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LUCIANO STRAMACCIA

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PRO-REFLECTIONS AND PRO-FACTORIZATIONS

by Luciano STRAMACCIA

RÉSUMÉ. Si C est une catégorie finiment complète, alors la catégorie Pro-C de ses pro-objets est complète et équilibrée. De plus cette catégorie a une structure de factorisation pour ses morphismes : (Epimorphisme, Monomorphisme extrêmal). Ce fait est utilisé pour étudier les sous-catégories pro-réflexives de C. On définit sur C une "structure de pro-factorisation".

INTRODUCTION.

After S. Mardešić [11, 12] generalized the concept of shape for an arbitrary category having a dense subcategory, some papers appeared - e.g. [4, 15, 18] - showing the existence of a certain analogy between the theory of epireflective subcategories and that of epidense ones (hereafter called "pro-epireflective"). Since epireflections are well studied by means of factorization structure, the natural conjecture is that an appropriate definition of "pro-factorization structure" would be the right tool to investigate pro-epireflections and, hence, shape theories.

All papers quoted above suggest that there is, in general, a certain advantage in passing from a category C to the category of inverse systems Pro-C, the crucial fact being that pro-epireflections in C become epireflections in Pro-C [15], Theorem 2.4. Actually, it turns out that Pro-C inherits some kind of "pre-properties" from the functor (meta-)category $[C, SET]$, which become effective as soon as C has some tiny structure enrichment. In Section 1 it will be seen that if C has finite limits, then Pro-C has all (small) limits, is balanced, is endowed with a nice (E, M)-factorization structure for morphisms, and the canonical functor (1.1)

$$L : \text{Pro-}C \to [C, SET]^o$$

preserves and reflects all those properties.

The fact that Pro-C is a complete (E, M)-category for morphisms, allows us to define K-perfect factorizations [7, 17] in Pro-C for every pro-epireflective subcategory K of C, and study their effect on C itself. In particular, looking at the trace on C of such factorization in Pro-C, one is led to a notion of pro-factorization structure, useful for handling pro-epireflections.

It is worth noting that most of the results are obtained under the mere assumption that C is finitely complete ; cowell poweredness of C is invoked in order to characterize pro-epireflective hulls.

Recently A. Tozzi [19, 20] has given some contributions on this subject, generalizing to pro-epireflective subcategories many results by Strecker, Herrlich and others (see the references). Her "factorizations in C with respect to Pro- C" are very close to our pro-factorizations.

1. SOME RESULTS CONCERNING Pro-C.

We will assume in what follows that C is a finitely complete category.

Recall that the category Pro-C [1, 2, 3] has as objects all inverse systems in C, indexed over directed sets, and described by the fully faithful functor

1.1. $$L : \text{Pro-}C \to [C, \text{SET}]^{o},$$

where, given $\underline{X} = (X_a, x_{aa'}, A)$ in Pro-C, then

$$L(\underline{X}) = \text{colim } C(X_a, -).$$

$L(\underline{X}) : C \to \text{SET}$ is the functor *pro-represented* by the inverse system $\underline{X}$. Any other functor $F : C \to \text{SET}$ which is naturally isomorphic to some $L(\underline{X})$, $\underline{X} \in$ Pro-C, is called a *pro-representable* functor [3, 15].

Later on we shall need a more explicit description of morphisms in Pro-C, like the one that can be found in [12].

1.2. Proposition. *For a functor* $F : C \to \text{SET}$ *the following are equivalent :*

(i) F *is pro-representable,*
(ii) F *is proper and preserves finite limits,*
(iii) The functor lim.Pro-F : Pro-$C \to$ SET *is representable.*

Proof. The equivalence of (i) and (ii) is stated in ([14], 10.7.6). The equivalence of (i) and (iii) is obvious but interesting to be noted. Pro-F is the extension of F to the pro-categories ; lim : Pro-SET $\to$ SET is the inverse limit functor. ◊

In view of the above proposition, the functor L (1.1) establishes an isomorphism between Pro-C and the category $l\{C, \text{SET}\}^{o}$ of all proper functors from C to SET, which preserve finite limits [13, 14] :

1.3. $$\text{Pro-}C \simeq l\{C, \text{SET}\}^{o}.$$

1.4. The following properties of the category $\{C, \text{SET}\}$ of proper functors and of the embedding $E : \{C, SET\} \to [C, SET]$, are well known; see ([13], pp. 149-154) :

(i) $\{C, SET\}$ is cocomplete and E preserves and reflects colimits.
(ii) E preserves and reflects monomorphisms and epimorphisms.
(iii) $\{C, SET\}$ is balanced ; each epimorphism (resp. monomorphism) is a strong and a strict one and, consequently, an extremal epimorphism.
(iv) $\{C, SET\}$ is an (Epi, Mono)-category for morphisms [9].

