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PARTIALLY-ADDITIVE MONOIDS

by Antonio BAHAMONDE

RÉSUMÉ. Les monoïdes partiellement additifs (pams) ont été introduits par Arbib et Manes pour fournir une approche algébrique à la sémantique de la récursion dans l'Informatique théorique. Dans cet article, les pams sont présentés comme des **G**-algèbres, où **G** est une théorie algébrique dans la catégorie des ensembles et applications partielles. Ensuite, on définit une topologie naturelle sur un pam, proche de la topologie de Scott pour les treillis continus. Les axiomes des pams sont alors classifiés selon leur naturalité topologique ou algébrique. On étudie aussi les sous-structures, surtout lorsque l'ensemble sous-jacent est fermé. Enfin, on obtient des relations entre structures de pam, ordres et topologie.

1. INTRODUCTION.

This paper deals with the algebraic and topological foundations of Partially Additive Monoids (pams). These structures were introduced by Arbib and Manes [1, 2, 3] in the denotational semantics of programming languages setting. In this introductory section we outline the motivations for the concepts of the paper.

In the denotational semantics of programming languages, we associate a suitable partially defined function $f_P : D \to D$ with each program P, where D is a space of states. This process exploits some algebraic properties of the family of maps f_P. The sum of those partial functions arises in a very natural way as follows.

Given a predicate B, and statements S_1 and S_2, assume that we already have a partial function interpretations

$$p : D \to \{T, F\}, \qquad f_1 : D \to D, \qquad f_2 : D \to D.$$

Then define two partial functions $p_T, p_F : D \to D$, where $p_T(d) = d$ with domain of definition $\{d \mid p(d) = T\}$, and $p_F(d) = d$ with domain $\{d \mid p(d) = F\}$. Therefore, the partial functions $f_1 p_T$ and $f_2 p_F$ have disjoint domains ; thus we can define a partial function

$$f_1 p_T + f_2 p_F : D \to D$$

given by

$$(f_1 p_T + f_2 p_F)(d) = \begin{cases} f_1 p_T(d) = f_1(d) & \text{if } d \in DD(f_1 p_T) \\ f_2 p_F(d) = f_2(d) & \text{if } d \in DD(f_2 p_F) \\ \text{undefined} & \text{else} \end{cases}$$

Thus, $f_1 p_T + f_2 p_F$ is a natural partial function interpretation for the statement "*if* B *then* S_1 *else* S_2" . The iterative statements provide a motivation for countable sums. The reason for restricting the sums to the countable case has to do with computability.

On the other hand, pams include a wide kind of countable-chain-complete posets, as shall be shown in § 6 below.

Pams are sets endowed with a partial operation Σ on countable (i.e., finite or denumerable) families of elements which satisfies, among other things, generalized commutative and associative laws. Our primary concern is to present pams as G-algebras, where G is an algebraic theory in the category of sets and partial functions, since Σ is a partial operation. However, in the literature two concepts of pams are considered. So, an additional topological criterion (the *limit axiom*) is specified in [1, 2, 3] (throughout the paper pam means this kind of structures), but this is not the case in [9] (here we call them *pam*(1, 2) or *positive partial monoids*). Our G-algebras here correspond to pams without the topological requirements ; that is to say, we just capture the algebraic structure of pams in our algebraic representation.

In the third paragraph, by means of the results obtained in the second, we associate with each pam a topology in a very natural way. In a certain sense this topology generalizes the Scott topology of continuous lattices [11, 6, 12]. Here the limit axiom of pams appears in its topological naturalness.

The algebraic representation of pams here presented has been very useful in order to provide the category of pams with a tensor product [4] following the Guitart's construction on algebraic categories [7]. On the other hand, a new approach to the information theory can be given from this representation of pams [5]. Here the topological aspects play a suggestive role in characterizing sequentially continuous information measures.

§ 4 is devoted to studying the pam-substructures (*sub-pams*), mainly when the underlying set is closed. In § 5, we study a particularly regular class of pams in the topological sense. We call them *continuous pams* following Scott.

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2. PAMS AS ALGEBRAS.

In this paragraph we will present partially-additive monoids (pams) [1, 2, 3] as algebras over an algebraic theory in the same way as groups or vector spaces are. That is to say, pams are sets endowed with some operations that satisfy some equations. These operations are partially defined and then we will take the category of sets and partially defined functions, Pfn, as the base category.

A morphism $f : A \to B$ in Pfn (partially defined function) is a total function $f : DDf \to B$, where DDf (domain of definition of f) is a subset of A. If $g : B \to C$ is a morphism in Pfn, the composition $gf : A \to C$ is defined by

$$(gf)a = g(fa) \qquad \text{for such } a \in DDf \text{ that } fa \in DDg .$$

If $f, g \in Pfn(A, B)$, we define $f \leq g$ to mean that

$$DDf \subset DDg \qquad \text{and} \qquad fa = ga \quad \text{for all} \quad a \in DDf .$$

(2.1) Definition [1, 2, 3]. A partially-additive monoid (pam) is a pair (A, Σ^A), where A is a non-empty set, and Σ^A is a partial operation on countable (i.e., finite or denumerable) families in A subject to the following axioms.

(1) Partition-associativity axiom : If the countable set I is partitioned into $(I_j \mid j \in J)$ (i.e., $(I_j \mid j \in J)$ is a countable family of pairwise disjoint sets whose union is I), then for each family $(x_i \mid i \in I)$ in A,

$$\Sigma^A(x_i \mid i \in I) = \Sigma^A(\Sigma^A(x_i \mid i \in I_j) \mid j \in J)$$

in the sense that the left side is defined iff the right side is defined, and then the values are equal.

(2) Unary sum axiom : For one-element families the sum is defined and $\Sigma^A a = a$.

(3) Limit axiom : If $(x_i \mid i \in I)$ is a countable family in A and if $\Sigma^A(x_i \mid i \in F)$ is defined for every finite $F \subset I$, then $\Sigma^A(x_i \mid i \in I)$ is defined.

We use the notation

$$x_1 + x_2 + \dots + x_n \qquad \text{for} \quad \Sigma^A(x_i \mid i = 1, 2, \dots, n).$$

And we drop the A from Σ^A when no confusion arises.

Notice that from the partition-associativity axiom, any subfamily of a summable family is summable. And since the unary sum axiom ensures that some sums exist, it follows that the empty sum is defined and provides an additive zero which we denote $\perp_A$ (or simply $\perp$ when no confusion arises). Moreover, since $(I_j \mid j \in N)$ is a partition of the empty sets if each I_j is empty, we infer that $\perp$ is even a denumerable zero.

Let $(x_i \mid i \in I)$, $(y_j \mid j \in J)$ be countable families in A, and let

ψ : I $\rightarrow$ J be a bijection such that

$$y_{\psi i} = x_i \qquad \text{for all} \quad i \in I$$

The partition-associativity and the unary sum axioms ensure that

$$\Sigma(x_i \mid i \in I) = \Sigma(y_j \mid j \in J)$$

in the sense that the left side is defined iff the right side is defined, and then they are equal. That is to say, Σ is countable commutative.

