto \Sigma_{n=0}^{\infty}(x_n - x_{n-1})\lambda^n$$

is an entire curve into E. (Use (1.5) and $a^n/n! \to 0$.) Since $f \circ c$ has to be holomorphic it admits a description

$$f(c(\lambda)) = \Sigma_{n=0}^{\infty} a_n \lambda^n .$$

On the other hand

$$f(c(\lambda)) = \Sigma_{n=0}^{k}(f(x_n) - f(x_{n-1}))\lambda^n + f(\Sigma_{n=k+1}^{\infty}(x_n - x_{n-1})\lambda^n)$$

(by the linearity of f). Hence

$$a_n = f(x_n) - f(x_{n-1}) \quad \text{and} \quad f(x) = f(c(1)) = \Sigma_{n=0}^{\infty} (f(x_{n+1}) - f(x_n)) = \lim_{k \to 0} f(x_k)$$

contradiction.

(3 ⇒ 2) is trivial.

(2 ⇒ 1) Let $c : D \to E$ be holomorphic, then

$$c_1 : \lambda \mapsto \frac{1}{\lambda}(\frac{c(\lambda) - c(0)}{\lambda} - c'(0))$$

is holomorphic, hence

$$\lambda \mapsto \frac{1}{\lambda}(\frac{f(c(\lambda)) - f(c(0))}{\lambda} - f(c'(0))) = (f \circ c_1)(\lambda)$$

is locally bounded. Therefore

$$\frac{f(c(\lambda)) - f(c(0))}{\lambda} \to f(c'(0)),$$

i.e., $f \circ c$ is holomorphic and $(f \circ c)'(0) = f(c'(0))$.

Now let E and F be arbitrary : Since scalarly bounded sets are bounded it is enough to show that $\ell \circ f$ is bounded. Since E_{born} is ultrabornological ($E_{born} = \lim_B E_B$ and the E_B can be choosen to be Banach spaces since E is c^{∞}-complete) it is enough to show that $\ell \circ f$ is bounded on Banach disks, i.e., $\ell \circ f|_{E_B} : E_B \to \mathbb{C}$ is bounded. But since E is assumed to be a Banach space this is a consequence of the special case.

Let us now show it for multilinear maps (or use the fact that all three statements can be checked in each variable separately (2.3)) :

(1 ⇒ 3) Let f be Fa-holomorphic ⇒ f is partially Fa-holomorphic $\underset{(2.6)}{\Longrightarrow}$ f is partially bounded, i.e., $f(z_1, \ldots, z_i, \ldots, z_n)$ is bounded in z_i

for all the other coordinates fixed. This implies that f is bounded. (The map

$$(z_1, \ldots, z_{n-1}) \mapsto (z_n \mapsto f(z_1, \ldots, z_n))$$

is a bounded multilinear map into $B(E_n, F)$, the space of bounded linear maps with its pointwise convergence, but since E_n is c^∞-complete this gives the same bornology as the uniform convergence on bounded subsets, hence f is bounded (cf. [Kriegl 82] and [Kriegl 83]).)

(3 ⇒ 2) is again trivial.

(2 ⇒ 1) since a bounded linear map commutes with the formation of difference quotients and M-limits, it is Fa-holomorphic. ◊

Remark. The example (1.4) shows that unlike the smooth case it is essential to have some kind of completeness of the lcs under consideration. This is another reason for assuming all spaces to be c^∞-complete.

Nevertheless it has to be mentioned that c^∞-completeness is still a concept stronger than necessary for the preceding theorem. The above proof works if E were such that its bornological coreflection E_{born} is ultrabornological (e.g., E is ultrabornological) and F could be totally arbitrary.

A consequence of this theorem is :

2.7. Proposition. *The category of Fa-holomorphic maps between c^∞-complete lcs is equivalent to the category of Fa-holomorphic maps between c^∞-complete bornological lcs. Hence every categorical internal statement we prove for the first category is equally true for the second.*

Proof. The functors that provide the equivalence are the inclusion functor and the bornological coreflection $(\ )_{born}$ (cf. [Jarchow 81, p. 276] under the name $(\ )^{bor}$). One composition is just the identity, the other is only a change of topology leaving the bornology and hence the holomorphic curves unchanged. ◊

Here we should mention some facts on holomorphic functions between Banach spaces that show that G-holomorphy is much closer to holomorphy than is true for the relationship of real G-differentiability to smoothness :

2.8. Proposition. *Let E and F be Banach spaces, U be an open set in E and $f : U \to F$ be a map, then :*

(1) f is Fa-holomorphic

⇔ (2) f is G-holomorphic and continuous

⇔ (3) f is G-holomorphic and locally bounded

⇔ (4) f is G-holomorphic and at every point the first derivative is bounded

⇔ (5) f is locally a convergent series of homogeneous continuous polynomials

⇔ (6) f is C-Fréchet differentiable

$\Longleftrightarrow$ *(7) f is G-holomorphic and in every component of* U *there is at least one point where all G-derivatives of f are continuous multilinear.*

Proof. (1 $\Rightarrow$ 4) (due to [Sebastião e Silva 53]). Since f is Fa-holomorphic so is its derivative (2.4) and by (2.6) it is bounded.

(4 $\Rightarrow$ 5) (due to [Zorn 46]). By (1.7) and the assumption $z \mapsto f'(z)$ is G-holomorphic and therefore

$$v \mapsto f^{(n+1)}(z)(v, v_1, \ldots, v_n) = (f'(.).v)^{(n)}(z).(v_1, \ldots, v_n) = (f')^{(n)}(z)(v_1,\ldots,v_n).v$$

is bounded, which shows that all derivatives are bounded. The rest follows from (1.10.3).

(5 $\Rightarrow$ 2) (due to [Taylor 37]). From (5) it can be concluded by a Baire category argument (cf. [Dineen 81, p. 68]) that these polynomials are locally uniformly continuous, and hence the corresponding series converges uniformly on this nbh (use (1.10.3)) and represents therefore a continuous map.

(2 $\Rightarrow$ 6) is first mentioned in [Graves 35] and first proved in [Taylor 37].

(3 $\Rightarrow$ 6) is due to [Hille 48].

(6 $\Rightarrow$ 1) is the chain rule for Fréchet-differentiable maps.

(2 $\Rightarrow$ 3) is obvious for normed spaces.

(3 $\Rightarrow$ 4) by the representation of f' as an integral in terms of f (1.10.1).

(2 $\Leftrightarrow$ 7) (due to [Zorn 45]). He considers the set

$$U_0 := \{z \in U \mid f \text{ is locally bounded around } z\}$$

and shows that this set is open and closed in U. ◊

2.9. Definition. It is therefore reasonable to use the term **holomorphic** instead of Fa-holomorphic for maps between Banach spaces.

This generalizes immediately to :

2.10. Proposition *(For parts see [Pizanelli 72] and [Colombeau 82]). Let* E *and* F *be* c^∞*-complete lcs,* U *be* c^∞*-open in* E *and* $f : U \to F$ *a map, then :*

(1) f is Fa-holomorphic

$\Longleftrightarrow$ *(2) for all* $\ell \in E'$ *and* B *a Banach disk the map* $\ell \circ f|_{E_B} : E_B \to \mathbb{C}$ *is holomorphic*

$\Longleftrightarrow$ *(3) f is G-holomorphic and* c^∞*-continuous*

$\Longleftrightarrow$ *(4) f is G-holomorphic and bounded on b-compact subsets*

$\Longleftrightarrow$ *(5) f is G-holomorphic and at every point the first derivative is bounded*

$\Longleftrightarrow$ *(6) f is* c^∞*-locally a convergent series of homogeneous bounded polynomials*

$\Longleftrightarrow$ *(7) f is G-holomorphic and in every* c^∞*-component of* U *there is at least one point where all derivatives are bounded multilinear.*

Proof. (1 $\Leftrightarrow$ 2) By (1.6) every holomorphic curve is locally holomorphic into some E_B and by (2.2) it is enough to test with functionals.
Hence all these statements can be reduced to corresponding ones about $\ell \circ f|_{E_B}$. Now the equivalences follow from the corresponding ones in (2.8). ◊

Remark. The last equivalence can be reformulated in the following way : The set of points in a c^∞-open set where a G-holomorphic function is locally Fa-holomorphic is c^∞-clopen. And the same is true for the lcs topology instead of the c^∞-topology (cf. [Pizanelli 72]), since an open connected set is c^∞-connected (qua polygonially connected (use radial nbhs)).

