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Proarrows II

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PROARROWS II
by R. J. WOOD

RÉSUMÉ. On continue une étude récente (cf. "Cahiers" XXIII-3, 1982) sur les homomorphismes de bicatégories $()_* : K \rightarrow M$ qui partagent certaines propriétés avec $()_* : CAT \rightarrow PROF$. Ici on impose sur $()_*$ des conditions d'exactitude sur des collages dans M , et la structure qui en résulte est assez riche. Notamment, on a un système de factorisation utile dans K , et les colimites locales existent dans M . Ces dernières donnent un calcul de matrices pour les morphismes de collages dans M . $()_* : TOP \longrightarrow TOPLEX^{co}$, "oublier l'image inverse", est encore un modèle pour le système élargi d'axiomes. Quelques propositions élémentaires sur les morphismes géométriques de topoi se retrouvent par ce formalisme simple.

0. INTRODUCTION.

This paper is a sequel to [WD1]. The objects of study are homomorphisms of bicategories, $()_* : K \rightarrow M$, which satisfy the following :

Axiom 1. The objects of M are those of K and $()_*$ is the identity on objects.

Axiom 2. $()_*$ is locally fully faithful.

Axiom 3. For every arrow f in K , f_* has a right adjoint f^* in M .

Such a homomorphism $()_*$ is said to *equip K with proarrows*, and is referred to as *proarrow equipment*.

There are many examples but two deserve particular attention. The paradigm is $()_* : CAT \rightarrow PROF$, where for $f : A \rightarrow B$,

$$f_*(b, a) = B(b, af).$$

A number of aspects of category theory are simplified by the introduction of profunctors - modulo their somewhat complicated composition. An axiomatic approach preserves the simplification while forgetting this complexity. When generalized category theory is contemplated in the form of $V-CAT$, $S-ind\ CAT$, or $cat(S)$ for monoidal V or left exact S this advantage becomes quite significant. Motivated chiefly by this, [WD1] pursued a development of a fragment of formal category theory along the lines set out in [S&W] for "Yoneda structures",

a related notion. Axioms 1, 2 and 3 do not take us as far as those for Yoneda structures. Missing is an ingredient which allows one to speak of "bijective on objects" arrows (here called "surjections", for reasons which will become clear shortly). In Section 2 we impose Axiom 5 which makes a strong claim about existence of Kleisli objects in M . Section 2 then quickly completes those aspects of the program in [S&W] which are compatible with proarrow equipment.

A second leading example is provided by "forget the inverse image", $()_* : TOP \rightarrow TOPLEX^{co}$, where TOP is topoi and geometric morphisms and $TOPLEX^{co}$ is topoi and left exact functors. It does not follow from Axioms 1, 2 and 3 that $()_* : K \rightarrow M$ enjoys a universal property. However $PROF$ is equivalent to codiscrete cofibrations in CAT and it transpires that $TOPLEX^{co}$ is equivalent to codiscrete cofibrations in TOP . The latter result was the main focus of [RW1] and the example was pursued further in [RW2]. Similar results for abelian categories and geometric morphisms were presented in [RW3].

Section 2 is independent of Section 1, which deals briefly with finite sums in K and M and concludes with a simple Axiom (4) that is assumed together with Axiom 5 for Sections 3 through 6. Local colimits, as described in [ST1], are met in Section 1 and subsequently. We remind the reader that a local colimit in M is a colimit in a hom category $M(B, A)$ that is preserved by composing with all $B' \rightarrow B$ and all $A \rightarrow A'$. Some of our results are similar to those in [ST1].

Section 3 establishes the existence of Artin glueing and a number of propositions that follow easily from it. A matrix calculus is employed and this is developed somewhat further in Section 5. Section 4, "Surjections Revisited", continues the theory of Section 2 using, by then available, local coequalizers in M . A monadicity theorem for M is the key result.

Section 6 deals exclusively and in some detail with the verification of Axioms 4 and 5 for the examples CAT and TOP . The reader may wish to consult it immediately after the introduction of Axiom 5. The latter is initially presented in a non-concise form that allows one to work with it immediately - at the expense of masking an overview structure. In Section 7 we introduce a formalism that reveals this structure and shows the common nature of Axioms 4 and 5.