1.5. *Proposition.* $I\{C, SET\}$ *is a complete category and the embedding*

$$E' : I\{C, SET\} \longrightarrow \{C, SET\}$$

preserves (and reflects) colimits. In particular E' *preserves (and reflects) epimorphisms.*

Proof. Let $D : I \dashrightarrow I\{C, SET\}$ be a diagram. It has a colimit F in $\{C, SET\}$ by 1.4 (i). To show that F preserves finite limits, one can restrict to the case in which F is a coproduct in $\{C, SET\}$. Since a coproduct is a direct limit of finite coproducts, and in SET direct limits commute with finite limits, the proof is complete. ◊

1.6. *Lemma.* *Let* $m' : G' \to G$ *be a monomorphism in* $\{C, SET\}$. *If* G *preserves finite limits, so does* G' .

Proof. Since m' is mono, then $(1_{G'}, 1_{G'})$ is the pullback of (m', m') in $\{C, SET\}$ and hence in $[C, SET]$, by 1.4 (ii). The assertion then follows from ([14], 7.6.4). ◊

1.7. *Proposition.* *The embedding* E' *preserves and reflects monomorphisms.*

Proof. Suppose $m: F \to G$ is a monomorphism in $I\{C, SET\}$. Let

$$F \xrightarrow{\ t\ } G' \xrightarrow{\ m'\ } G$$

be its (E, M)-factorization in $\{C, SET\}$ by 1.4 (iv). By the Lemma, $m = m'.t$ is also a factorization in $I\{C, SET\}$. Since m is mono, then t is also a mono in $I\{C, SET\}$; on the other hand t is a strict epi (1.4 (iii)) in a category having pushouts ([9], 34 J (h)), so t is a regular epi in $\{C, SET\}$, hence in $I\{C, SET\}$ by 1.5. It follows that t must be an isomorphism, so that $m = m'.t$ is a mono in $\{C, SET\}$. ◊

Since Pro-$C \to I\{C, SET\}^{o}$, we can state the following theorem which summarizes the previous results.

1.8. *Theorem.* Pro-*C* *is a complete category and the functor (1.1)* L : Pro-$C \simeq \{C, SET\}^{o}$ *is such that :*

(i) L *preserves and reflects limits and consequently monomorphisms.*

(ii) L *preserves and reflects epimorphisms.*

In particular, Pro-*C* *is balanced and inherits the morphisms* (E, M)-*factorization structure from* $\{C, \mathrm{SET}\}^o$. ◊

1.9. Remarks. a) The fact that, for a finitely complete *C*, Pro-*C* is complete, was already stated in ([3], 8.9.5) in a dual form, that is, concerning the category Ind-*C* = (Pro-$C^o)^o$. However, no one seems to have used this result till now.

b) The effect of Johnstone and Joyal's [6] concept of "continuous category" on the subject and results above seems to be very interesting, although not clear to me at the present state. I feel however that it is worth to be explored and plan to do in a future paper.

1.10. Definition. A subcategory *K* of *C* is called *pro-epireflective* in *C* if every *C*-object X has a *K-expansion*, that is, there is a $\underline{K} = (K_i, p_{ij}, I)$ in Pro-*K* and a (Pro-*K*)-morphism $\underline{p} : X \to \underline{K}$ with every $p_i : X \to K_i$ a *C*-epimorphism, such that every time an $f : X \to H$, $H \in K$ is given, then there is a unique

$$\underline{g} : \underline{K} \to H \quad \text{with} \quad \underline{g}.\underline{p} = f .$$

2. PERFECT FACTORIZATIONS IN Pro-*C*.

In ([7], VI and V2 (4)) it is stated that a finitely complete (Epi, Extremal mono) category *C* has, for every epireflective subcategory *K*, a related *K*-perfect factorization structure. A similar correspondence holds between pro-epireflective subcategories of *C* and certain "pro-factorization" structures, provided *C* is a finitely complete category, as we continue to assume.