In the real case the finest lcs topology coarser than the topology generated by the smooth curves is the bornological coreflection. It is perhaps surprising that this remains true for holomorphic curves even though they generate a weaker topology than the smooth curves.

2.11. *Corollary.* *Let* E *be a* c^∞*-complete lcs. Let us denote by* hE *the final topology with respect to all holomorphic curves* $c : \mathbb{D} \to E$. *Then the finest lcs topology coarser than* hE *is just the bornological coreflection of* E.

Proof. Given a topology on a vector space E with the property that the vector operations are separately continuous (as it is the case with hE), then there exists a finest lcs topology coarser than this topology. A 0-nbh basis is given by all open absolutely convex sets. As an lcs-topology it is initial with respect to the continuous linear functionals, which have obviously to be continuous for the given topology. So consider the initial topology generated by all linear functionals continuous for the given topology. This is a coarser lcs-topology but has to be finer than the finest locally convex one. Hence it is the finest one.
It only remains to show that the linear functionals continuous with respect to hE are the bounded ones. This follows from (2.6). ◊

There is a difference to the smooth case. Namely it is important that the lcs under consideration be c^∞-complete. Otherwise the conclusion is wrong as example (1.4) shows.
And there is another important difference. The topology of every real Fréchet space is final with respect to the smooth curves. This is no longer true for complex lcs and holomorphic curves (see example (2.5) and also [Ancel 83] for a similar statement).

We now want to provide another tool for carrying over the real theory to the complex case. The main clue consists in the equivalence of Fa-holomorphy to smoothness between the associated real vector spaces plus an algebraic condition, which amounts to saying that the Cauchy-Riemann equations should be fulfilled. We will follow this approach, since it stresses the fact that holomorphy is only an algebraic refinement of smoothness, and since that way we are not bored by recycling proofs of the real theory. Of course, in doing so we will make strong use of the theory already developed for smooth functions (cf. [Kriegl 82 and 83]).

Let us start right away :

2.12. Theorem *(cf. [Nachbin 74, p. 71]). Let E and F be c^∞-complete lcs, U be c^∞-open in E, and $f : U \to F$ a map, then :*

(1) f is Fa-holomorphic

$\Leftrightarrow$ *(2) $f : E_{\mathbf{R}} \supset U \longrightarrow F_{\mathbf{R}}$ is smooth and $f'(z)$ is* **C***-homogeneous for all z.*

Proof. (1 $\Leftarrow$ 2) Let $c : D \to U \subset E$ be holomorphic $\Rightarrow$ c is smooth and $c'(z)$ is **C**-linear (1.2). Hence $f \circ c$ is smooth and

$$(f \circ c)'(z) = f'(c(z)) \circ c'(z)$$

is **C**-linear as composition of two **C**-linear maps, i.e., $f \circ c$ is holomorphic (1.2).

(1 $\Rightarrow$ 2) By [Kriegl 83, Chapter 3] it is enough to show that all real G-derivatives exist and are c^∞-continuous : But since f is G-holomorphic they have to exist (1.10), and they are c^∞-continuous (2.10.2) since they are Fa-holomorphic (2.4) as well. ◊

Now we are able to prove the chain rule for Fa-holomorphic maps :

2.13. Theorem *(Chain rule). Let E, F, G be c^∞-complete lcs, U and V be c^∞-open in E and F, $f : U \to V \subset F$ and $g : V \to G$ be Fa-holomorphic maps, then $g \circ f$ is Fa-holomorphic and*

$$(g \circ f)'(z).v = g'(f(z)).f'(z).v.$$

Proof. Since f and g are Fa-holomorphic, they are smooth between the underlying real lcs, hence the chain rule follows from the corresponding one for smooth maps (cf. [Kriegl 83]). ◊

As a consequence we can prove the exponential law for the category of Fa-holomorphic maps :

2.14. Theorem. *The category of Fa-holomorphic maps between c^∞--complete lcs is cartesian closed.*

Proof. We consider H(E, F), the space of Fa-holomorphic maps from E to F, as an lcs-subspace of $C^\infty(E_{\mathbf{R}}, F_{\mathbf{R}})$ defined in [Kriegl 83]. It is obviously closed, since it is defined by the algebraic equations

$$f'(z)(\lambda\, v) = \lambda\, f'(z)(v).$$

Therefore H(E, F) is c^∞-complete if F is.

Now we have to show that for a map $f : E_1 \times E_2 \to F$ the following are equivalent : $f : E_1 \times E_2 \to F$ is Fa-holomorphic $\Leftrightarrow$ $f^\dagger : E_1 \to H(E_2, F)$

exists as a Fa-holomorphic map.

(1 ⇒ 2) $f^{\dagger}(z) = f(z, .)$ is holomorphic, hence $f^{\dagger}$ has values in the right space and is smooth as such. Furthermore $(f^{\dagger})'(z) = (\partial_1 f)^{\dagger}(z)$ and is therefore **C**-linear. Hence $f^{\dagger}$ is Fa-holomorphic by (2.12).

(1 ⇐ 2) Since $f^{\dagger}$ is Fa-holomorphic it is smooth into $H(E_2, F)$ and hence into $C^{\infty}(E_2, F)$. Therefore f is smooth on the product. Furthermore since the derivative of f is just the sum of its partial derivatives and

$$\partial_1 f(z, y) = (f^{\dagger})'(z)(y) \quad \text{and} \quad \partial_2 f(z,y) = (f^{\dagger}(z))'(y)$$

it is **C**-linear. ◊

See [Kriegl 83] for several results that follow quite easily from this, e.g., that the composition map is Fa-holomorphic.

For the next theorem we have to define F_V for every 0-nbh V in the lcs F (cf. [Jarchow 81, p.]). With F_V one denotes F modulo the kernel of the Minkowsky functional on V. Supplied with this functional F_V is a normed space.

2.15. *Theorem* [*Colombeau 74, p. 146*]. *Let E and F be c^{∞}-complete lcs, U be c^{∞}-open in E and $f : U \to F$ be a map, then :*

(1) f is Fa-holomorphic

⇔ (2) f is Silva holomorphic in the extended sense (after [Sebastiao e Silva 57]), i.e., for every Banach disk B and 0-nbh V, the map $f|_{E_B}: E_B \to F_V$ is locally holomorphic between normed spaces

⇔ (3) f is G-holomorphic and continuous from $c^{\infty}E$ into $c^{\infty}F$

⇔ (4) f is Silva holomorphic with respect to the bornology formed by the b-compact subsets of E, i.e., for every b-compact disk B there is another one K such that $f|_{E_B}: E_B \to F_K$ is holomorphic.

Proof. This was proved in [Colombeau 74, p. 146] under the assumption of sequential completeness. Let us sketch how to obtain the same result for c^{∞}-completeness.

(1 ⇒ 2) is obvious.

(2 ⇒ 1) Use property (2) in (2.10) and define $V := \ell^{-1}(\mathbf{D})$ for the linear functional ℓ we test with. Then $F_V = \mathbf{C}$ and the projection $F \to F_V$ is just ℓ.

(3 ⇒ 1) since $c^{\infty}F \to F$ is continuous.

(1 ⇒ 3) since f is **R**-smooth, hence carries the generating family of $c^{\infty}E$ into the generating family of $c^{\infty}F$.

(1⇔ 4) since by (2.12) we have to consider only **R**-smooth functions for which this was proved in [Kriegl 83, § 3]. ◊

We have used c^{∞}-completeness so far without giving definite reasons for doing so. Now we will show that this is really the appropriate concept :

2.16. *Theorem.* *Let E be an lcs, then :*

(1) E is c^∞-complete
$\Leftrightarrow$ *(2) E is h-closed in every lcs it is contained*
$\Leftrightarrow$ *(3) every holomorphic curve c into a surrounding lcs having all $c^{(n)}(0) \in E$ lies completely in E (and is therefore holomorphic into E)*
$\Leftrightarrow$ *(4) every power series $\Sigma_{n=0}^{\infty}\lambda^n a_n$ with all $a_n \in E$ that converges scalarly (i.e., the images of the finite subsums under linear continuous functionals converge in* **C***) converges in E (and defines therefore a holomorphic curve)*
$\Leftrightarrow$ *(5) Let b_n be bounded in E , then $\Sigma_{n=0}^{\infty}\lambda^n b_n$ converges in E for all $\lambda \in$* **D** *(and defines therefore a holomorphic curve into E).*

Proof. ($1 \Rightarrow 2$) since E is c^∞-complete iff it is c^∞-closed in every lcs it is contained.