A pam satisfies the following "positivity property" :

$$\text{if} \quad \Sigma(x_i \mid i \in I) = \perp \quad \text{then each} \quad x_i \quad \text{is} \ \perp.$$

That is why pams are called positive partial monoids in [9]. To see this, let $i \in I$ and set $y = \Sigma(x_j \mid j \in I\text{-}\{i\})$. Then

$$x_i = x_i + (y+x_i) + (y+x_i) + \ldots = (x_i+y) + (x_i+y) + \ldots = \perp .$$

(2.2) Some examples [1, 2, 3]. The basic example (alluded to in the introduction) is the set Pfn(A, B) with the sum defined for families $(f_i \mid i \in I)$ such that their domains of definition are pointwise disjoint as follows :

$$[\Sigma^{di}(f_i \mid i \in I)]\, a = \begin{cases} f_i a & \text{if} \quad a \in \mathrm{DD} f_i \\ \text{undefined else.} \end{cases}$$

It is possible to give the same definition when the family is overlap-summable ; that is to say, if whenever $a \in \mathrm{DD} f_i \cap \mathrm{DD} f_j$, then $f_i a = f_j a$. Endowed with the overlap-sum, (Pfn(A, B), Σ^{ov}) is a pam too.

Let Rel(A, B) be the set of relations from a set A to a set B. That is, functions $f: A \rightarrow PB$, where PB is the set of subsets of B. Rel(A, B) has a pam structure in which every countable family is summable : define

$$[\Sigma(f_i \mid i \in I)]\, a = \cup(f_i a \mid i \in I).$$

let $(L, \leq)$ be a complete lattice, then (L, sup) is a pam, where $\sup(l_i \mid i \in I)$ is the supremum of $L' = \{ l_i \mid i \in I\}$. Actually, we only need partially ordered sets (posets) L for which some countable subsets of L have a supremum. We shall not bother to spell out this now, as it will arise naturally in § 6.

(2.3) Let A be a set. A family $(x_i \mid i \in I)$ in A can be seen as a function $x : I \rightarrow A$ such that $x\, i := x_i$. Two families in A, $x : I \rightarrow A$,

$y : J \to A$ are said to be *equivalent* iff there exists a bijection $\psi : I \to J$ such that $y\psi = x$. Note that this defines an equivalence relation R_A in the set of countable families in A, CF(A), such that

$$\Sigma x = \Sigma y \quad \text{whenever} \quad x \, R_A \, y ,$$

and (A, Σ) is a pam (2.1).

Let us now remark that for any countable family $x : I \to A$ there exists a bijection $\psi : I \to \bar{I}$, where $\bar{I} \subset N$ (where N is the set of natural numbers), and a partial function $\bar{x}: N \to A$, where $DD\bar{x} = \bar{I}$, such that $xi = \bar{x}\psi i$ for all $i \in I$. Thus, we always can take partial functions from N to A as representative elements of the equivalence classes in $CF(A)/R_A$. That is, to say,

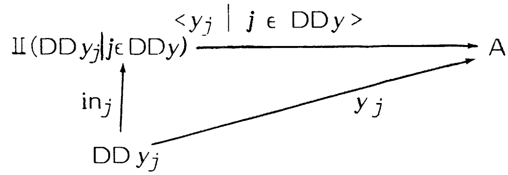
$$CF(A)/R_A = Pfn(N, A)/R_A .$$

For this reason, in order to compute sums of countable families in A, throughout this paper, we consider such families as partial functions from N to A.

A partition of a family $x : N \to A$ is a family of families

$$y : N \to Pfn(N, A) \quad \text{such that} \quad x \, R_A \, \langle y_j \mid j \in DDy \rangle,$$

where $(y_j \mid j \in DDy)$ is the unique morphism in Pfn(N, A) defined by the commutativity for all $j \in DDy$ of the diagram



where $\amalg(DD y_j \mid j \in DDy)$ means a subset of N isomorphic to the disjoint union (coproduct in Pfn) of the sets DDy_j, and in_j are the natural injections.

(2.4) Taking account of these remarks we are going to introduce an algebraic theory in Pfn in order to capture pams(1, 2) as algebras of this theory.

Definition ([8], p. 32). An *algebraic theory in extension form* in a category H is a triple $\mathbf{T} = (T, \eta, (\ )^{\#})$ where T is an object function, assigning to each object A of H another object TA, η is an assignment to each object A of H of a map $\eta A : A \to TA$, and with $(\ )^{\#}$ assigning to each $f : A \to TB$ an "extension" $f^{\#} : TA \to TB$, subject to the following three axioms :

$$(1)\ f^{\#}\eta A = f, \qquad (2)\ (\eta A)^{\#} = 1_{TA} , \qquad (3)\ (f^{\#}g)^{\#} = f^{\#}g^{\#}.$$

Let $GA = Pfn(N, A)/R_A$ be for any set A the quotient set alluded to in (2.3) above. Denote by $[f]$ the equivalence class of $f \in Pfn(N, A)$. Define

$$\eta A : A \to GA \qquad \text{by} \qquad (\eta A)a := [a] ,$$

where "a" denotes the element of A and the one-element family.

As in (2.3), if $(f_i \mid i \in I)$ is a family of families in A, we will denote by $\langle f_i \mid i \in I\rangle$ the family in A given by the universal property of the coproduct, where

$$DD\langle f_i \mid i \in I\rangle = \amalg(DDf_i \mid i \in I).$$

Note that if $f_i \, R_A \, g_i$ for all $i \in I$, then

$$\langle f_i \mid i \in I\rangle \, R_A \, \langle g_i \mid i \in I\rangle .$$

On the other hand, if $x : DDx \to A$ is a family in A, then

$$x \; R_A \langle x_i \mid i \in DDx\rangle .$$

Given $f : A \to GB$ in Pfn, define $f^{\#} : GA \to GB$ as follows. Let $[(x_i \mid i \in I)] \in GA$ such that $x_i \in DDf$ for all $i \in I$, and let y_i be any representative element of $fx_i \in GB$ for all $i \in I$. Then define

$$f^{\#}[(x_i \mid i \in I)] = [\langle y_i \mid i \in I\rangle] .$$

Conversely, if there exists $i \in I$ such that $x_i \notin DDf$, then $f^{\#}[(x_i \mid i \in I)]$ is undefined.

(2.5) *Proposition.* $G = (G, \eta, (\;)^{\#})$ *is an algebraic theory in* Pfn.

Proof. The proof is straightforward and can be safely left to the reader.

Corollary *([8], (1.3)).* $G : Pfn \to Pfn$ *is a functor,* $\eta : 1_{Pfn} \to G$ *is a natural transformation, and for each set* A *the map* $\mu A = (1_{GA})^{\#}$: $GGA \to GA$ *defines a natural transformation* $\mu : GG \to G$.

Moreover, the triple (G, η, μ) is an *algebraic theory in* Pfn *in monoid form* ; that is to say, for every set A,

(1) $$(\mu A)(\eta GA) = 1_{GA} = (\mu A)(G\eta A),$$

(2) $$(\mu A)(G\mu A) = (\mu A)(\mu GA).$$

(2.6) Let (A, Σ^A) be a pam. Then Σ^A can be seen as a morphism in Pfn, $\Sigma^A : GA \to A$, but not every partially defined function from GA to A defines a pam structure on A. The next aim is to characterize such morphisms.