2.1. We need some notation. Let Ω be a class of objects of Pro-*C* ; then

a) $\underline{E}(\Omega)$ is the class of all (Pro-*C*)-epimorphisms which are Ω-*extendable* ; this means that $\underline{e} : \underline{X} \to \underline{Y}$ belongs to $\underline{E}(\Omega)$ iff $\underline{e}$ is an epimorphism and, for every $\underline{f} : \underline{X} \to \underline{K}$, $\underline{K} \in \Omega$, there is some

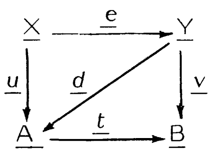
$$\underline{g} : \underline{Y} \to \underline{K} \quad \text{such that} \quad \underline{g}.\underline{e} = \underline{f}.$$

b) $\underline{P}(\Omega)$ is the class of all Ω-*perfect* (Pro-*C*)-morphisms. $\underline{t} : \underline{A} \to \underline{B}$ is in $\underline{P}(\Omega)$ iff, given morphisms $\underline{u}$, $\underline{v}$, $\underline{e}$, where

$$\underline{e} \in \underline{E}(\Omega) \quad \text{and} \quad \underline{v}.\underline{e} = \underline{t}.\underline{u},$$

then there exists a (unique) *diagonal* $\underline{d}$ such that

$$\underline{d}.\underline{e} = \underline{u} \quad \text{and} \quad \underline{t}.\underline{d} = \underline{v} .$$



As usual we write $\underline{P}(\Omega) = \underline{\Lambda}\underline{E}(\Omega)$, where $\underline{\Lambda}$ denotes the "lower diagonalization" operator ([17], §2).

From now on let K be a (full, isomorphism-closed) pro-epireflective subcategory of C, whence Pro-K is an epireflective subcategory of Pro-C (see [15]).

2.2. Proposition. *(i)* $\underline{E}$(Pro-K) = $\underline{E}(K)$.
(ii) $\underline{P}$(Pro-K) = $\underline{P}(K)$.

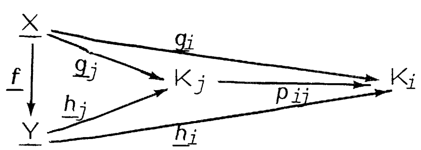
Proof. (ii) is a consequence of (i). As for (i) : since $K \subset$ Pro-K, $\underline{E}$(Pro-K) is included in $\underline{E}(K)$. Let $\underline{f} : \underline{X} \to \underline{Y}$ be a (Pro-C)-morphism which is K-extendable and let $\underline{g} : \underline{X} \to \underline{K}$, $\underline{K} \in$ Pro-K. Since

$$\underline{g} = \{ \underline{g_i} : \underline{X} \to K_i \mid i \in I \}$$

is a natural source in Pro-C [9] and since every K_i is in K, then for every $i \in I$, there is an

$$\underline{h_i} : \underline{Y} \to K_i \qquad \text{with} \qquad \underline{h_i} . \underline{f} = \underline{g_i} \ .$$

If $i \leq j$ in I, look at the diagram



where

$$p_{ij} \cdot \underline{g_j} = \underline{g_i} \ , \quad \underline{h_i} . \underline{f} = \underline{g_i} \quad \text{and} \quad \underline{h_j} . \underline{f} = \underline{g_j} .$$

Then one has

$$(p_{ij} . \underline{h_j}) . \underline{f} = p_{ij} . (\underline{h_j} . \underline{f}) = p_{ij} . \underline{g_j} = \underline{g_i} = \underline{h_i} . \underline{f},$$

hence

$$p_{ij} . \underline{h_j} = \underline{h_i} \ ,$$

since $\underline{f}$ is an epimorphism. It follows that $(\underline{h_i})_I$ gives a (Pro-C)-morphism $\underline{h} : \underline{Y} \to \underline{K}$ such that $\underline{h} . \underline{f} = \underline{g}$. ◊

2.3. Proposition. *If K is a pro-epireflective subcategory of C, then it induces on* Pro-C *a factorization structure* $(\underline{E}(K), \underline{P}(K))$ *for morphisms.*