($2 \Rightarrow 1$) Let x_n be Mackey-convergent towards x with $x_n \in E$ and $x \in F$. Then by choosing a subsequence we might assume that $2^n e^{3n}(x_{n+1} - x_n)$ is bounded. Now consider

$$\Sigma_{n=0}^{\infty} c_n(\lambda)(x_{n+1} - x_n)$$

where $c_n(\lambda) := \cos(3^n\lambda)$ is an entire map. This series converges in F (that can be assumed to be c^∞-complete) since

$$|c_n(\lambda)| \leqq (e^{3^n im(\lambda)} + e^{-3^n im(\lambda)})\tfrac{1}{2} \leqq e^{3n}.$$

And we call the limit $c(\lambda)$. Furthermore since $c_n(\pi i 3^{-k}/2) = 0$ for $n \geqq k$ we have that $c(\pi i 3^{-k}/2) \in E$. Finally from (2) we conclude that $x = c(0) \in E$.

($1 \Rightarrow 5$) is Lemma 1.5.

($5 \Rightarrow 4$) Let $\Sigma_{n=0}^{\infty}\lambda^n a_n$ be scalarly convergent, i.e., $\Sigma_{n=0}^{\infty}\lambda^n \ell(a_n)$ is convergent in **C**. Hence $\ell(r^n a_n)$ is bounded for all $0 < r < 1$ by (1.5), and therewith so is $r^n a_n$. Therefore by (5) the series converges in E for $\lambda < r < 1$.

($4 \Rightarrow 3$) Let c be holomorphic into F (without loss of generality c^∞-complete), then

$$c(\lambda) = \Sigma_{n=0}^{\infty}\frac{\lambda}{n!}c^{(n)}(0)$$

and by assumption $c^{(n)}(0) \in E$. Now $\ell \circ c$ converges and hence by (4) the sum converges in E, i.e., $c(\mathbf{D}) \subset E$.

($3 \Rightarrow 1$) Let x_n be Mackey-convergent towards x with $x_n \in E$ and $x \in F$ which can be assumed to be (c^∞-)complete without loss of generality. Then without loss of generality $n!(x_{n+1} - x_n)$ is bounded, hence

$$\lambda \mapsto \Sigma_{n=0}^{\infty} \lambda^n (x_{n+1} - x_n)$$

represents a holomorphic curve c in F. Since $c^{(n)}(0) \in E$ it follows from (3) that $x = c(1) \in E$. ◊

2.17. Remark. There are natural statements similar to the ones above but slightly weaker like :

(1') Every curve $c : \mathbf{D} \to E$ which is holomorphic in a surrounding lcs F has all derivatives in E (and is therefore holomorphic into E).

(2') Let $c : \mathbf{D} \to F$ be a holomorphic map in a surrounding lcs, with $c(\mathbf{D} \setminus \{0\}) \subset E$, then $c(0) \in \mathbf{D}$.

(3') Every holomorphic curve c in E has an antiderivative.

(4') Every scalarly holomorphic map (curve) is Fa-holomorphic.

Furthermore there are three important properties that depend also on some form of completeness :

(6) Let f be linear and commuting with convergent power series, then f is bounded.

(6') Let f be linear and Fa-holomorphic, then f is bounded.

(6") Every Fa-holomorphic map with domain in U is $\mathbf{R}$-smooth.

All these statements are true under the assumption of c^∞-completeness :

($1 \Rightarrow 1'$) since the derivatives are M-limits of certain difference-quotients.

($2 \Rightarrow 2'$) is trivial since $c^{-1}(E)$ is closed.

($1' \Rightarrow 2'$) Let $c : \mathbf{D} \to F$ be holomorphic with $c(\lambda) \in E$ for all $\lambda \neq 0$, then $c_1(\lambda) := \lambda\, c(\lambda)$ is holomorphic, and has values in E. Hence by (1'), $c_1'(0) = c(0) \in E$.

($2' \Rightarrow 1'$) Let $c: \mathbf{D} \to E \subset F$ be holomorphic, then

$$\lambda \mapsto c_1(\lambda) := \frac{c(\lambda) - c(0)}{\lambda}$$

is holomorphic, and $c_1(\mathbf{D} \setminus \{0\}) \subset E$. Hence $c'(0) = c_1(0) \in E$ by (2').

($3 \Rightarrow 3'$) Let $c : \mathbf{D} \to E$ be holomorphic, consider an antiderivative c_1 in the completion of E. Then

$$c_1(0) := 0 \in E \quad \text{and} \quad c_1^{(n+1)}(0) = c^{(n)}(0) \in E$$

(by assumption). Hence $c_1(\mathbf{D}) \subset E$ by (3).

($2' \Rightarrow 4'$) Let c be scalarly holomorphic, then c is holomorphic into the completion. Then c is holomorphic into E, since $c^{(n)}(\mathbf{D}) \in E$ by (2').

($4' \Rightarrow 2'$) Let c be holomorphic into a surrounding lcs F with values in E, then by the Hahn-Banach Theorem c is scalarly holomorphic. Hence c is holomorphic into E by (4'), and therefore $c^{(n)}(0) \in E$.

($1 \Rightarrow 6'$) was shown in (2.6) and is true for ultrabornological spaces instead of c^∞-completeness.

($6 \Rightarrow 6'$) since there are more Fa-holomorphic curves then convergent power series.

($6' \wedge 1' \Rightarrow 6$) since in this case every Fa-holomorphic curve defined on $\mathbf{D}$ is a convergent power series.

($6' \Leftrightarrow 6''$) since the proof of (2.12) and all other necessary results are still valid under this assumption.

But unlike the case of R-smooth maps they do not characterize c^∞-completeness. And indeed we give now an example that proves all these properties (together) being strictly weaker than c^∞-completeness :

2.18. Example. There exists a non c^∞-complete lcs E that nevertheless fulfills (1'), (2'), (3'), (4'), (6') and (6"). Let

$$E := \{x \in \mathbf{C}^{\mathbf{N}} \mid \text{density}(\text{supp}\, x) := \lim_{n\to\infty} \frac{\text{card}\{k \leq n \mid x_n = 0\}}{n} \neq 0\}$$

as subspace of $\mathbf{C}^{\mathbf{N}}$. Since E includes the finite sequences it is M-dense in $\mathbf{C}^{\mathbf{N}}$ hence not c^∞-complete.

Now let us consider a curve $c : \mathbf{D} \to E$ that is scalarly holomorphic. Then all $pr_n \circ c =: c_n$ are holomorphic. Suppose $A := \{n \mid c_n \neq 0\}$ has density $\neq 0$. Let

$$Z_n := \{\lambda \in \tfrac{1}{2}\mathbf{D} \mid c_n(\lambda) = 0\}.$$

If $n \in A$ then Z_n is finite. Hence $\bigcup_{n\in A} Z_n$ is countable. Let $\lambda_0 \in \frac{1}{2}\mathbf{D}\setminus\bigcup_n Z_n$ Then $c_n(\lambda_0) \neq 0$ for all $n \in A$, which is a contradiction to $c(\lambda_0) \in E$. This shows that (4') ($\Leftrightarrow$ (1') $\Leftrightarrow$ (2')) as well as (3') is fulfilled.

Since E is ultrabornological as has been proved by Valdivia (6') is true, and together with (1') this implies that (6) is true.

It should be mentioned that there is a relationship to T(aylor) S(eries) completeness as defined in [Dineen 81, p. 128]. He calls a topology τ on H(U, F), for E and F lcs and U balanced open in E, TS-complete iff for every sequence of continuous n-homogeneous polynomials p_n for which $\Sigma_{n=0}^{\infty} |p_n|_q < \infty$ for every τ-seminorm $|\ |_q$ the series $\Sigma_{n=0}^{\infty} p_n$ converges to an element in H(U, F).

Let us show now that : $H(\mathbf{D}, F)_{co}$ is TS-complete $\Leftrightarrow$ F is c^∞-complete. First mark that (continuous) n-homogeneous polynomials from $\mathbf{C}$ into F are exactly $\lambda \mapsto \lambda^n a_n$. And the seminorms in the co-topology are

$$f \mapsto \sup\{|f(\lambda)|_q \mid |\lambda|<r\}$$

with $0 < r < 1$ and q a seminorm of F. Hence

$$\Sigma_{n=0}^{\infty} \sup\{|\lambda^n a_n|_q \mid |\lambda|<r\} < \infty \quad \text{for all } 0< r< 1$$

iff $|r_n a^n|$ is bounded for all $0 < r < 1$, i.e. $r_n a^n$ is bounded.