Moreover, an *additive map* $f : (A, \Sigma^A) \to (B, \Sigma^B)$ [1, 2, 3] is a total function $f : A \to B$ such that $f\Sigma^A \leq \Sigma^B G f$. In diagram

$$\begin{array}{ccc} GA & \xrightarrow{\ \Sigma^A\ } & A \\ {\scriptstyle Gf}\downarrow & \geqq & \downarrow{\scriptstyle f} \\ GB & \xrightarrow[\ \Sigma^B\ ]{} & B \end{array}$$

Note that if we constrict $f\Sigma^A$ and $\Sigma^B G f$ to $DD\Sigma^A$, then we have the equality of the two maps above.

Definition [8]. An *algebra of an algebraic theory* $\mathbf{T} = (T, \eta, \mu)$ in a category H is a pair (A, δ), where A is an H-object and $\delta : TA \to A$ is an H-morphism such that

$$\delta\eta_A = 1_A \qquad \text{and} \qquad \delta T\delta = \delta\mu A.$$

A *morphism of* $\mathbf{T}$*-algebras* $f : (A, \delta) \to (A', \delta')$ is an H-morphism

$$f : A \to A' \qquad \text{such that} \qquad f\delta = \delta' T f .$$

The category of $\mathbf{T}$-algebras of H will be denoted by $H^{\mathbf{T}}$.

Definition. A pair (A, Σ^A) is a *pam*(1, 2) *(positive partial monoid* in [9]), provided that it meets the partition-associativity and the unary sum axioms.

Proposition. *A pair* (A, Σ^A) *where* $\Sigma^A : GA \to A$ *is a* Pfn*-morphism is a pam*(1, 2) *iff it is a* **G***-algebra.*

Notice that if $f : (A, \Sigma^A) \to (B, \Sigma^B)$ is a morphism of **G**-algebras and is total, then it is an additive map of pams(1, 2). But the opposite is by no means true, just look at the identity map from (Pfn(A, B), Σ^{di}) to (Pfn(A, B), Σ^{ov}) (2.2).

(2.7) Let A be a set. Define " $\leqq$" in GA by

$$[x] \leqq [y] \quad \text{iff there exist } x' R_A x \text{ and } y' R_A y \text{ such that } x' \leqq y' \text{ in Pfn}(N, A).$$

The relation "$\leqq$" is well defined in GA and is actually the least order in GA that makes the canonical projection $p_A : \mathrm{Pfn}(N, A) \to GA$ monotone (isotone).

Proposition. *A pair* (A, Σ^A)*, where* $\Sigma^A : GA \to A$ *is a* Pfn*-morphism, is a pam iff it is a* **G***-algebra such that* $DD\Sigma^A$ *is a countable-chain-complete subset of* $(GA, \leqq)$ *(that is, every countable chain in* $DD\Sigma^A$ *has a supremum in* $DD\Sigma^A$ *).*

Proof. Just notice that the limit axiom is equivalent to the following property in a pam(1, 2) : if $(x_i \mid i \in N)$ is a family such that

$$(x_i \mid i = 0, 1, \dots, n) \quad \text{for all} \quad n \in N$$

is summable (its class belongs to $DD\Sigma^A$), then $(x_i \mid i \in N)$ is summable.

(2.8) In general, let (A, Σ^A) be a pam, and let $a, b \in A$. Define

$$a \leqq b \quad \text{iff there exists} \quad c \in A \text{ such that } \quad a + c = b .$$

This relation is reflexive and transitive, but in general, it is not anti-symmetric (a counter-example can be found in [3]). Anyway, the sum of the empty family $\Sigma^A \emptyset = \perp_A$ is the minimum, because if $a \leqq \perp_A$ then there exists $b \in A$ such that $a + b = \perp_A$, and therefore $a = b = \perp_A$ (2.1).

Let A be a set. The "$\leqq$" relation defined above for the free pam $(GA, \mu A)$ over A coincides with the one defined in (2.7).

(2.9) The category Pfn is isomorphic to the category of pointed sets Set_*. Therefore it has equalizers, coequalizers, products, and coproducts. Moreover Pfn has a factorization system (E, M), where we take for E the class of all partially defined surjective functions, and for M all injective functions. Thus we have the following

Proposition. *The category of G-algebras,* Pfn^G , *has small colimits ([8], p. 276), and is a regular category ([8], p. 239).*

Let us recall that a regular category is a triple (H, E, M) where H is a locally small category with small limits, (E, M) is a factorization system in H, and H is E-cowellpowered. Therefore, in particular, Pfn^G has a factorization system (E^G, M^G) where a map of algebras $f \in E^G$ (resp. M^G) just in case $f \in E$ (resp. M) as Pfn-morphism.

Anyway, some of these results can be extended to the category Pam of pams and additive maps (2.6).

Proposition. *The category* Pam *has :*

(i) a factorization system (E^*, M^*), where E^* (resp. M^*) are the surjective additive maps (resp. injective additive maps).
(ii) a zero object : $(\{0\}, \sup)$ where $\{0\}$ is the one-element poset.
(iii) products [9].

Proof. (iii) Given a family $((A_k, \Sigma_k) \mid k \in K)$ of pams, their product is the pam (A, Σ) as follows. As a set, let $A = \Pi(A_k \mid k \in K)$ be the cartesian product of the sets A . Given a family $(x_i \mid i \in I)$ in A, $x_i = (x_{ik} \mid k \in K)$, say that $(x_i \mid i \in I)$ is summable in A iff for each k

there exists $x_k = \Sigma_k(x_{ik} \mid i \in I)$, and then define $x = \Sigma(x_i \mid i \in I)$ by $x = (x_k \mid k \in K)$.

(ii) Let (A, Σ) be a pam. The only additive maps

$$¡ : (A, \Sigma) \to (\{0\}, \sup), \qquad ! : (\{0\}, \sup) \to (A, \Sigma)$$

are given by

$$¡a = 0, \quad !0 = \bot \quad \text{for all} \quad a \in A.$$

3. NATURAL TOPOLOGY OF PAMS.

(3.1) Let (A, Σ) be a pam. A sequence $(a_n \mid n \in N)$ in A is said to be an *ascending chain* iff

$$a_n \leqq a_{n+1} \text{ for each } n \in N.$$

That is to say, iff there exists a sequence (not necessarily unique) $(x_i \mid i \in N)$ in A such that

$$a_n = \Sigma(x_i \mid i = 0, ..., n).$$

Therefore $(x_i \mid i \in N)$ is a summable family in A.

Let $a = \Sigma(x_i \mid i \in N)$; the set of all such sums will be called the set of *Arbib-Manes limits of* (a_n) ([2], p. 599 ; [3]), in symbols AM-lim(a_n) . Notice that this is nonempty but may have more than one element.

In Pfn(A, B), $f \in$ AM -lim(f_n) iff f is the least supper bound of the ascending chain (f_n). Therefore, f is unique.