Proof. Since Pro-K is epireflective in Pro-C [15] and since Pro-C is a (finitely) complete (Epi, Extremal mono)-category, one can use ([7], V2 (4)) to show that Pro-C is an ($\underline{E}$(Pro-K), $\underline{P}$(Pro-K))-category ; then apply the proposition above. ◊

2.4. Lemma. *Let $\underline{e} : \underline{X} \to \underline{Y}$ be in $\underline{E}(K)$; then, for every $j \in J$, $\underline{e}_j : \underline{X} \to Y_j$ is in*

$\underline{E}_r(K) = \underline{E}(K) \cap \{$ (Pro-C)-morphisms with rudimentary codomain$\}$.

The converse is also true.

Proof. Let $\underline{g} : \underline{X} \to K$, $K \in K$; by assumption there is an $\underline{h} : \underline{Y} \to K$ with $\underline{g} = \underline{h}.\underline{e}$. The assertion then follows from the fact that we can assume that $\underline{h}$ is a full morphism ([16], 1.8). The converse depends on the fact that $\underline{e}$ is an epimorphism. ◊

2.5. Proposition. *$\underline{P}(K) = \underline{P}_r(K)$, where $\underline{P}_r = \underline{\Lambda}\,\underline{E}_r$.*

Proof. Since $\underline{E}_r(K) \subset \underline{E}(K)$, then $\underline{P}(K) \subset \underline{P}_r(K)$. Let $\underline{m} : \underline{X} \to \underline{Y}$ be in $\underline{P}_r(K)$ and consider the following commutative square

2.5.1.
$$\begin{array}{ccc} \underline{H} & \xrightarrow{\underline{e}} & \underline{K} \\ {\scriptstyle \underline{s}}\downarrow & & \downarrow{\scriptstyle \underline{t}} \\ \underline{X} & \xrightarrow{\underline{m}} & \underline{Y} \end{array}$$

where

$\underline{e} \in \underline{E}(K)$, $\underline{X} = (X_a,\ x_{aa'}, A)$, $\underline{Y} = (Y,\ y_{jj'},\ J)$ and $\underline{K} = (K,\ k_{ii'}, I)$.

For every $j \in J$ it is $\underline{t}_j.\underline{e} = \underline{m}_j.\underline{s}$. Assuming $\underline{t}_j$ to be full, for every $i \in I$ one has another commutative square

2.5.2.
$$\begin{array}{ccc} \underline{H} & \xrightarrow{\underline{e}_i} & K_i \\ {\scriptstyle \underline{s}}\downarrow & & \downarrow{\scriptstyle t^i_j} \\ \underline{X} & \xrightarrow{\underline{m}_j} & Y_j \end{array}$$

Holding $i \in I$ fixed, one obtains a family of C-morphisms

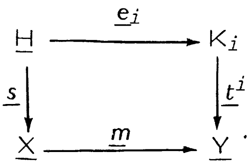
$$\{t^i_j : K_i \to Y_j \mid j \in J\}$$

which is natural in $j \in J$. In fact, given $j \leq j_o$, the morphisms t^i_j and $y_{jj_o} \cdot t^i_{j_o} : K_i \to Y_j$ are both representatives of $\underline{t}_j : \underline{K} \to Y_j$ and (2.5.2) continues to commute with t^i_j replaced by $y_{jj_o}.t^i_{j_o}$. Hence

$$t^i_j.\underline{e}_i = (y_{jj_o}.t^i_{j_o}).\underline{e}_i ,$$

and, since we could have assumed from the beginning that $\underline{e}_i$ is an epimorphism ([16], 3.2), it follows that $t_j^i = y_{jj_o} \cdot t_{j_o}^i$. In other words, $\underline{t}^i = (t_j^i)_J : K_i \to \underline{Y}$ is a (Pro-C)-morphism which makes the following diagram commutative

2.5.3.