Some remarks about notation and terminology are in order. $\Phi\Psi$ denotes the composite

$$\cdot \xrightarrow{\Phi} \cdot \xrightarrow{\Psi} \cdot \cdot$$

We wrote $\Phi \otimes \Psi$ for it in [WD1]. Whenever possible we write f for f_* and refer to such (and their isomorphs) as *representables*. For the most part we ignore coherence and explicit descriptions of bi-phenomena,

saying "pushout" etc. when we mean bi-pushout etc. We hope that the loss of precision is off-set by a gain in clarity. For $f : A \rightarrow B$ in K we write $\tilde{f}$ for the unit and $\tilde{L}$ for the counit of the $f \dashv f^*$ adjunction in M . If $\tilde{f}$ is an isomorphism we say that f is an *inclusion*. The term *fully faithful* was employed in [WD1].

The author wishes to thank R.D. Rosebrugh, R. Paré and F.W. Lawvere for useful discussions. Lawvere and Paré have suggested independently that Axiom 1 be dispensed with. The theory that results will be developed elsewhere; however, some preliminary remarks as to a further unification of the two leading examples above seem appropriate.

Call a homomorphism satisfying Axioms 2 and 3 *proequipment*. Write cat for the bicategory of small categories and TOT for the bicategory of total objects in CAT (relative to $P = set^{(\)^{op}}$, see [S&W] or [WD1]) and cocontinuous functors. Then $cat \rightarrow TOT$ given by

$$(f : A \rightarrow B) \mapsto (f_! : set^{A^{op}} \rightarrow set^{B^{op}}),$$

where $f_!$ is left Kan extension along f , is *proequipment*. In [WD1] we pointed out that taking the full image of a homomorphism satisfying Axioms 2 and 3 produces proarrow equipment. Applying this to $cat \rightarrow TOT$ we recover $(\)_* : cat \rightarrow prof$. Any cocontinuous functor between total categories has a (CAT) right adjoint, so $TOT \rightarrow CAT$ is also *proequipment*.

Write LEX (respectively lex) for the bicategory of left exact objects, left exact arrows and arbitrary transformations in CAT (resp. cat). Noting that the left Kan extension of a left exact functor is left exact and observing dualities, we see that $lex^{co} \rightarrow TOP$ given by

$$(f : A \rightarrow B) \mapsto (f^! : set^{A^{op}} \rightarrow set^{B^{op}}),$$

where $f^!$ is *right* Kan extension along f , is *proequipment*. Clearly $TOP \rightarrow LEX^{co}$ is *proequipment* too.

It is interesting to note these examples together :

$$cat \rightarrow TOT \rightarrow CAT, \quad lex^{co} \rightarrow TOP \rightarrow LEX^{co}.$$

Certainly, general questions about change of universe are provoked. Moreover, the second row is essentially lex^{co} of the first. This is because TOP could be replaced by Grothendieck topoi and Freyd has shown that the latter are precisely the total objects in LEX .

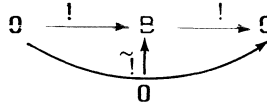
1. FINITE SUMS.

Proposition 1. *If K has an initial object, 0 , which is preserved by $(\)_* : K \rightarrow M$ and $(\)^* : K^{coop} \rightarrow M$ preserves the terminal object of*

$K^{\text{coop}}(\mathbf{0})$, then

- (i) For all B in K , $! : \mathbf{0} \rightarrow B$ is an inclusion,
- (ii) M has initial objects locally.

Proof. (i)



is an isomorphism since $\mathbf{0}$ is initial in M .