(3.2) If $(a_n \mid n \in N)$ is such that $a_n = a$ for all $n \in N$, then $a \in$ AM-lim(a_n).
If $a \in$ AM-lim(a_n) and $(a_{nk} \mid k \in N)$ is a subsequence of (a_n), then : $a \in$ AM-lim(a_{nk}). To prove this, let

$$a = \Sigma(x_i \mid i \in N), \quad \text{where} \quad a_n = \Sigma(x_i \mid i = 0, ..., n),$$

then define

$$y_0 = \Sigma(x_i \mid i = 0, ..., n_0) = a_{n_0},$$

$$y_{j+1} = \Sigma(x_i \mid i = (nj)+1, ..., n(j+1)) \quad \text{for all} \quad j \geqq 0,$$

thus

$$a_{nk} = \Sigma(y_j \mid j = 0, ..., k), \qquad a = \Sigma(y_j \mid j \in N).$$

Proposition. *If the* AM-lim (a_n) *is one-element for every ascending chain in a pam* (A, Σ), *then* A *is a poset with the "$\leqq$" relation defined in (2.8).*

Proof. Let $a, b \in A$ such that $a \leqq b$ and $b \leqq a$. Then $(a, b, a, b, ...)$ is an ascending chain ; let $c \in$ AM-lim$(a, b, a, b, ...)$. Therefore

$$c \in \text{AM-lim}(a, a, a, ...) \quad \text{and} \quad c \in \text{AM-lim}(b, b, b, ...)$$

so that $a = c = b$.

(3.3) Let (A, Σ^A), (B, Σ^B) be pams. A total function $f : A \to B$ is said to be AM-*continuous* ([2], P. 599 ; [3]) iff

$$f(\text{AM-lim}\,(a_n)) \subset \text{AM-lim}\,(fa_n)$$

for every ascending chain (a_n) in A. Note that f must be monotone (isotone).

Of course, once we have a definition of convergence and continuity, we must wonder if there exists a topology in which these notions make topological sense. We are going to define such a topology in an arbitrary pam. That topology shall make topologically continuous the AM-continuous maps. The opposite shall hold in a special kind of pams that will be called continuous pams following Scott.

(3.4) Definition. Let (A, Σ) be a pam. A subset $U \subset A$ is called *additive-open* iff it satisfies the following two axioms :

(AO.1) $y \in U$ whenever $x \in U$ and $x \leqq y$.
(AO.2) if $\Sigma(x_i \mid i \in N) \in U$, then there exists $n \in N$ such that

$$\Sigma(x_i \mid i = 0, \ldots, n) \in U.$$

It is clear that the family of additive-open subsets of A is a topology τ_A in A, and in this topology the AM-limits of ascending chains are topological limits.

Proposition. *Let (A, Σ) be a pam, and let $(a_n \mid n \in N)$ be a sequence in A that converges to a in τ_A : $(a_n) \underset{\tau_A}{\to} a$. Then if $b \leqq a$, $(a_n) \underset{\tau_A}{\to} b$.*

Proposition. *Let U be a subset of A. The following statements are equivalent :*

(i) U is additive-open.
(ii) If $(x_i \mid i \in I)$ is summable, then $\Sigma(x_i \mid i \in N) \in U$ iff there exists $n \in N$ such that $\Sigma(x_i \mid i = 0, \ldots, n) \in U$.

Proof. If $x \leqq y$ and $x \in U$, then there exists $c \in A$ such that $x + c = y$. Therefore $(x_i \mid i \in N)$, where

$$x_o = x, \quad x_1 = c, \quad x_n = \bot \qquad \text{for all } n \geqq 2$$

is summable, and for $n = 0$,

$$\Sigma(x_i \mid i = 0, \ldots, n) = x \in U,$$

then $\Sigma(x_i \mid i \in N) = y \in U$.

Proposition. *Let C be a subset of A. The following statements are equivalent :*

(i) C is additive-closed.
(ii) C satisfies :
(ii.1) $x \leqq y$, $y \in C$ implies $x \in C$.
(ii.2) $\Sigma(x_i \mid i = 0, \ldots, n) \in C$ for all n implies $\Sigma(x_i \mid i \in N) \in C$.
(iii) $\Sigma(x_i \mid i = 0, \ldots, n) \in C$ for all n iff $\Sigma(x_i \mid i \in N) \in C$.
(iv) No sequence in C can converge to a point of the complement of C.
(v) C contains every topological limit of any ascending chain $(a_i \mid i \in N)$ with $a_i \in C$ for all $i \in N$.

(3.5) Proposition. *Let (A, Σ^A), (B, Σ^B) be pams, and let $f : A \to B$ be an AM-continuous function. Then f is additive-continuous.*

Proof. Let $U \subset B$, $U \in \tau_B$, and $a \in f^{-1}U$. Then, if $a \leqq b$, $fa \leqq fb \in U$; therefore, $b \in f^{-1}U$.

Moreover, if $\Sigma(x_i \mid i \in N) \in f^{-1}U$, as

$$\Sigma(x_i \mid i \in N) \in \text{AM-}\lim(\Sigma(x_i \mid i = 0, \ldots, n)),$$

we have that

$$f(\Sigma(x_i \mid i \in N)) \in \text{AM-lim}(f\,\Sigma(x_i \mid i = 0, \ldots, n)) \cap U.$$

But AM-limits are additive limits, so there exists $n \in N$ so that

$$f(\Sigma(x_i \mid i = 0, \ldots, n)) \in U, \quad \text{i.e.,} \quad \Sigma(x_i \mid i = 0, \ldots, n) \in f^{-1}U.$$

(3.6) Now, we are going to achieve the same topology in another way. Let (A, Σ^A) be a pam. We have the maps

$$\text{Pfn}(N, A) \xrightarrow{\;p_A\;} GA \xrightarrow{\;\Sigma^A\;} A.$$

Let us call $S^A = \Sigma^A p_A$. Then $DDS^A \subset \text{Pfn}(N, A)$ fulfills :

(i) $g \in DDS^A$ whenever $g \leqq f \in DDS^A$.

(ii) If $D \subset DDS^A$ is a countable directed set, then

$$\amalg D = \sup\{d \mid d \in D\} \in DDS^A.$$

In Pfn(N, A) we have the well known Scott countable topology τ_S, where $U \subset \text{Pfn}(N, A)$ is said to be a Scott countable open ($U \in \tau_S$) whenever it satisfies the following

(SO.1) $g \in U$ whenever $f \in U$ and $f \leqq g$.
(SO.2) If D is countable directed, and $\sup(D) \in U$, then $D \cap U \neq \emptyset$.

Thus, DDS^A is closed in this topology, and $U \subset DDS^A$ is a Scott countable open in the relative topology iff it satisfies (SO.1) and (SO.2) for all f, g and D in DDS .

Theorem. *Let* (A, Σ^A) *be a pam. Then the additive topology* τ_A *is the quotient topology given by* $S^A : DDS^A \to A$.

Proof. Let $U \subset A$ such that $(S^A)^{-1} U = V$ is an open in DDS^A, and let $a \in U$. Then there exists a family

$$(x_i \mid i \in I) \quad \text{such that} \quad S^A(x_i \mid i \in I) = a.$$

If $a \leqq b$ there exists $c \in A$ such that $a + c = b$. Then

$$(x_i \mid i \in I) \leqq (c, x_i \mid i \in I) \in V \quad \text{and} \quad S^A(c, x_i \mid i \in I) = b \in U.$$

Moreover, if $\Sigma(x_i \mid i \in N) \in U$, the family $(x_i \mid i \in N)$ is in V. Let

$$D = \{(x_i \mid i = 0, \ldots, n) \mid n \in N\}.$$

D is countable directed in DDS^A, then there exists $n \in N$ such that $(x_i \mid i = 0, \ldots, n) \in V$, and

$$S^A(x_i \mid i = 0, \ldots, n) = \Sigma^A(x_i \mid i = 0, \ldots, n) \in U.$$

Thus $U \in \tau_A$.