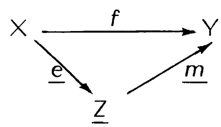
Now, since $\underline{e}_i \in \underline{E}_r(K)$ by the Lemma (2.4), and $\underline{m} \in \Lambda\underline{E}_r(K)$, then there exists the diagonal $\underline{d}^i$ in (2.5.3). Observe now that, for every $a \in A$, $d_a^i : K_i \to X_a$ represents a (Pro-C)-morphism $\underline{d}_a : \underline{K} \to X_a$ and, $\underline{e}$ being an epimorphism, $\underline{d} = (\underline{d}_a)_A : \underline{K} \to \underline{X}$ is a (Pro-C)-morphism and a diagonal for (2.5.1). This concludes the proof. ◊

2.6. Corollary. *Every pro-epireflective subcategory K of C induces a factorization for morphisms of the form $(\underline{E}^r(K), \underline{P}_{rr}(K))$, where :*

$\underline{E}^r(K)$ = {*Morphisms in $\underline{E}(K)$ having rudimentary domain*},

$\underline{P}_{rr}(K)$ = {*Morphisms in $\underline{P}_r(K)$ having rudimentary codomain*}. ◊

The corollary does not say that $(\underline{E}^r(K), \underline{P}_{rr}(K))$ is a factorization "structure" on C ; it claims that every C-morphism $f : X \to Y$ may be factorized (in Pro-C) as illustrated below



with

$\underline{e} \in \underline{E}^r(K)$, $\underline{m} \in \Lambda\underline{E}_r(K)$, $\underline{Z} \in$ Pro-C.

2.7. Proposition. *The morphisms in $\underline{E}^r(K)$ are exactly the K-expansions of C-objects.*

Proof. It is clear that the K-expansion of any object in C is in $\underline{E}^r(K)$. On the other hand, if $\underline{e} : X \to \underline{K}$ is in $\underline{E}^r(K)$ and $\underline{K} \in$ Pro-K, then $\underline{e}$ is a K-expansion of X. The assertion then follows by the equality

$$\text{Pro-}K = D\underline{P}(K),$$

which means that the objects of Pro-K are exactly those which are domain of some morphism in $\underline{P}(K)$ (this is the implication a ⇒ b of Theorem 3.10 of [17], Pro-K being epireflective in Pro-C). ◊

With the above notations, in the $(\underline{E}^r(K), \underline{P}_{rr}(K))$-factorization of a C-morphism $f : X \to Y$ the first morphism $\underline{e} : X \to \underline{K}$ is the K-expansion of X, in particular it is a pointwise epimorphism (cf. the definition of K-expansion and [15], under the name of "strong (Pro-K)-epimorphism").

2.8. Examples. (a) Let HLC be the category of locally compact Hausdorff spaces. It is pro-epireflective in the category TYCH of Tychonoff spaces. If $X \in$ TYCH, then its HLC-expansion $\underline{p} : X \to \underline{K}$ is formed by all dense embeddings of X into open sets $K_i \subset \beta X$, containing it. Every continuous map $f : X \to Y$ between Tychonoff spaces, may be factorized as

$$X \xrightarrow{\underline{p}} \underline{K} \xrightarrow{\underline{m}} Y$$

where $\underline{m}$ is HLC-perfect. To see this, let

$$X \xrightarrow{e} Z \xrightarrow{t} Y$$

be the usual factorization of f with e dense and compact-extendable and t perfect. Observe now that each p_i is also a dense and compact-extendable map ; then, by the construction of e as a cointersection ([17], Th. 2.8), it follows that there exists a (Pro-TYCH)-morphism

$$\underline{h} : \underline{K} \to Z \qquad \text{such that} \qquad \underline{h} . \underline{p} = e.$$

Setting $\underline{m} = t . \underline{h}$, one has the ($\underline{E}$(HLC), $\underline{P}$(HLC))-factorization of f.

It is worth noting that $\underline{h}$, and thus $\underline{m}$, is a full morphism [16] ; this says that, although in general f cannot be factorized through its Stone-Čech compactification βX, it is factorizable through any open neighborhood of X in βX :

$$X \xrightarrow{f} Y = X \xrightarrow{p_i} K_i \xrightarrow{m_i} Y,$$

where p_i is dense and compact-extendable, and m_i has a perfect "component".

(b) Let PM be the category of pseudometric spaces. PM is probireflective in TOP, the PM-expansion of a space X, $\underline{p} : X \to \underline{K}$, being formed by identity functions onto pseudometric spaces whose topology is less than that of X. It follows that every continuous map $f : X \to Y$ in TOP may be factorized as

$$X \xrightarrow{p_i} K_i \xrightarrow{m_i} Y$$

in many ways, but where the K_i's form an inverse system in PM, which is uniquely determined.