Hence the TS-completeness is equivalent to (3) ($\Leftrightarrow$ c^∞-complete).

We will now give reasons why Fa-holomorphy behaves better than other concepts of holomorphy. Let us first mention those different concepts we are going to discuss :

2.19. Definition. (1) **S-holomorphy** in the restricted sense (after [Sebastiao e Silva 56]), i.e., G-holomorphic and b-locally bounded. (That is for every B and z there exists an $\varepsilon > 0$ such that $f(z + \varepsilon B)$ is bounded.)

For this concept see especially [Colombeau 82 and 74] and [Lazet 73].

(2) **Hy-holomorphy** (cf. [Dineen 81, p. 60]), i.e., G-holomorphic and continuous on compact subsets.

(3) **T-holomorphy** (after [Taylor 37], cf. [Zorn 45] and [Pizanelli 72]), i.e., G-holomorphic and continuous in the lcs-topology. For this concept see e.g. [Dineen 81] under the name of holomorphy.

(4) **H-holomorphy** (after [Hille 48]), i.e., G-holomorphic and locally bounded.

2.20. Remark. As we have already stated in Theorem (2.8) in Banach spaces these concepts are all equivalent to Fa-holomorphy. But in general they are all different, although we obviously have the simplications

$$H \Rightarrow T \Rightarrow Hy \Rightarrow Fa, \qquad H \Rightarrow S \Rightarrow Fa$$

(for $(H \Rightarrow T)$ see [Dineen 81, p. 59]). Examples that the converse directions are wrong can be found at the following sources (for this and what follows let E and F be c^∞-complete lcs, U an open subset of E and $f: U \to F$ a map ; let us furthermore denote with kE the topology generated by the compact subsets of E) :

(Fa,T$\Rightarrow$S) [Colombeau 82, p. 100].

(Fa,S $\Rightarrow$ Hy) Use an lcs E for which $kE = E$ but $E_{born} \neq E$ like a strong dual of a non-Schwartz Fréchet-Montel space. Then $E \to E_{born}$ provides an example.

(Hy $\Rightarrow$ T) [Pizanelli 72] Take any bi-linear bounded non-continuous map.

(T $\Rightarrow$ H) [Pizanelli 72] Use the identity on non-normable space ; or see [Bochnak & Siciak 71, p. 98] for a map

$$g : \ell^2 \to \mathbf{C}^{\mathbf{N}}, \quad x \mapsto (f(kx))_k$$

where $f : \ell^2 \to \mathbf{C}$ is a holomorphic map that is not bounded on the unit ball, like $f(x) := \sum n x_n^n$.

(S $\Rightarrow$ T) Use the counterexample to (Fa $\Rightarrow$ T) and the fact that (S $\Leftrightarrow$ Fa) for codomain spaces with countable base of bornology [Colombeau 82, p. 91].

Let us mention the most important cases where some of these concepts coincide :

(H $\Leftrightarrow$T) for F normed [Dineen 81, p. 58].

(T $\Leftrightarrow$ Hy) if $E = kE$, as is the case for duals of Fréchet-Montel space or metrizable lcs.

(Hy $\Leftrightarrow$Fa) if $kE = c^\infty E$, as is the case for duals of Fréchet Schwartz spaces or strict LF-spaces.

(H $\Leftrightarrow$ S) for E normed (trivial).

(S $\Leftrightarrow$ Fa) if F has a countable base of bornology [Colombeau 82, p. 91].

2.21. Remarks. The counterexample for (T $\Rightarrow$ H) shows already that

H-holomorphy is definitively not a good concept, since these maps do not even form a category.

Although the other classes do form categories they still have severe drawbacks, and most of the special situations where some of the theorems were proved to be still valid are those where the spaces under consideration force these concepts to be equivalent to Fa-holomorphy :

The most common used concept is that of T-holomorphy. But. as in the case of real smooth functions it has essential disadvantages, since many naturally occuring maps are multilinear and bounded (hence Fa-holomorphic) but not continuous (hence not T-holomorphic). The most important example of such a situation is ev : $E'\times E \to \mathbf{C}$. For non-normable E there does not exist an lc-topology on E' making ev continuous. This shows at the same time that Hartogs' Theorem cannot be true for this class in full generality since obviously ev is separately continuous and linear, hence separately T-holomorphic (even H-holomorphic) but not T-holomorphic. And consequently it is not possible to achieve cartesian closedness for such a category.

That the points where a G-holomorphic map is locally T-holomorphic is not closed can be seen as follows [Pizanelli 72] : Let $f : \ell^2 \to \mathbf{C}$ be a holomorphic map that is not bounded on the unit ball and

$$g(x, y) := \Sigma_{n=0}^{\infty} x_n f(x_0^n z)$$

be defined on $\mathbf{C}^{(\mathbf{N})} \times \ell^2$, then g is G-holomorphic and continuous on a 0-nbh but not on the whole space.

That the continuity of the derivatives of a G-holomorphic map is not sufficient to imply T-holomorphy of the map can be easily seen as follows. Take an arbitrary Fa-holomorphic map defined on a bornological lcs. Then at every point the derivative is continuous. But the map need not be T-holomorphic.

Now one possibility to get rid of at least some of these drawbacks is to use the concept of S-holomorphy as it has been studied in [Lazet 73]. There multilinear bounded maps are S-holomorphic. But for normed domain spaces this is obviously equivalent to the bad behaved T-holomorphy.

Furthermore one has the following counter-examples. Most of them are also counter-examples for T-holomorphy :

- In [Colombeau 82, p. 125]) a counter-example was given that shows that Hartogs' Theorem is no longer true : Let

$$f((x_n), (y_n)) := (\Sigma_{n=0}^{\infty} (k x_n y_n)^n)_k \ , \quad c_0 \times c_0 \to \mathbf{C}^{\mathbf{N}} \ .$$

Then f is G-analytic, separately S-holomorphic but not locally bounded. But, in [Lazet 73, p. 26] it was proved that the theorem is true if the codomain has a countable base of bornology (but this implies already Fa-holomorphy).

- In [Colombeau 82, p. 121] a counter-example is given that Zorn's first Theorem is false in this setting : Let t_n be a strictly increasing sequence in [0, 1],

$$f : c \mapsto \Sigma_{n=0}^{\infty}(c(t_{2n+1}) - c(t_{2n}))^n,\ C([0, 1], \mathbf{C}) \to \mathbf{C}. \qquad I(c) := \int_0^1 c(t)dt$$

and finally

$$F : c \mapsto (\dots, (f(I(c^i)c), \dots)_i,\quad C([0, 1],\ \mathbf{C}) \to \mathbf{C}^{\mathbf{N}}$$

Then F is G-analytic and bounded in a 0-nbh, but not bounded in any nbh of $t \mapsto 2$. In [Lazet 73, p. 28] it is proved that the theorem becomes true if the codomain has a countable base of bornology (but then S $\Leftrightarrow$ Fa).

- Furthermore [Colombeau 82, p. 100] showed that the derivative $f' : U \to B(E, F)$ of a S-holomorphic map f need not be S-holomorphic any more (although the derivatives $f^{(n)} : U\times E\times \dots \times E \to F$ are).

- Finally it is still not possible to test the holomorphy via the continuous linear functionals [Colombeau 82, p. 117], since any Fa-holomorphic map into an lcs with countable base of bornology (like **C**) is already S-holomorphic.

It is important to mention that the concept of S-holomorphy depends only on the bornology of the domain space and is quite easy to generalize to convex bornological spaces (cf. [Lazet 73]). The above examples show that if we are concerned with maps between lcs, then it is not a good idea to use for S-holomorphy the von Neumann bornology. If one uses instead the bornology of b-compact subsets, then one obtains exactly Fa-holomorphy. Or more general : S-holomorphy in the extended sense is equivalent to S-holomorphy on the associated Schwartz cbs [Colombeau 82, p. 90].

Another possibility to get rid of these disadvantages is to use the concept of Hy-holomorphy. This corresponds to the differential calculus developed in [Seip 79 and 81] for maps between real lcs using the continuity of all derivatives on compact subsets. Since we do not know any non trivial example of a Fa-holomorphic map between bornological lcs that is not Hy-holomorphic we cannot decide which of the nice properties of Fa-holomorphic maps fail to be true for Hy-holomorphic ones (cf. [Colombeau 82, p. 425]).