Let now $U \in \tau_A$, and $V = (S^A)^{-1} U$. If $(x_i \mid i \in I) \in V$ and

$$(x_i \mid i \in I) \leqq (x_i \mid i \in J) \in DDS^A$$

then

$$x_I = \Sigma^A(x_i \mid i \in I) \leqq \Sigma^A (x_i \mid i \in J) = x_J .$$

As $x_I \in U$, $x_J \in U$, hence $(x_i \mid i \in J) \in V$.

Let $D = \{f_i \mid i \in I\}$ be a countable directed set in DDS^A such that

$$\sup(D) = (x_j \mid j \in J) \in V.$$

Then $\Sigma^A(x_j \mid j \in J) \in U$ and there exists a finite subset $J' \subset J$ such that $\Sigma^A (x_j \mid j \in J') \in U$. Then, as J' is finite and D directed, there must exist

$$f_k \in D \quad \text{such that} \quad (x_j \mid j \in J') \leqq f_k ,$$

and so $S^A f_k \in U$ and $f_k \in V$.

So the additive topology τ_A in A is the largest topology for A such that the sum is continuous.

Corollary. *If* $f : (A, \Sigma^A) \to (B, \Sigma^B)$ *is additive, then it is additive continuous*

Proof. The following diagram is commutative, where

$$\bar{f}(x_i \mid i \in I) = (fx_i \mid i \in I)$$

is continuous.

$$\begin{array}{ccc} DDS^A & \xrightarrow{S^A} & A \\ \big\downarrow \bar{f} & & \big\downarrow f \\ DDS^B & \xrightarrow{S^B} & B \end{array}$$

Then fS^A is continuous and so is f because the topology in A is the quotient one given by S^A.

(3.7) Let (A, ≦) be a poset such that (A, sup) is a pam (2.2). Then A is ω-chain-complete (ω is the first infinite ordinal), that is, every countable chain has a supremum. Therefore all countable directed subsets D of A have a supremum ([10], p. 55).

Proposition. *Let* (A, ≦) *be a poset such that* (A, sup) *is a pam. Then the Scott countable topology (3.6) coincides with the additive topology. Moreover, the relation (2.8) is the given one.*

Corollary. *The additive topology of* (Rel(A, B), ∪) *and* (Pfn(A, B), Σ^{ov}) *(2.2) is the Scott countable topology.*

Proposition. *In the pam* (Pfn(A, B), Σ^{di})*, where* Σ^{di} *is the disjoint sum (2.2), the additive topology is the Scott countable topology, and then the additive topology of* (Pfn(A, B), Σ^{ov}).

Proof. It is routine to prove that the Scott countable topology is contained in the additive one. To show the opposite let us remark that $f \in$ Pfn(A, B) can be seen as a subset of AxB, namely

$$\{(a, fa) \mid a \in DD\, f\},$$

and the directed or disjoint union and difference of such sets can be interpreted as an element of Pfn(A, B). Actually the directed (resp. disjoint) union corresponds with the supremum (resp. sum) in Pfn(A, B).

Thus if $D = \{f_n \mid n \in N\}$ is a directed subset of Pfn(A, B) such that $\amalg D \in U$, where U is an additive open, define

$$g_o = f_o, \qquad g_{n+1} = f_{n+1} - \cup(g_i \mid i = 0, \ldots, n).$$

Then the family $(g_n \mid n \in N)$ is disjoint-summable and

$$\Sigma^{di}(g_n \mid n \in N) = \cup(g_n \mid n \in N) = \amalg(f_n \mid n \in N) = \amalg D \in U,$$

so there exists $n \in N$ such that

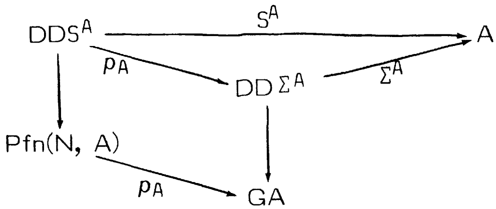
$$\Sigma^{di}(g_i \mid i = 0, \ldots, n) = f_n \in U.$$

Thus, the additive topology is quite a natural one at least in these basic examples of pams.

(3.8) *Proposition.* *Let* A *be a set. The additive topology of the pam* (GA, μA) *(§ 2) is the quotient topology given by* p_A : Pfn(N, A) ⟶ GA *Moreover,* p_A *is open.*

Proof. is straightforward.

(3.9) Let (A, Σ^A) be a pam. Then the diagram



is commutative. Moreover,

$$DDS^A = p_A^{-1}(DD\Sigma^A),$$

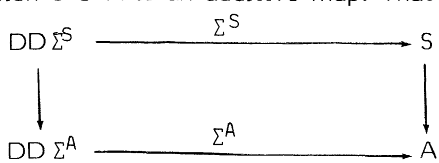
so DD Σ^A is closed in GA and its relative topology is the quotient one given by p_A : $DDS^A \to DD\Sigma^A$. Therefore Σ^A: $DD\Sigma^A \to A$ is an identification and gives once more the additive topology of A.

Corollary. *Let* $f : (A, \Sigma^A) \to (B, \Sigma^B)$ *be a morphism of G-algebras. Then* DDf ⊂ A *is additive closed.*

4. SUBPAMS.

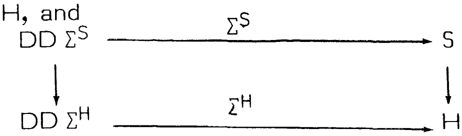
(4.1) Let (A, Σ^A) be a pam. A subobject of it in Pam is said to be a *subpam* of (A, Σ^A).

Given that Pam has a factorization system (2.9) subpams can be identified with subsets S of A endowed with a pam structure (S, Σ^S) such that the inclusion S ⊂ A is an additive map. That is,



is commutative.

If (S, Σ^S) and (H, Σ^H) are subpams, we define (S, Σ^S) ≦ (H, Σ^H) to mean that S ⊂ H, and



is commutative.

(4.2) Given a pam (A, Σ^A), we define Sub(A) to be the set of all subsets $Z \subset DD\Sigma^A$ such that

(i) Z is closed in $DD\Sigma^A$ (equivalently in GA) (3.9),
(ii) if $[f] \in Z$, then $[(\Sigma^A f)] \in Z$,
(iii) if f_i, $g_i \in Z$ and $\Sigma^A f_i = \Sigma^A g_i$ for all $i \in I$ (where I is countable), then $\langle f_i \mid i \in I\rangle \in Z$ implies that $\langle g_i \mid i \in I\rangle \in Z$.

Sub(A) is ordered by inclusion and is closed under arbitrary intersections. Note that the intersection of the empty family is $DD\Sigma^A$. Therefore, Sub(A) is a complete lattice.

Theorem. *There is an isomorphism of posets between* Sub(A) *and the set of subpams of* (A, Σ^A).

Proof. If (S, Σ^S) is a subpam, then $DD\Sigma^S \in$ Sub(A) since it is closed in GS and therefore in GA. The conditions (ii) and (iii) are fulfilled by $DD\Sigma^S$ due to the unary and the partition-associativity axioms (2.1).