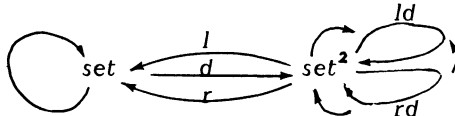
- (ii) $(B \xrightarrow{0} A) \simeq (B \xrightarrow{!*} \mathbf{0} \xrightarrow{!} A)$. For given $\Phi : B \rightarrow A$,

$$\frac{!*\Phi : B \rightarrow A}{! \rightarrow !\Phi : \mathbf{0} \rightarrow A} \quad *$$

Hence $\mathbf{0} : B \rightarrow A$ is initial in $M(B, A)$. For any $\Phi : A \rightarrow A'$, $\mathbf{0} \Phi \simeq \mathbf{0}$ since $(!_A)\Phi \simeq !_A'$. (Observe that thus far we have not used the hypothesis concerning $(*)$.) For any $\Phi : B' \rightarrow B$, $\mathbf{0}$ being terminal in M yields

$$\Phi (!_B)^* \simeq (!_{B'})^* \quad \text{and hence} \quad \Phi \mathbf{0} \simeq \mathbf{0}. \quad \diamond$$

Counterexample 2. If K has an initial object $\mathbf{0}$ which is preserved by $(*)_* : K \rightarrow M$, it does not follow that $\mathbf{0}$ is also terminal in M . Let K be the subcategory of CAT , $d : \text{set} \rightarrow \text{set}^2$, where d is the diagonal. (So K is isomorphic to the locally discrete bicategory $\mathbf{2}$ and set is initial in K .) Let M be the subcategory of CAT :



where $l \dashv d \dashv r$ in CAT , and the non-identity transformations shown are those arising from the adjointness. Then, we still have $l \dashv d \dashv r$ in M , the inclusion of K in M is proarrow equipment, set is initial in M but set is not terminal in M .

Proposition 3. If K has finite sums which are preserved by $(*)_* : K \rightarrow M$ and $(*)^* : K^{\text{coop}} \rightarrow M$ preserves finite products, then

- (i) M has finite sums locally ;

for every sum diagram, $i : A \rightarrow A+B \leftarrow B : j$, in K , with $A+B$ denoted $A \oplus B$ when regarded as an object of M ,

- (ii) the injections are inclusions,
- (iii) $\mathbf{0} \xrightarrow{\sim} ij^*$ and $\mathbf{0} \xrightarrow{\sim} ji^*$,
- (iv) $A \oplus B \simeq i^*i + j^*j$.

Proof. (i) If $\Phi, \Psi : B \rightarrow A$, then

$$(B \xrightarrow{\Phi + \Psi} A) \simeq (B \xrightarrow{c^*} B \oplus B \xrightarrow{\Phi \oplus \Psi} A \oplus A \xrightarrow{c} A),$$

where $c: A+A \rightarrow A$ is the codiagonal. For given $\Gamma: B \rightarrow A$:

$$\frac{c^*(\Phi \oplus \Psi)c \rightarrow \Gamma: B \rightarrow A}{(\Phi \oplus \Psi)c \rightarrow c\Gamma: B \oplus B \rightarrow A}$$

$$\frac{[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}] \rightarrow [\begin{smallmatrix} \Gamma \\ \Gamma \end{smallmatrix}]: B \oplus B \rightarrow A}{\Phi \rightarrow \Gamma, \Psi \rightarrow \Gamma: B \rightarrow A}$$

Hence $M(B, A)$ has finite sums. For any $\Gamma: A \rightarrow A'$,

$$(\Phi \oplus \Psi)c_A \Gamma \simeq [\begin{smallmatrix} \Phi \Gamma \\ \Psi \Gamma \end{smallmatrix}] \simeq (\Phi \Gamma \oplus \Psi \Gamma)c_{A'}.$$

So $(\Phi + \Psi)\Gamma \simeq \Phi\Gamma + \Psi\Gamma$. (Again, observe that thus far we have not used the hypothesis concerning $(\)^*$; it does not follow from the other hypotheses, as Counterexample 2 surely suggests.) $c^*: B \rightarrow B \oplus B$ is the product diagonal. For any $\Gamma: B' \rightarrow B$,

$$\Gamma(c_B)^*(\Phi \oplus \Psi) \simeq [\Gamma\Phi, \Gamma\Psi] \simeq (c_B)^*(\Gamma\Phi \oplus \Gamma\Psi).$$

So $\Gamma(\Phi + \Psi) \simeq \Gamma\Phi + \Gamma\Psi$.