Let us describe a bit the problems in finding counter-examples to this equivalence :

First we want to prove that Hy-holomorphy for bornological lcs is equivalent to G-holomorphy + boundedness on compact subsets (for a counter-example in non-bornological lcs, cf. [Dineen 81, p. 61]). So let f be G-holomorphic and bounded on compact sets ; then the same is true for f', since

$$f'(z)v = \frac{1}{\pi i}\int_{\partial D} f(z + \lambda v)\lambda^{-2}\, d\lambda\ .$$

For fixed v and compact K the set

$$\{ z + \lambda vz \in K \mid \lambda \in \partial D\}$$

is compact as well, and therefore $\{ f'(z)v \mid z \in K\}$ is bounded. Now consider

$$f(y) - f(x) = \int_0^1 f'(x + t(y-x))(y-x)\, dt.$$

Since $(t, x, y) \mapsto x + t(y-x)$ is continuous,

$$\{x + t(y-x) \mid x, y \in K, t \in I\}$$

is compact and therefore f' is bounded on this set. Hence $\{f'(x + t(y-x))\}$ is equicontinuous. Let now V be an arbitrary closed disked 0-nbh in F, then there is a 0-nbh W in E such that $(f'(x + t(y-x))\, W \subset V$. Therefore

$$f(y) - f(x) \in V \quad \text{for} \quad y-x \in W \quad \text{and} \quad x, y \in K,$$

i.e., f is continuous on K.

Now one reason for the above mentioned difficulties is that every Fa-holomorphic (hence bounded) polynomial is Hy-holomorphic (since it is obviously bounded on compact sets as well). Another reason is that (2.2) is equally true for Hy-holomorphic mpas : Let $\ell \circ f$ be Hy-holomorphic for all ℓ, then $\ell(f(K))$ is bounded for all compact K, hence $f(K)$ is bounded for all compact K and by the statement above is f Hy-holomorphic.

Now Fa-holomorphy is equivalent to G-holomorphy + continuity on $c^\infty E$, and Hy-holomorphy is per definition equivalent to G-holomorphy + continuity on kE. Hence if $c^\infty E = kE$ (like in a metrizable space), then Fa-holomorphy and Hy-holomorphy coincide.

The easiest example with $c^\infty E \neq kE$ is a uncountable product $\mathbf{C}^\Gamma$. But for this space the Fa-holomorphic functionals are continuous [Frölicher & Kriegl 83] and therefore Hy-holomorphic, i.e., Fa-holomorphy and Hy-holomorphy are equivalent for such a domain space as well.

The other class of spaces we know with $c^\infty E \neq kE$ are the duals of non-Schwartz Fréchet-Montel spaces [Frölicher & Kriegl 83]. But [Dineen 77, p. 163] conjectures that for such spaces again Fa $\Leftrightarrow$ Hy, and for the standard example of such a space (cf. [Jarchow, p. 233]) he even proved this.

3. REMARKS ON CONVENIENT BORNOLOGICAL VECTOR SPACES.

The "natural reflexiveness" $E = E'^{\times}$ of certain Schwartz spaces plays an important role in holomorphy (see [Colombeau 82]). In this section we point out that this external (ad hoc) reflexiveness is equivalent in our setting to a canonical reflexiveness: $E \simeq E^{**}$, where $E^* = [E, \mathbf{C}]$ is the dual space with respect to an internal hom functor. The background to be provided for this is relevant also for the next section.

We have seen in the preceding sections that Mackey complete bornological separated locally convex $\mathbf{C}$-vector spaces (henceforth called

convenient locally convex spaces) provide a useful setting in which to pursue the study of holomorphy. Let *CLC* denote the category formed by these spaces and continuous linear maps between them. As is well known, a linear function u between such spaces is bornological (i.e., carries bounded sets to bounded sets) iff u is continuous. Moreover, in 2.6 we saw that u is bornological iff u is Fa-holomorphic. Thus the category *CLC* emerges as the meeting point of a number of important concepts. We elaborate on this in the remainder of the paper, building larger categories with canonical formalisms also for non-linear bornological maps and for non-linear Fa-holomorphic maps, but whose linear parts coincide with *CLC*.

Let us begin by recalling the isomorphism between *CLC* and a certain category of bornological vector spaces. *Borno* will denote the category of bornological spaces and bornological maps (sets structured with postulated bounded subsets and functions which preserve boundedness). The scalar field $\mathbb{K}$ (real or complex) carries the obvious bornology, so the vector operations are bornological maps. One now forms in the usual way the category *BV* of bornological $\mathbb{K}$-vector spaces and its subcategory *SBV* of functionally separated *BV*-spaces (i.e., spaces having enough bornological linear functionals to separate points ; such spaces are called regular in [Hogbe-Nlend 77]). Note that "subcategory" means "full subcategory" in this paper. Important among *SBV*-spaces are the canonical function spaces $\ell_\infty(X ; E)$ $(X \in Borno , E \in SBV)$ formed by all bornological maps $f : X \to E$ and carrying the natural bornology.

By a **convenient bornological vector space** [Nel 84b] is meant a Mackey closed bornological vector subspace of some canonical function space $\ell_\infty(X ; \mathbb{K})$. The category *CBV* of these spaces is fully embedded into the category of complete *SBV*-spaces which plays a prominent role in [Hogbe-Nlend 77]. The essential difference between *CBV* and complete *SBV*-spaces lies in the fact that every *CBV* space carries the initial bornology induced by its linear functionals. This has far reaching consequences, as the following shows.

3.1. *Theorem* [*Nel* 84c]. *The categories CLC and CBV are isomorphic.* ◊

The isomorphism 3.1 facilitates the study of *CLC* in several ways. To begin with, it provides a simple way of seeing that *CLC* is a nice category. This is not readily established directly : the genesis of *CLC* makes it a reflexive subcategory (Mackey complete) of a coreflective subcategory (bornological) of a reflective subcategory (separated) of $LC := \{\text{locally convex spaces}\}$. Thus neither its limit not its colimit constructions are in general formed as in the parent category *LC* (where this is simple) and the latter does not even have the wanted internal hom spaces $[E, F]$. The genesis of *CBV* on the other hand is in a sense prototypical of how "nice" categories for functional analysis are formed. Let us elaborate on this, since it will serve also as preparation and motivation for the parallel holomorphic enrichment of

CLC to follow in Section 4.

Let *XV* be a category of **K**-vector spaces, formed over some cartesian closed topological category *X* in the usual way : *X*-spaces structured with vector operations which are *X*-maps and so on. Such *XV* are extremely nice categories (see properties 1.0 through 1.15 in [Nel 84b]). A subcategory *F* of such *XV* is called a **functional analytic category** if : (1) the scalar field **K** lies in *F* , (2) *F* is reflective, and (3) the canonical function spaces [X ; E] (spaces of all *X*-maps X → E) lie in *F* whenever E does. Two features are noteworthy. Firstly, all the mentioned categorical properties of *XV* are inherited by functional analytic subcategories except possibly the creation of regular factorizations by the underlying functor *XV* → *X*. Secondly, one readily forms new functional analytic categories out of given ones by application of the following result.

3.2. *Upgrading Theorem* *[Nel 84b]. Suppose F is a functional analytic subcategory of XV and M is a class of monomorphisms in F which contains all kernels, is preserved by all functors [X ;] and is closed under compositions, intersections and preimages. Then the subcategory* F_M *of all E which admit an M-map E → [X ; K] for some X is again a functional analytic subcategory and every canonical map* ρ_E*: E → E** belongs to M.* ◊

As a standard application we may take *F* to be all of *XV* , *M* to be the "upgrading class" of all monomorphisms. Then F_M is nothing but the subcategory *SXV* of all functionally separated *XV*-spaces. Thus in particular, the above category *SBV* of separated bornological vector spaces is functional analytic. But now we can upgrade it further by choosing

$$M = \{\text{Mackey closed bornological embeddings}\}$$

as upgrading class. This has the properties required of *M* [Nel 84b]. The upgraded category F_M is by definition nothing but *CBV*, and so the latter is functional analytic too.

A second way in which *CBV* facilitates the study of *CLC* arises from the fact that all cartesian products, all projective limits, all cotensor products [X ; **K**] (= ℓ_∞(X ; **K**) here) and all internal hom spaces [E, F] are formed precisely as in the parent category *BV* , where they are transparently simple. For example, [E, F] just carries the natural bornology, but to describe the structure of its counterpart in *CLC* in terms of semi-norms or 0-nbhs is rather complicated.