2.9. *Theorem.* *Let A be a subcategory of C and let K be its pro-epireflective hull (= the least pro-epireflective subcategory of C containing*

A). Then Pro-K *is the epireflective hull of* Pro-A *in* Pro-C.

Proof. Suppose G is the epireflective hull of Pro-A in Pro-C. Then Pro-$A \subset G \subset$ Pro-K. Since both G and Pro-K are epireflective in Pro-C, they induce perfect factorizations $(\underline{E}(G), \underline{P}(G))$ and $(\underline{E}(K), \underline{P}(K))$ in Pro-C, respectively. Now

$$\underline{E}(K) \subset \underline{E}(G) \quad \text{and} \quad \underline{P}(G) \subset \underline{P}(K).$$

Let $\underline{p} : X \to \underline{K}$ be a K-expansion of X and let

$$X \xrightarrow{\underline{e}} \underline{G} \xrightarrow{\underline{m}} Y$$

be its $(\underline{E}(G), \underline{P}(G))$-factorization. By Proposition (2.7), $\underline{G} \in G \subset$ Pro-K, so there must be a morphism

$$\underline{n} : \underline{K} \to \underline{G} \qquad \text{such that} \qquad \underline{n}.\underline{p} = \underline{e} .$$

It is easily realized that $\underline{m}$ and $\underline{n}$ are inverse isomorphisms, hence $\underline{e} : X \to \underline{G}$ is a K-expansion of X. It follows that $\underline{e} \in \underline{E}(K)$, so

$$\underline{E}(K) = \underline{E}(G) \quad \text{and} \quad \underline{P}(K) = \underline{P}(G),$$

and, from Proposition (2.7), $G =$ Pro-K. Note that Proposition (2.7) holds in general with K replaced by any reflective subcategory of Pro-C, in this case G. ◊

3. PRO-FACTORIZATIONS IN C.

C is always a finitely complete category.

The results of the preceding section lead to the following definition.

3.1. Definition. A *pro-factorization structure* (for morphisms) in C is a pair $[\underline{E}, \underline{P}]$ such that :

(i) $\underline{E} = \underline{E}^r \cup \underline{E}_r$ is an isocompositive class of (Pro-C)-morphisms, with the following properties :

a) $\underline{E}^r$-elements are pointwise epimorphisms ;

b) if $(\underline{e}^i : A_i \to \underline{Y}^i)_I$ is an inverse system in $\underline{E}$, then its limit $\underline{e} : \underline{A} \to \underline{Y}$ is such that $\underline{e}_j : \underline{A} \to Y_j \in \underline{E}_r$, for all $j \in J$.

(ii) $\underline{P}$ is an isocompositive class of (Pro-C)-morphisms with rudimentary codomain (i.e., an object of C).

(iii) Every C-morphism $f: X \to Y$ may be decomposed as

$$X \xrightarrow{\underline{e}} \underline{K} \xrightarrow{\underline{m}} Y$$

where $\underline{e} \in \underline{E}^r$ and $\underline{m} \in \underline{P}$.

(iv) $$\underline{P} = \underline{\Lambda} \underline{E}_r .$$

3.2. Remarks. (1) We call a full subcategory K of C pro-$\underline{E}^r$-reflective iff every C-object X has a K-expansion which belongs to $\underline{E}^r$. The importance of condition (3.1 (i, b)) then depends on the fact that if K is pro-$\underline{E}^r$-reflective in C, then Pro-K is $\underline{E}$-reflective in Pro-C, in the sense that the (Pro-K)-reflection of every (Pro-C)-object $\underline{X}$, $\underline{p} : \underline{X} \to \underline{Y}$, is such that $\underline{p}_j : \underline{X} \to Y_j \in \underline{E}_r$, for all $j \in J$ (cf. [15], 2.6).

(2) Note that, by arguments similar to that of Proposition (2.5), one shows that (3.1 (iv)) implies $\underline{P} = \Lambda\,\underline{E}$. From this follows that every C-morphism has an essentially unique $[\underline{E}, \underline{P}]$-pro-factorization, that is, up to isomorphisms of the intermediate inverse system. Moreover, $\underline{E} \cap \underline{P}$ is a class of isomorphisms.