On the other hand, any $Z \in$ Sub(A) gives rise to an evident subpam

$$\begin{array}{ccc} Z & \xrightarrow{\Sigma^S} & S \\ \downarrow & & \downarrow i \\ DD\Sigma^A & \xrightarrow{\Sigma^A} & A \end{array}$$

where (Σ^S, i) is the image factorization of the composition $Z \subset DD\Sigma^A \to A$.

Corollary. *The ordered set of subpams of a given pam* (A, Σ^A) *is a complete lattice.*

Moreover, the forgetful functor U : Sub(A) $\to$ P(A) *(where* P(A) *is the complete lattice of all subsets of* A, *and* $UZ = \Sigma^A(Z)$ *), has a left adjoint* ACL *(algebraic closure) given by*

$$ACL(S) = (S \cup \{\bot_A\}, \Sigma^*),$$

where Σ^* *is just defined for one-element families or infinite* $(x_i \mid i \in N)$ *where* $x_i = \bot_A$ *for all* $i \in N$ *but at most one.*

(4.3) ***Proposition.*** *Let* $C \subset A$. *Then* C *is closed iff* $Z = (\Sigma^A)^{-1} C \in$ Sub(A).

Proof. Z is closed since Σ^A is continuous. Conditions (ii) and (iii) follow from the unary and partition associativity axioms.

Conversely, if $Z = (\Sigma^A)^{-1} C \in$ Sub(A), then C is closed since the additive topology is the quotient one given by Σ^A (3.9).

Let C be a subset of A, define

$$Str(C) = \{ Z \mid Z \in Sub(A),\ UZ = \Sigma^A(Z) = C \}.$$

If C is closed, then

$$(\Sigma^A)^{-1} C = \sup\ [Str(C)] = \max\ [Str(C)].$$

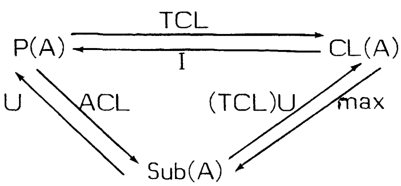
Actually, Str(C) is then a complete lattice.

Corollary. *Let* CL(A) *be the set of closed sets in* τ_A. *The functor*

$$\max : CL(A) \to Sub(A), \qquad \max(C) = (\Sigma^A)^{-1} C,$$

is the right adjoint to (TCL)U : Sub(A) → CL(A) , *where* TCL *is the topological closure functor from* P(A) *to* CL(A).

(4.4) Let I : CL(A) → P(A) be the inclusion functor. Then I is right adjoint to TCL : P(A) →CL(A). Thus, we have the diagram



where

$$TCL \dashv I, \qquad ACL \dashv U, \qquad (TCL)U \dashv \max.$$

Therefore,

$$(TCL)U(ACL) \dashv U\max = I$$

and then

$$TCL = (TCL)U(ACL).$$

Furthermore, since the topological closure of any subset contains $\perp_A$, we have that

$$I(TCL) = U(ACL)I(TCL).$$

Hence, if we call

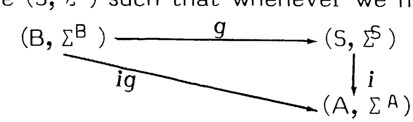
$$F = I(TCL) : P(A) \to P(A) \quad \text{and} \quad H = U(ACL) : P(A) \to P(A)$$

we have that

$$F = HF = FH.$$

In this sense we can say that the algebraic and topological closures commute.

(4.5) Definition ([8], p. 147). An *optimal lift* of an inclusion $i : S \subset A$ is a pam substructure (S, Σ^S) such that whenever we have a diagram



g is additive iff ig is.

Optimal lifts, when they exist, are unique.

Proposition. *Let* C *be a closed subset of* A. *Then the inclusion* $C \subset A$ *has an optimal lift : the structure given by* $(\Sigma^A)^{-1}C$ (4.3).

But optimal lifts do not need to be closed. To see this let us consider (A, sup), where $A = \{0, 1, 2\}$ $(0 \leq 1 \leq 2)$. Then $S = \{0, 2\}$ is not closed (since $1 \leq 2$, but $1 \notin S$). Nevertheless, (S, sup) is an optimal lift since there exists $\sup(X) \in S$ for all $X \subset S$.

The remainder of this section is devoted to relating the topologies of subpams.

(4.6) Let (S, Σ^S) be a subpam of (A, Σ^A). Then, the additive topology in S given by Σ^S, $\tau_\Sigma S$, contains the relative topology in S as a subset of A, τ_{rel}. To see this, let $C \subset A$ be a closed set, then

$$\begin{array}{ccccc} H & \longrightarrow & S \cap C & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ DD\Sigma^S & \xrightarrow{\Sigma^S} & S & \longrightarrow & A \end{array}$$

is a pullback, and then H is closed in $DD\Sigma^S$ (equivalently in $DD\Sigma^A$). Hence, $S \cap C$ is closed in $\tau_\Sigma S$, because $(\Sigma^S)^{-1}(S \cap C) = H$ is closed.

Proposition. *Let* $Z_0, Z_1 \in \mathrm{Sub}(A)$ *such that* $UZ_0 = UZ_1 = S$. *If* $Z_0 \subset Z_1$, *then* $\tau_{Z_1} \subset \tau_{Z_0}$.

Proposition. *Let* C *be a closed subset of* A. *The subpam structure* $Z = (\Sigma^A)^{-1} C$ *(4.3) induces the relative topology ; that is*

$$\tau_Z = \tau_{(\Sigma^A)^{-1} C} = \tau_{rel}.$$

Proof. If C_1 is closed in τ_Z, then its pullback over $\Sigma^A/Z : Z \to C$, Z_1, is closed. Therefore

$$\begin{array}{ccc} Z_1 & \longrightarrow & C_1 \\ \downarrow & & \downarrow \\ Z & \longrightarrow & C \\ \downarrow & & \downarrow \\ DD\Sigma^A & \xrightarrow[\Sigma^A]{} & A \end{array}$$

is a pullback, and then C_1 is closed in A.

5. CONTINUOUS PAMS.

In this paragraph we shall introduce a special kind of pams in order to get the opposite of (3.5). We call them continuous pams following Scott. These pams satisfy a very natural axiom given below. Of course, the basic examples given in (2.2), and the free pam (GA, μA) for all A, are continuous pams.

(5.1) Definition. A pam (A, Σ) is said to be a *continuous pam* provided that whenever

$$\Sigma(x_i \mid i = 0, \dots, n) \leqq a \qquad \text{for all} \qquad n \in N,$$

then $\Sigma(x_i \mid i \in N) \leqq a$.

Equivalently, given a family $(x_i \mid i \in I)$ in A and $a \in A$, we have $\Sigma(x_i \mid i \in F) \leqq a$ for all finite subsets F of I iff $\Sigma(x_i \mid i \in I) \leqq a$.

(5.2) Proposition. *Let (A, Σ) be a pam. The following statements are equivalent :*

(i) (A, Σ) is continuous.

(ii) The sets $Ra = \{b \mid b \nleqq a\}$ are in τ_A for all $a \in A$.

(iii) $\downarrow b = \{c \mid c \leqq b\}$ is the topological closure of $\{b\}$ for all $b \in A$. That is, $\downarrow b = \overline{\{b\}}$.