A third way in which *CBV* facilitates the study of *CLC* stems from the long known fact that for certain purposes bornology provides a more natural setting than topology, as the studies in [Hogbe-Nlend 77] and [Colombeau 82] convincingly show. Let us recall in this connection the internal and external exponential laws [Nel 84b] (applicable to all X in X and all E, F and G in a functional analytic $F \subset XV$):

$$[E\otimes F, G] \simeq [E, [F, G]] \simeq [F, [E, G]],$$
$$[X\odot E, F] \simeq [E, [X ; F]] \simeq [X; [E, F]].$$

In the case of *CBV* these laws provide a canonical formalism particularly favorable for the study of integration spaces. In fact, the external exponential laws can be written in more suggestive notation as

$$[\ell_1(X, E), F] \simeq [E, \ell_\infty(X ; F)] \simeq \ell_\infty(X ; [E, F]),$$

where the functors ℓ_1 and ℓ_∞ generalize the ones familiar from Banach space theory. By using these laws, one derives generalized Riesz representations

$$[L_1(S, E), F] \simeq [E, M(S, F)] \simeq M(S, [E, F])$$

(see [Nel 84c] for background and further references). These results are not as readily proved in the topological context of *CLC*-spaces.

Let us now consider how the external duality

$$LC \rightleftarrows SBV$$

(studied in [Hogbe-Nlend 77]) relates to the intrinsic duality based on the canonical dual spaces E^* in *CBV*. Recall that for a given *LC*-space E, the external dual E' is defined to be the *SBV*-space formed by all continuous linear functionals on E and endowed with the equicontinuous bornology : polars V^o of 0-nbhs are the basic bounded sets. For *SBV*-spaces F the external dual F^x is the *LC*-space formed by all bounded linear functionals and equipped with polars B^o of bounded $B \subset F$ as its basic 0-nbhs. Recall also that under the isomorphism (3.1), a *CLC*-space E is transformed into a *CBV*-space by taking the usual von Neumann bornology and that E^* carries the natural bornology.

3.3. Proposition. *For a CLC-space* E *the external (bornological) dual* E' *carries the same bornology as* E^*, *with* E *transported to CBV via 3.1. For a CBV-space* F *the external (topological) dual* F^x *has the same bounded sets as* F^*. *Hence for CBV-spaces,* $\rho_E : E \to E^{**}$ *is an isomorphism iff* $E = E'^x$.

Proof. For $A \subset E'$ we have (taking E as *CBV*-space when calculating E^*) the following implications. A is bounded in E^* $\Rightarrow$ for all bounded $B \subset E$, A(B) is bounded $\Rightarrow A^o$ is a bornivorous disk in E $\Rightarrow A^o$ is a 0-nbh in E $\Rightarrow A^{oo}$ is bounded in E' $\Rightarrow$ A is bounded in E'. Conversely, for a subset H of E' we have : H is bounded in E'$\Rightarrow H \subset V^o$ for some bornivorous disk V in E $\Rightarrow$ H(B) is bounded for all bounded $B \subset E$ $\Rightarrow$ H is bounded in E^*. It follows that E' and E^* carry the same bornology. For a *CBV*-space F and $H \subset F^x$ we have : H is bounded in $F^x \Leftrightarrow$ for every bounded $B \subset F$, B^o absorbs H $\Leftrightarrow$ for every bounded

$B \subset F$, $H(B)$ is bounded $\Leftrightarrow$ H is bounded in F^*. This shows that F^x and F^* carry the same bornology. ◊

Thus for convenient bornological vector spaces one can replace the external $'^x$-reflexiveness by an equivalent concept of canonical reflexiveness in a functional analytic category, which has the nice properties given in [Nel 84b]. Most of the special spaces considered in bornological studies already lie in the smaller category *CBV*.

4. AN INTRINSIC HOLOMORPHY STRUCTURE.

It was established in 2.14 that the category formed by all Fa-holomorphic maps between *CLC*-spaces is cartesian closed. While this is useful, one ultimately needs a more general category : Riemann surfaces or complex manifolds are usually not even open subspaces of ambient locally convex vector spaces. The category of holological spaces studied in this section provides a comprehensive framework fulfilling these needs. It includes all the Fa-holomorphic maps just mentioned as a fully embedded subcategory while providing the nice stability and closure properties of a topological universe. It is favorable for holomorphic differential calculus in the same way that the topological universe of diffeological spaces is favorable for smooth calculus (see [Nel 84c]).

Recall that

$$\mathbf{D} := \{\lambda \in \mathbf{C} \mid |\lambda| < 1\}.$$

By an affine covering for $\mathbf{D}$ will be meant a family $(\mu_i)_{i \in I}$ of affine maps $\mu_i(\lambda) = \alpha_i \lambda + \beta_i$ with $\alpha_i \neq 0$ ($i \in I$, α_i, $\beta_i \in \mathbf{C}$), such that

$$\mathbf{D} = \cup_{i \in I} \ \mu_i(\mathbf{D}).$$

4.1. Definition. Holological space means a set X structured with functions $\mathbf{D} \to X$, to be called (holomorphic) **imprints** into X, such that the following axioms are satisfied :

4.3 a) Every constant function $\mathbf{D} \to X$ is an imprint.

4.3 b) If $f : \mathbf{D} \to X$ is an imprint and $g : \mathbf{D} \to \mathbf{D}$ is a holomorphic map, then $f \circ g$ is an imprint.

4.3 c) If $f : \mathbf{D} \to X$ is a function such that for some affine covering $\mu_i : \mathbf{D} \to \mathbf{D}$ ($i \in I$) all compositions $f \circ \mu$ are imprints, then f is an imprint.

Holological maps $f : X \to Y$ between holological spaces are functions f such that for every imprint g into X, $f \circ g$ is an imprint into Y. We thus obtain the category *Holo* (holological spaces, holological maps).

4.2. ***Proposition*** *[Nel 84, 2.10 b]. Holo is a topological universe.* ◊

Topological universes are categories of well structured sets, having all initial and final structures, moreover with the pleasant feature that final covering families are stable under pullbacks. They have canonical mapping spaces (X, Y) with exponential law

$$\dagger: (W\times X, Y) \to (W, (X, Y)), \quad \dagger(f)(w)(x) = f^{\dagger}(w)(x) = f(w, x).$$

The inverse of $\dagger$ is denoted $\ddagger$ ($g^{\ddagger}(w, x) = g(w)(x)$). There is always an evaluation map

$$\mathrm{ev} : (X, Y)\times X \to Y, \quad \mathrm{ev}(f, x) = f(x) .$$

In all topological universes the formation of initial structures, final structures and canonical mapping space structures proceeds in the "same" predictable way [Nel 84]. So a reader familiar with these structures from the study of special topological universes such as bornological spaces or convergence spaces should quickly feel at home with holological spaces. We state for convenient reference how products and hom spaces are formed in *Holo*.

4.3. Proposition *(cf. [Nel 84]). (a) The holomorphic imprints into a holological product* W×X *are those functions* $h : \mathbf{D} \to W\times X$ *for which the projections compose to give imprints* $pr_1 \circ h$ *and* $pr_2 \circ h$ *into* W *and* X *respectively.*

(b) The hom space (X, Y) *in Holo consists of all holological maps* X → Y *and its holomorphic imprints are all functions* $h : \mathbf{D} \to (X, Y)$ *such that for every imprint* (g, b) *into* D×X *, the composition* $h^{\ddagger} \circ (g, b)$ *is an imprint into* Y *(or equivalently, such that h is a holological map* D×X → Y*).* ◊

We will suppose every open subset Ω of **C** to be the holological space having all holomorphic maps $\mathbf{D} \to \Omega$ as its imprints. Notice that all imprints become holological maps under this convention and every holological space carries the final holological structure induced by its imprints.

The structure defined for **C** is such that the arithmetical operations of addition, subtraction, multiplication are holological maps **C**×**C** → **C**. So we can routinely form the associated category *HV* of **C**-vector spaces over *Holo*. Note that in this section [X ; F] denotes the canonical mapping space in *HV*. Every *HV*-space E carries an **intrinsic bornology** : A ⊂ E is bounded means $\ell(A)$ is bounded in **C** for every holological linear functional ℓ . Clearly, every holological linear map is bornological with respect to the intrinsic bornologies. Bornological concepts, particularly Mackey convergence and Mackey completeness for *HV*-spaces, will always be with reference to this intrinsic bornology unless explicitly otherwise stated.

We now come to the central concept of this section. We define a

convenient holological vector space to be a *HV*-space which admits holological embedding into some function space $[X ; \mathbf{C}]$ in such a way that it is also a Mackey complete bornological subspace. *CHV* denotes the subcategory of *HV* formed by these spaces. The two main objectives of this section are to show :

(a) that *CHV* is a functional analytic subcategory of *HV* ,
(b) that *CLC* is isomorphic to *CHV* (hence to *CBV*).