(3) It is clear that every pro-epireflective subcategory K of C induces on C a pro-factorization structure for morphisms of C. We shall call such a pro-factorization a *K-perfect* one.

3.3. *Proposition.* *Let $[\underline{E}, \underline{P}]$ be a pro-factorization structure for morphisms of C. If C is cowellpowered, then $[\underline{E}, \underline{P}]$ may be extended to a pro-factorization structure $[\underline{E}, \underline{P}']$ for C-sources ([8], 1.1).*

Proof. The definition of pro-factorization structures for sources is the obvious one. If C is cowellpowered, then every C-object admits only a set of $\underline{E}^r$-quotients in Pro-C, depending on the fact that $\underline{E}^r$-elements are all pointwise epimorphisms (3.1 (ii)). Then all details go as in ([8], 1.3.2) ; $\underline{P}'$ is formed by all compositions of $\underline{P}$-morphisms with products (in Pro-C). ◊

3.4. Remark. Note that our $[\underline{E}, \underline{P}']$ pro-factorization for sources is quite similar to the "($\underline{E}$, $\underline{M}$)-factorization on C with respect to Pro-C" defined by A. Tozzi, in particular see Remark 1.10, Definition 2.1 and Proposition 2.3 of [20].

We point out that we were led to the definition of pro-factorization by the analysis of the intrinsic properties of Pro-C, at least when C is finitely complete ; hence this seems to be the right way to study pro-epireflections.

3.5. *Lemma.* *Let C have an $[\underline{E}, \underline{P}]$ pro-factorization structure for morphisms (sources). The following hold :*

(i) Every (Pro-C)-morphism of the form $f : \underline{X} \to Y$ may be factorized as

$$\underline{X} \xrightarrow{\;\underline{e}\;} \underline{K} \xrightarrow{\;\underline{m}\;} Y$$

where $\underline{e}_i : \underline{X} \to K_i \in \underline{E}_r$ for all $i \in I$, and $\underline{m} \in \underline{P}$.

(ii) $\underline{f} : \underline{X} \to Y \in \underline{P}$ iff for every factorization as in (i), with $\underline{e}_i \in \underline{E}_r$, $i \in I$, $\underline{e}_i$ must be an isomorphism for all i.

Proof. This proof may be done rephrasing that of Proposition 2.3 in [20], by virtue of (3.1 (i)), (3.2 (2)), and the fact that Pro-C is closed under the formation of inverse limits [3]. ◇

3.6. Lemma. *Let* $[\underline{E}, \underline{P}']$ *be a pro-factorization structure for sources in* C. *A source*

$$(\underline{X} \xrightarrow{\underline{m}^\lambda} H^\lambda)_\Lambda \quad , \qquad \underline{X} = (\underline{X}_a, x_{aa'}, A)$$

is in $\underline{P}'$ *iff, for every* $a \in A$, *the source* $(m_a^\lambda : X_a \to H^\lambda)_\Lambda$ *is in* $\underline{P}'$, *too.* ◇

3.7. Theorem. *Let* C *be cowellpowered and have an* $[\underline{E}, \underline{P}']$ *- pro-factorization for sources. If* K *is a full, isomorphism-closed, subcategory of* C, *the following are equivalent :*

(i) K *is pro-* $\underline{E}^r$*-reflective in* C.
(ii) K *is stable under* $\underline{P}'$ *-sources (this means that, given*

$$(m^\lambda : \underline{X} \to H^\lambda)_\Lambda \in \underline{P}' \qquad \text{with} \qquad H^\lambda \in K, \qquad \lambda \in \Lambda ,$$

then $\underline{X} \in$ Pro-K.

<u>Proof.</u> (i) ⇒ (ii). Let $(\underline{m}^\lambda : \underline{X} \to H^\lambda)_\Lambda$ be as in (ii) and let

$$\underline{p} : \underline{X} \to \underline{K} , \qquad \underline{K} = (K_i , k_{ii'} , I),$$

be the (Pro-K)-reflection of $\underline{X}$. For every $\lambda \in \Lambda$ there is a morphism $\underline{g}^\lambda : \underline{K} \to H^\lambda$ such that $\underline{g}^\lambda . \underline{p} = \underline{m}^\lambda$, $\lambda \in \Lambda$, hence

$$g_i^\lambda . \underline{p}_i = \underline{m}^\lambda \qquad \text{for every} \quad i \in I.$$

By Lemma 3.5, it follows that each $\underline{p}_i$ must be an isomorphism ; then $\underline{X} \in$ Pro-K.