(iv) $a \leqq b$ iff $b \in U$ whenever $U \in \tau_A$ and $a \in U$.

Proof. The statements (ii) and (iii) are obviously equivalent.
(i) implies (ii) : Let $(x_i \mid i \in N)$ such that $\Sigma(x_i \mid i \in N) \in Ra$. If

$$\Sigma(x_i \mid i = 0, \dots, n) \leqq a \qquad \text{for all} \qquad n \in N,$$

then $\Sigma(x_i \mid i \in N) \leqq a$.

(ii) implies (iv) : Let $a, b \in A$ such that $b \in U$ whenever $U \in \tau_A$ and $a \in U$. if $a \nleqq b$, then $a \in Rb$, but $b \notin Rb$.

(iv) implies (i) : Let $(x_i \mid i \in N)$ be a family in A such that

$$\Sigma(x_i \mid i = 0, \dots, n) \leqq a \qquad \text{for all} \qquad n \in N.$$

If $\Sigma(x_i \mid i \in N) \in U$, $U \in \tau_A$, then there exists $n \in N$ such that

$$\Sigma(x_i \mid i = 0, \dots, n) \in U,$$

hence $a \in U$, and $\Sigma(x_i \mid i \in N) \leqq a$.

Corollary. *Let $(a_n \mid n \in N)$ be an ascending chain in a continuous pam (A, Σ). Then $(a_n) \underset{\tau_A}{\longrightarrow} b$ iff $b \leqq a$ for every $a \in$ AM-lim (a_n).*

Proof. It is due to (3.4), and (iv) above.

Corollary. *Let $(a_n \mid n \in N)$ be an ascending chain in a continuous pam (A, Σ), and $a, b \in$ AM-lim (a_n). Then $a \leqq b$ and $b \leqq a$.*

(5.3) Proposition. *Let (A, Σ) be a continuous pam. Then the following statements are equivalent :*

(i) (A, Σ) is a T_0-space.
(ii) (A, Σ) (2.8) is a poset.
(iii) The AM-limits are unique.

Proof. By (5.2) and (3.2).

Remark. The partial order " $\leqq$" defined on a T_0-space by

$$x \leqq y \quad \text{iff} \quad x \in \{y\}^-$$

is called the *specialization order* ([6], p. 123).

If (A, Σ) is a continuous pam that fulfills any of the equivalent statements above, the partial order defined in (2.8) is the specialization order ; that is to say, the additive topology determines the partial ordering by means of a purely topological definition.

Corollary. *Let (A, Σ) be a continuous pam that fulfills any of the equivalent statements of the last theorem. Let $(a_n \mid n \in N)$ be any ascending chain in A. Then the AM-limit of (a_n) is*

$$\max\{a \mid (a_n) \xrightarrow[\tau_A]{} a\} = \sup\{a_n \mid n \in N\}$$

(5.4) Theorem. *Let (A, Σ^A), (B, Σ^B) be continuous pams and posets, and let $f : A \to B$ be a function. The following statements are equivalent :*

(i) f is additive-continuous.
(ii) f is AM-continuous (3.3).
(iii) f preserves sup of countable ascending chains.
(iv) f preserves sup of countable directed sets.

Proof. The equivalence of (iii) and (iv) is in ([10], p. 56). The statements (ii) and (iii) are the same because of (5.3). And (ii) implies (i) was proved in (3.5).

To prove that (i) implies (ii), we shall need the following

Lemma. If $f: A \to B$ is additive continuous and B is a continuous pam, then f is isotone.

Proof of the Lemma. Let $a, b \in A$ such that $a \leqq b$ and let $U \in \tau_B$ so that $fa \in U$. Then $a \in f^{-1}U \in \tau_A$, hence $a \leqq b \in f^{-1} U$; that is to say $fb \in U$. Thus $fa \leqq fb$ by (5.2.iv).

(i) implies (ii) : Let $(a_n \mid n \in N)$ be an ascending chain in A. Then

$$(a_n) \xrightarrow[\tau_A]{} \sup\{a_n \mid n \in N\}.$$

Hence

$$(fa_n) \xrightarrow[\tau_B]{} f \sup\{a_n \mid n \in N\},$$

and then

$$f \sup\{a_n \mid n \in N\} \leqq \sup\{fa_n \mid n \in N\}$$

because (fa_n) is an ascending chain in B by the lemma and in virtue of (5.2).

On the other hand, due to the lemma

$$fa_n \leqq f \sup\{a_n \mid n \in N\} \quad \text{for all} \quad n \in N.$$

Therefore

$$\sup\{fa_n \mid n \in N\} \leqq f \sup\{a_n \mid n \in N\}.$$

(5.5) Let A, B be sets. Then (GA, μA) (§ 1) and all the pam examples given in (2.2) are continuous pams.

6. COMPARISON BETWEEN THE ORDER AND THE SUM IN A PAM.

We now return to compare the order and the sum in a pam (A,Σ). What we are going to study is the question arisen in (2.2). That is, some posets $(L, \leqq)$ are pams endowed with sup ; we shall describe the posets thereby obtained. Conversely, we shall characterize the pams which come from such posets.

To this end we shall build a new algebraic theorey in Pfn : $\mathbf{P}_N$. In a certain sense the construction of (L, sup)-pams from $\mathbf{P}_N$-algebras is parallel to the way we got pams from **G**-algebras. Furthermore, we shall find a close relation between $\mathbf{P}_N$ and **G**-algebras.

(6.1) Let A be a set. Define $P_N A$ to mean the set of all countable subsets of A. Thus we have got a functor P_N: Pfn → Pfn,

$$P_N(f : A \to B) = (P_N f : P_N A \to P_N B),$$

where

$$(P_N f)A' = \begin{cases} \{fa \mid a \in A'\} & \text{if } A' \subset DDf \\ \text{undefined else} \end{cases}$$

for all $A' \in P_N A$.

Define $UA : P_N P_N A \to P_N A$ by

$$UA\,[\{A'_i \mid i \in I\}] = \cup(A'_i \mid i \in I),$$

and

$$\eta A : A \to P_N A \quad \text{by} \quad (\eta A)a = \{a\}.$$

It is easy to check that $\mathbf{P}_N = (P_N, \eta, U)$ is an algebraic theory in Pfn in monoid form.

Moreover, if $\sup : P_N A \to A$ is an arbitrary partial function, the

P_N-algebra equations (2.5) are clearly equivalent to

(i) $\sup\{x\} = x$ for all $x \in A$,
(ii) $\sup(\cup(A_i' \mid i \in I)) = \sup(\sup A_i' \mid i \in I)$,

where the equal sign in (ii) is in the sense of the equality in the partition-associativity axiom (2.1.1) ; that is, the left side is defined iff the right side is defined, and then they are equal.

It is a well known result that if (A, sup) satisfies (i) and (ii) above, and sup is total, then, via

$$a \leqq b \quad \text{iff} \quad \sup\{a, b\} = b,$$

A is a poset, and sup A' is the supremum of A' for all $A' \in P_N A$ ([8], p. 57). No extra difficulties arise if sup is a Pfn-morphism.