To attain (a) is a lengthy matter. We were not able to succeed via applications of the Upgrading Theorem alone (as in the case of *CBV*) because we could not show Mackey closed embeddings (see below) to be closed under composition in the absence of Mackey completeness. So we had to do some upgrading from scratch, patterned after the proof of the Upgrading Theorem. This ad hoc upgrading requires all spaces in sight to be both functionally separated and "derivative complete". Therefore we upgrade *HV* in stages, carefully contrived to let all spaces have these desired properties while keeping the resulting category functional analytic.

We dispose of the first stage quickly : a routine application of the Upgrading Theorem is all that is needed to form the functional analytic category *SHV* of separated *HV*-spaces (see the discussion in Section 3).

The next stage is similar in approach, but technically more complicated. A space E in *SHV* is called **derivative complete** if there exists a map

$$\mathrm{Slope}_E : [\mathbf{D} ; E] \to [\mathbf{D} \times \mathbf{D} ; E]$$

in *SV*, necessarily unique, such that

$$f(\lambda) - f(\mu) = (\lambda - \mu)\,\mathrm{Slope}(f)(\lambda, \mu).$$

Let *DHV* denote the subcategory of *SHV* formed by all spaces E which admit holological embedding into some function space $[X ; \mathbf{C}]$ as a derivative complete subspace. If E is a *DHV*-space, every holological map $f : \mathbf{D} \to E$ has an intrinsic derivative defined by $f'(\lambda) := \mathrm{Slope}(f)(\lambda, \lambda)$.

4.4. Theorem [*Nel* 84c]. *DHV is a functional analytic subcategory of HV.* ◊

Some clarification is needed here since the proof of 4.4 was given for the smooth case in the setting of diffeological spaces. These spaces are defined just like holological spaces : just substitute **R** for **C** and "smooth" for "holomorphic" in the definition. Now 4.4 was derived by purely categorical argument from four purely categorical properties [Nel 84c, 1.1, 1.4] which hold in *Holo* too, by well known classical results (the known analytic proofs of these categorical facts are however quite different in the two cases). The proof of 4.4 (which occupies Section 2 of the cited paper) then draws on properties like completeness and the external exponential law for *SHV* : it applies verbatim to the present context (it was written with such application in mind). One could formulate without problem a result in an abstract category yielding a smooth and holomorphic version of 4.4 as special cases, but this does not seem worth while for such a relatively small, transient part of the theory.

We now embark on the final stage of upgrading towards a functional analytic subcategory of *DHV* in which all spaces are Mackey complete. Note that in this section E* denotes the canonical dual space formed in *HV*. When E lies in a functional analytic subcategory, then so does its dual E* [Nel 84b].

4.5. Proposition. *Every canonical function space* [X ; C] *in HV is Mackey-complete.*

Proof. Take a Mackey Cauchy sequence (f_n) in [X ; C]. Then there is a bounded set M ⊂ [X ; C] and a positive double sequence ε_{nm} convergent to 0 such that $f_n - f_m = \varepsilon_{nm} b_{nm}$, with b_{nm} in M. For each ℓ in [X ; C]*, $\ell(f_n)$ is a Cauchy sequence in **C** whose convergence gives a function

$$f_\infty : [X ; C]^* \to C , \qquad \text{namely } f_\infty(\ell) = \lim_n \ell(f_n).$$

Identifying points $x \in X$ with the linear functionals ρx, we thus have also $f_\infty(x) = \lim_n f_n(x)$ for all x. We have to verify two things :

(a) that the function $f_\infty : X \to C$ is holological, and

(b) that f_n is Mackey convergent to f_∞.

For (a) we compose with imprints h into X and note that

$$(f_\infty \circ h)(\lambda) = \lim_n f_n(h(\lambda)).$$

To get the required holomorphy of $f_\infty \circ h$ it is enough (by classical scalar variable theory) to show the preceding limit to be uniform on compact disks Q ⊂ **C**. To this end we note that

$$\sup_{\lambda \in Q} | f_n(h(\lambda)) - f_m(h(\lambda)) | \leq \varepsilon_{nm} . \sup_\lambda | b_{nm}(h(\lambda)) | .$$

Since the continuous function $|b_{nm} \circ h|$ attains a maximum value on Q at μ (say) we can use $\sup_{nm} |b_{nm}(h(\mu))|$ to conclude uniform Cauchyness on Q. Towards (b), we note first that by the classical theory of dual pairs, applied to [X ; C] and [X ; C]*, we can replace M by its bipolar M^{oo}, a bounded weakly closed disk containing M. Put :

$$\delta_m := \sup_{n \geq m} \varepsilon_{nm}$$

to get a positive sequence convergent to 0 with $\varepsilon_{nm}/\delta_m \leq 1$. The sequence $a_m := (f_m - f_\infty)/\delta_m$ can be verified to lie in the bounded disk M^{oo} and (b) follows at once. ◊

4.6. Proposition. *Every space* E *in SHV carries the initial holological* *structure induced by its linear functionals $\ell \in E^*$.*

Proof. For a function $h: D \to [D ; C]$ we have the following equivalences: h is holological $\Leftrightarrow$ $h^{\ddagger}: D\times D \to C$ is holomorphic $\Leftrightarrow h^{\ddagger}(\ ,\lambda)$ and $h^{\ddagger}(\mu,\ )$ are holomorphic for all $\lambda, \mu \Leftrightarrow h^{\ddagger}(\ ,\lambda)$ is holomorphic for all λ $\Leftrightarrow \rho(\lambda) \circ h$ is holomorphic for all λ (where

$$\rho: D \to [(D, C) ; C], \quad \rho(\lambda)(f) = f(\lambda)).$$

We conclude that $[D ; C]$ carries the initial holological structure induced by the linear functionals of the form $\ell = \rho(\lambda)$. But $[X ; C]$ carries the initial structure of the family

$$[h ; C] : [X ; C] \to [D ; C],$$

where h varies through the imprints into X; this follows because such h form a final covering and the functor $[\ ; C]$ transforms finality into initiality. It follows that $[X ; C]$ carries the initial structure induced by the maps $\rho(\lambda) \circ [h ; C]$ $(\lambda \in D,\ h \in (D, X))$. Since every *SHV*-space is embedded into some $[X ; C]$, the result follows. ◊

The next batch of lemmas prepare the way for the proof that *CHV* is functional analytic. They are mainly concerned with **M-closed embeddings** in *SHV*, i.e., holological embeddings $m : E \to F$ of which the image $m(E)$ is Mackey closed in F.

4.7. Lemma. *Every kernel $k : E \to F$ (equalizer) in SHV is an M-closed embedding.* ◊

4.8. Lemma. *If $n \circ m$ is a composition of M-closed embeddings and n is bornologically initial, then $n \circ m$ is an M-closed embedding.* ◊

4.9. Lemma. *If $m \circ n = j \circ k$ is a pullback diagram in SHV and m is an M-closed embedding, then so is k.*

Proof. A subset is Mackey closed iff if it is closed with respect to the associated Mackey closure topology [Hogbe-Nlend 77] and j (qua bornological linear map) is continuous under this topology. Therefore the preimage of a Mackey closed subspace under j is Mackey closed. The stated result now follows by straight-forward verification. ◊

4.10. Lemma. *If $m : E \to F$ is an M-closed embedding, then so is*

$$[X ; m] : [X ; E] \to [X ; F]. \qquad ◊$$

4.11. Lemma. *If a is a bounded sequence in a Mackey complete SHV-space F, then $\Sigma_{n=0}^{\infty} \lambda^n a_n$ is a Mackey convergent series in F for all λ in D and the sum $f(\lambda)$ defines a holological map $D \to F$.*

Proof. One uses the argument in the proof of Lemma 1.5 in conjunction with 4.6. ◊

4.12. *Lemma.* *If $m : E \to F$ is an M-closed embedding between DHV-spaces and F is Mackey complete, then m is bornologically initial.*