(ii) ⇒ (i). Let $X \in C$ and let $(f^\lambda : X \to H^\lambda)_\Lambda$ be a representative set of quotients of X with all H^λ in K. Let

$$(X \xrightarrow{\underline{e}} \underline{K} \xrightarrow{\underline{m}^\lambda} H^\lambda)_\Lambda$$

be its pro-factorization ; then $\underline{K} \in$ Pro-K by (ii). If $g : X \to M$, $M \in K$ and

$$X \xrightarrow{\underline{e}'} \underline{N} \xrightarrow{\underline{m}'} M$$

is its pro-factorization, then $\underline{N}$ is also in Pro-K and $\underline{e}'$ is pointwise epi. If $\underline{N} = (N_c, n_{cc'}, C)$, then each $\underline{e}'_c : X \to N_c$ must be an $f^\lambda : X \to H^\lambda$. It follows that

$$\underline{g} = \underline{m}'.\underline{e}' = \underline{m}'.\underline{m}^\lambda.\underline{e},$$

and this completes the proof.

3.8. Corollary. *Let C be as above. Every subcategory A of C has a pro-$\underline{E}^r$-reflective hull D(A), whose objects are all X ∊ C such that there exists a $\underline{P}'$-source with domain X and codomains in A.*

Proof. That D(A) is the smallest pro-$\underline{E}^r$-reflective subcategory of C which contains A follows from the theorem and Lemma 3.6. ◊

3.9. Lemma. *Let K be a full, isomorphism-closed, subcategory of C, then :*

(i) K is closed under finite limits iff Pro-*K is closed under limits.*

(ii) If C is cowellpowered and has an [$\underline{E}$, $\underline{P}$] pro-factorization structure for morphisms, then K is stable under $\underline{P}$-morphisms iff it is closed under $\underline{P}'$-sources.

Proof. Part (i) follows from a combination of (1.8) and ([3], 5.0). (ii) follows from (3.3) and (3.6). ◊

3.10. Proposition. *Let C be cowellpowered with an [$\underline{E}$, $\underline{P}$] pro-factorization structure for morphisms. A full, isomorphism-closed, subcategory K of C is pro-$\underline{E}^r$-reflective in C iff it is stable under $\underline{P}$-morphisms.*

3.11. Proposition. *Let C as above. Every subcategory A of C has a pro-$\underline{E}^r$-reflective hull D(A), whose objects are all X ∊ C which are domain of some $\underline{P}$-morphism with codomain in A.* ◊

The following result generalizes the analogous one ([8], 1.1 (10)), concerning factorization structures.

3.12. Proposition. *If C is cowellpowered and has an [$\underline{E}$, $\underline{P}$] pro-factorization structure for morphisms, then $\underline{E}$ = $\underline{E}$(K), $\underline{P}$ = $\overline{\underline{P}(K)}$, K being the pro-reflective subcategory of C generated by all $\underline{E}$-injective objects.*

3.13. Remark. The results contained in Theorem 3.7 and Proposition 3.11 are the generalization to the pro-reflective case of the classical one ([8, 10, 17], see also [5]).

The proofs of Propositions 3.10 and 3.11 are almost easy, being obtained by diagonalization (3.1 (iv) and (3.2 (2)).

Many examples may be found illustrating the matter, at least in the case of perfect pro-factorizations. For instance, the following subcategories are all pro-epireflective in TOP : metrizable spaces, sequentially compact spaces, first countable spaces, second countable spaces, separable spaces, finite dimensional compact spaces, etc..

We note that, because of their complex nature, pro-factorizations seem to be less manageable than factorizations ; hence, it would be very interesting to know the relations between the two concepts. Indeed this is an argument for further study. By the way, recall that, in constructing the pro-reflection of HCL in TYCH (2.8 (a)), we used the (dense compact-extendable, perfect)-factorization in TYCH ; similarly, we needed the (bimorphism, initial)-factorization in TOP to define the pro-reflection of PM in TOP (2.9 (b)).

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Dipartimento di Matematica
Universita' di Perugia
Via Pascoli
I-06100 PERUGIA. ITALY