(6.2) *Theorem.* *Let* A *be a set, and let* $\sup : P_N A \to A$ *be a* P_N*-morphism. Define*

$$a \leqq b \quad (a, b \in A) \qquad \textit{iff} \quad \sup\{a, b\} = b.$$

The following statements are equivalent :

(i) (A, sup) is a P_N*-algebra.*
(ii) (A, $\leqq$*) is a poset such that if* $X \subset A$ *(*$X \in P_N A$*) has a supremum and* $X' \subset X$ *, then X' has a supremum.*
(iii) (A , $\leqq$*) is a poset and every countable subset of A with upper bounds has a supremum.*

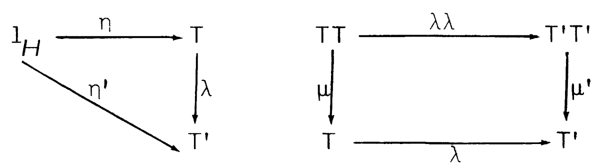
Proof hints. (i) implies (ii) : By (6.1.ii)

$$\sup(X) = \sup(X' \cup (X-X')) = \sup(\sup(X'), \sup(X-X')).$$

(ii) implies (iii) : If a is an upper bound of X', then the supremum of $X' \cup \{a\}$ is a ; therefore X' has a supremum.

(iii) implies (i) : (6.1.ii) is satisfied in every poset provided that $\sup(A_i')$ exists whenever $\sup(\cup(A_i' \mid i \in I))$ exists

(6.3) Definition ([8], p. 209). Let $\mathbf{T} = (T, \eta, \mu)$, $\mathbf{T'} = (T', \eta', \mu')$ be algebraic theories in a category H in monoid form. A *theory map* $\lambda : \mathbf{T} \to \mathbf{T'}$ is a natural transformation $\lambda : T \to T'$ such that the following two diagrams commute :



Each theory map $\lambda : \mathbf{T} \to \mathbf{T'}$ induces a functor $F_\lambda : H^{\mathbf{T'}} \to H^{\mathbf{T}}$ defined by

$$F_\lambda(X, \xi' : T'X \to X) = (X, \xi'(\lambda X) : TX \to X).$$

Define the natural trtansformation $\alpha : G \to P_N$ (where G is the functor defined in (2.4)) given by

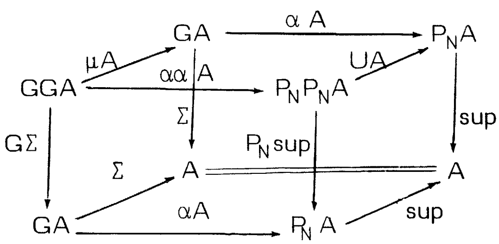
$$\alpha A : GA \to P_N A, \qquad \alpha A(a_i \mid i \in I) = \{a_i \mid i \in I\} .$$

It is not hard to check that α is a theory map from **G** to $\mathbf{P}_N$. Thus we have a functor

$$F_\alpha : \mathrm{Pfn}^{P_N} \to \mathrm{Pfn}^{G} .$$

(6.4) Theorem. *Let A be a set, and let Σ: GA $\to$ A, sup : $P_N A \to A$ be partial functions such that $\Sigma = (\sup)(\alpha A)$. Then (A, Σ) is a G-algebra iff (A, sup) is a $\mathbf{P}_N$-algebra.*

Proof hints. Let us consider the diagram



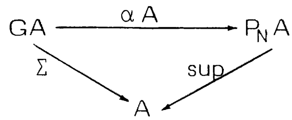
The commutativity of the right-most facet is equivalent to the commutativity of the left-most facet, since $\alpha\alpha A$ is an epimorphism and the rest of the cube is commutative. The opposite is obviously true since the identity A = A is an isomorphism.

Corollary. *A poset (A, $\leqq$) gives rise to a pam (A, sup) iff it is chain ω-complete and fulfills any of the equivalent statements (6.2.i), (6.2.ii), or (6.2.iii).*

This Corollary could be obtained with a straightforward proof. But in the present approach we have additionally got an algebraic representation of these posets parallel to the one given for pams. Moreover, the conditions here required to posets have arisen in a natural way.

The remainder of this paragraph is devoted to searching necessary and sufficient conditions to ensure that a pam (A, Σ) is of the form (A, sup) for a given ordering (A, $\leqq$). Notice that this ordering must be the one given in (2.8), since in the last Corollary the pam-ordering (in the sense of (2.8)) of (A, sup) is the original one.

(6.5) In virtue of the Theorem (6.4), a necessary condition is that $\Sigma : GA \to A$ must be factorizable through αA.



Thus, Σ must be independent of the number of occurrences of an element in a family. This idea is formalized in the next definition.

Definition. A pam (A, Σ) is said to be *idempotent* provided that

$$\Sigma(a_i \mid i \in I) = a \quad \text{if} \quad a_i = a \text{ for all } i \in I \text{ and for all countable } I.$$

Theorem. *Let (A, Σ) be a pam. The following statements are equivalent :*

(i) (A, Σ) is idempotent.

(ii) There is a $\mathbf{P}_N$-algebra (A, ϑ) such that $F_\alpha(A, \vartheta) = (A, \Sigma)$.

Proof. Obviously (ii) implies (i). To see the opposite let us notice that $\Sigma : GA \to A$ can be factorizable through αA. Let $\vartheta : P_N A \to A$ be the only Pfn-morphism such that $\vartheta(\alpha A) = \Sigma$. Then (A, ϑ) is a $\mathbf{P}_N$-algebra (6.4), and $F_\alpha(A, \vartheta) = (A, \Sigma)$.

Corollary. *Let (A, Σ) be an idempotent pam. Then $(A, \leqq)$ (2.8) is a poset.*

Let us now characterize, in the additive sense, the idempotent pams.

(6.6) *Proposition.* *Let (A, Σ) be a pam such that $(A, \leqq)$ (2.8) is a poset. The following statements are equivalent :*

(i) (A, Σ) is idempotent.

(ii) For all $x \in A$, $(x_i \mid i \in N) = x$ if $x_i = x$ for all $i \in N$.

(iii) (A, Σ) is continuous and $x+x = x$ for all $x \in A$.

Proof. (ii) implies (iii) : Let $x \in A$, if $x_i = x$ for all $i \in N$,

$$x+x = \Sigma(x_i \mid i \in N) + \Sigma(x_i \mid i \in N) = \Sigma(x_i \mid i \in N) = x.$$

To see that (A, Σ) is continuous, let

$$\Sigma(x_i \mid i = 0, \ldots, n) \leqq a \qquad \text{for all} \quad n \in N.$$

Then there exists a family $(y_i \mid i \in N)$ such that $x_i + y_i = a$, thus

$$a = \Sigma(a_i \mid i \in N) = \Sigma(x_i + y_i \mid i \in N) = \Sigma(x_i \mid i \in N) + \Sigma(y_i \mid i \in N),$$

where $a_i = a$ for all $i \in N$; therefore $\Sigma(x_i \mid i \in N) \leqq a$.

(iii) implies (i) : Let I be a countable set and let $x_i = x$ for all $i \in I$. Then $\Sigma(x_i \mid i \in F) = x$ for all $F \subset I$ finite. Then

$$x \leqq \Sigma(x_i \mid i \in I) \leqq x .$$

To end the paragraph let us collect some of the results here obtained in the following Corollary.

Corollary. *Let (A, Σ) be an idempotent pam. Then (A, Σ) is continuous and $(A, \leqq)$ (2.8) is a poset.*

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