Proof. Suppose contrariwise that there exists $A \subset E$ such that $m(A)$ is bounded in F while A is unbounded in E. Then we can choose an unbounded sequence a_n in A such that for some functional $w \in E^*$ we have

$$|w(a_n)| > n^n.$$

Use 4.11 to define the holological map $f : D \to F$ by

$$f(\lambda) := \Sigma_{n \geqq 0} \lambda^n m(a_n).$$

Since

$$s_k(\lambda) := \lambda^0 a_0 + \dots + \lambda^k a_k$$

lies in E and $f(\lambda)$ is the Mackey limit of the sequence $m(s_k(\lambda))$ in F, it follows by the assumed M-closedness that $f(\lambda) = m(h(\lambda))$ for unique $h(\lambda)$ in E. Since m is a holological embedding, $h : D \to E$ is holological and we conclude that for every functional $v \in E^*$ the series

$$\Sigma_n \lambda^n v(c_n) \qquad \text{where} \qquad c_n := h^{(n)}(0)/n!$$

is convergent to $(v \circ h)(\lambda)$ and $|v(c_n)|^{1/n}$ is a bounded sequence. But for every $u \in F^*$ we have

$$(u \circ f)(\lambda) = \Sigma_{n \geqq 0} \lambda^n u(m(a_n)) = u(m(h(\lambda))) = \Sigma_n \lambda^n (u \circ m)(c_n).$$

By comparing coefficients and cancelling we conclude that $a_n = c_n$. But this means $|w(a_n)|^{1/n}$ is a bounded sequence, a contradiction. ◊

4.13. *Theorem.* *CHV is a functional analytic subcategory of SHV.*

Proof. In view of 4.4 it is enough to show *CHV* is a functional analytic subcategory of *DHV*. (1) The scalar field **C** obviously admits a Mackey closed embedding into [1 ; **C**]. (2) Let us show *CHV* is reflective in *DHV*. Take any space E in *DHV*. Its bidual E^{**} admits by definition a regular monomorphism into [E^* ; **C**] [Nel 84b], so E^{**} lies in *CHV* (4.5, 4.7) ; in particular, E^{**} is Mackey complete. Similarly, E^* lies in *CHV*. Let R(E) denote the intersection (in *DHV*) of all Mackey complete subspaces of E^{**} which contain the image $\rho(E)$. Thus we obtain a factorization

$$E \to R(E) \to E^{**}$$

of ρ_E , say $\rho_E = m \circ \bar{\rho}_E$ where $m : R(E) \to E^{**}$ is a Mackey closed embedding. By 4.5, 4.12 and 4.8, R(E) lies in *CHV*. Now consider any map $u : E \to F$ in *DHV*, with F in *CHV*. Since **C** is reflexive in *DHV* [Nel 84b] and ρ_F is a natural transformation, every ℓ in F^* extends over ρ_F. Hence ρ_F is always bornologically initial. It follows that F is isomorphic to R(F) and $\rho_F : F \to F^{**}$ is an M-closed embedding. By Lemma 4.9, the pullback of ρ_F and u^{**} yields maps $k : G \to E^{**}$ and j

such that $u^{**} \circ k = \rho_F \circ j$ and k is an M-closed embedding. By the universal properties of pullbacks and intersections, u must factor through j and m must factor through k, say $m = k \circ h$. Then $\bar{u} := j \circ h$ furnishes the looked for map such that $\bar{u} \circ \bar{\rho}_E = u$. For uniqueness, suppose we also have $v \circ \bar{\rho}_E = u$. By forming the kernel $w : W \to R(E)$ of v and $\bar{u}$, by applying 4.12 and 4.8, we deduce that w is an isomorphism and $v = \bar{u}$. Thus *CHV* is reflective.
(3) Suppose F is *CHV*. Then ρ_F is an M-closed embedding as we have seen, hence by 4.10 the same holds for $[X ; \rho_F] : [X ; F] \to [X ; F^{**}]$, where X is any holological space. But

$$[X ; F^{**}] = [X ; [F^*, \mathbf{C}]] \simeq [X \otimes F^*, \mathbf{C}] = (X \otimes F^*)^*,$$

by the external exponential law of functional analytic categories. Since every dual space H^* lies in *CHV*, we have $[X ; F^{**}]$ in *CHV*. It now follows, by 4.12 that $[X ; F]$ lies in *CHV*. ◊

In the smooth case, where **R** is the scalar field, Mackey completeness is equivalent to derivative completeness (for *SHV*-spaces). In the complex case it follows as in the smooth case (cf. 4.5 in [Nel 84c]) that Mackey complete implies derivative complete. But the converse fails, as Example 2.18 shows.

4.14. *Theorem.* *CHV* ≃ *CLC* ≃ *CBV*.

Proof. There is, by 2.6, an obvious faithful functor *CLC* → *HV* which restructures every *CLC*-space E into a *HV*-space by specifying its imprints $\mathbf{D} \to E$ to be all holomorphic functions in the classical sense. By 2.6 this functor is in fact full. Therefore the old space and the new space have the same linear functionals and the same bounded sets. Thus E, qua *HV*-space, is functionally separated. Moreover, the *SHV*-space E is again Mackey complete, since this depends only on the bounded sets. Let us show E is embedded into some $[X ; \mathbf{C}]$. As noted in the proof of 4.13, every linear functional $u : E \to \mathbf{C}$ extends over ρ_E. By 4.6, ρ_E must be an embedding. Since $E^{**} = [E^*, \mathbf{C}]$ is embedded into $[E^* ; \mathbf{C}]$, we arrive at an embedding of E into some $[X ; \mathbf{C}]$, automatically Mackey closed by Mackey completeness of E and bornologically initial by 4.12. So we have a functor *CLC* → *CHV*. In the reverse direction, there is an obvious functor which restructures E into a *CLC*-space via its linear functionals, the Mackey completeness deriving again automatically from the common bounded sets. In view of 4.6 it is readily seen that the two functors are mutual inverses. ◊

4.15. *Proposition.* *A function* $f : E \to F$ *between CLC-spaces is Fa-holomorphic iff it is holological between the corresponding CHV-spaces.* ◊

Let us now show that the basic cotensor products of *CHV* over *Holo* reduce to well-known spaces.

Let $H(\Omega, \mathbf{C})$ (Ω open in $\mathbf{C}$) denote the Fréchet space formed by all holomorphic maps $f : \Omega \to \mathbf{C}$ and equipped with the seminorms

$$\|f\|_Q := \sup_{\lambda \in Q} |f(\lambda)|$$

where Q varies through compact disks contained in Ω.

4.16. Proposition. *Under the isomorphism $CLC \simeq CHV$, the Fréchet space $H(\Omega, \mathbf{C})$ in CLC corresponds to the canonical mapping space (cotensor product) $[\Omega ; \mathbf{C}]$ in CHV.*

Proof. The two CHV-spaces in question have, by definition, the same underlying vector spaces. Let us check their holological structures. For functions $f : \mathbf{D} \to H(\Omega, \mathbf{C})$ we have :

f is holomorphic $\Leftrightarrow$ for all $\lambda \in \Gamma$,

$$\lim_{\tau \to \lambda} (f(\tau) - f(\lambda))/(\tau - \lambda) = f'(\lambda)$$

exists in $H(\Omega, \mathbf{C}) \Leftrightarrow$ for all λ and Γ,

$$\lim_{\tau\to\lambda} \sup_{\mu \in \Gamma} \left| [f(\tau)(\mu) - f(\lambda)(\mu)]/(\tau - \lambda) - f'(\lambda)(\mu) \right| = 0 \qquad (1^*)$$

$\Leftrightarrow$ $f^{\ddagger} : \mathbf{D} \times \Omega \to \mathbf{C}$ is holomorphic (2^*) $\Leftrightarrow$ $f : \mathbf{D} \to [\Omega; \mathbf{C}]$ is a *Holo*-map.

Statement (2*) follows from (1*) by the Hartogs' Theorem ; (2*) implies (1*) by an argument based on continuity of the partial derivatives and local compactness of $\mathbf{D}$, which shows the limit to be locally uniform in the remaining variable. ◊

4.17. Holological Hartogs' Theorem. *Let W and X be holological spaces, E a CHV-space and $f : W \times X \to E$ a function. Then f is holological iff f is holological in each variable separately.*

Proof. For the non-trivial implication, assume that f is separately holological. The imprints $(g, h) : \mathbf{D} \to W \times X$ form a final covering hence likewise the family $g \times h : \mathbf{D} \times \mathbf{D} \to W \times X$, where g and h vary through imprints into W and X respectively. For every λ and μ in $\mathbf{D}$ and every linear functional u we have that

$$u \circ f \circ (g \times h)(\lambda, \ ) = u \circ f(g(\lambda), \ ) \circ h$$

and similarly $u \circ f \circ (g \times h)(\ , \mu)$ are holological maps. By the classical Hartogs' Theorem all $u \circ f \circ (g \times h)$ are holological, hence (initiality) so are all $f \circ (g \times h)$, hence (finality) so is f. ◊

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