(ii), (iii) and (iv). Consider the diagram

$$A \xleftarrow[\begin{smallmatrix} A \\ 0 \end{smallmatrix}]{i} A \oplus B \xleftarrow[\begin{smallmatrix} 0 \\ B \end{smallmatrix}]{j} B.$$

For any $\Gamma: A \rightarrow C$ and any $[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}]: A \oplus B \rightarrow C$ we have:

$$\frac{[\begin{smallmatrix} A \\ 0 \end{smallmatrix}]\Gamma \rightarrow [\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}]}{[\begin{smallmatrix} \Gamma \\ 0 \end{smallmatrix}] \rightarrow [\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}]}$$

$$\frac{\Gamma \rightarrow \Phi}{\Gamma \rightarrow i[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}]}$$

Hence $i \dashv \begin{smallmatrix} A \\ 0 \end{smallmatrix}$ and thus $i^* \simeq \begin{smallmatrix} A \\ 0 \end{smallmatrix}$. Similarly, $j^* \simeq \begin{smallmatrix} 0 \\ B \end{smallmatrix}$. So

$$A \xrightarrow{\sim} ii^*, \quad B \xrightarrow{\sim} jj^*, \quad 0 \xrightarrow{\sim} ij^* \quad \text{and} \quad 0 \xrightarrow{\sim} ji^*.$$

This proves (ii) and (iii). For any $[\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}]: A \oplus B \rightarrow A \oplus B$ we have:

$$\frac{i^*i + j^*j \rightarrow [\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}]: A \oplus B \rightarrow A \oplus B}{[\begin{smallmatrix} A \\ 0 \end{smallmatrix}]i + [\begin{smallmatrix} 0 \\ B \end{smallmatrix}]j \rightarrow [\begin{smallmatrix} \Phi \\ \Psi \end{smallmatrix}]: A \oplus B \rightarrow A \oplus B}$$

$$\begin{array}{c}
 \frac{\frac{[i]_0 + [0]_j \rightarrow [\frac{\Phi}{\Psi}]}{A \oplus B \rightarrow A \oplus B} \quad \frac{[i]_0 \rightarrow [\frac{\Phi}{\Psi}], [0]_j \rightarrow [\frac{\Phi}{\Psi}]}{A \oplus B \rightarrow A \oplus B}}{i \rightarrow \Phi : A \rightarrow A \oplus B, \quad j \rightarrow \Psi : B \rightarrow A \oplus B} \\
 \hline
 [i]_j \rightarrow [\frac{\Phi}{\Psi}] : A \oplus B \rightarrow A \oplus B \\
 \hline
 A \oplus B \rightarrow [\frac{\Phi}{\Psi}] : A \oplus B \rightarrow A \oplus B .
 \end{array}$$

So $i*i + j*j \simeq A \oplus B$ as claimed. $\diamond$

From $i*i \simeq A$ and $ij* \simeq 0$ we see that $i \simeq [A, 0]$. Similarly, $j \simeq [0, B]$. Indeed, if $(\)_*$ and $(\)^*$ satisfy the hypotheses of proposition 3, then M admits a matrix calculus for proarrows

$$B_1 \oplus \dots \oplus B_m \rightarrow A_1 \oplus \dots \oplus A_n$$

which is analogous to that available for a category with finite direct sums and abelian monoid valued homs. This will become most useful when we discuss glueing and local finite colimits. A transformation between matrices is a matrix of transformations and will be denoted with the help of brackets. On the other hand, transformations $\varphi : \Phi \rightarrow \Gamma$, $\psi : \Psi \rightarrow \Gamma : B \rightarrow A$ give rise to a transformation $\Phi + \Psi \rightarrow \Gamma : B \rightarrow A$ which will be denoted $(\frac{\Phi}{\Psi})$. This brackets-global versus parentheses-local convention will be extended without further ado to other tuples of transformations in the sequel.

Beginning in Section 3 we will assume :

Axiom 4. K has finite sums which are preserved by $(\)_*$ and $(\)^*$ preserves finite products. $\diamond$

2. THE KLEISLI CONSTRUCTION,

We recall some basic terminology concerning monads $(A, \Phi) = (A, \Phi, \eta, \mu)$ in a bicategory M . An *opalgebra* for (A, Φ) is an arrow $\Theta : A \rightarrow X$, together with a transformation

$$\begin{array}{ccc}
 A & & \\
 \downarrow \Phi & \searrow \Theta & \\
 & \omega \uparrow & \\
 A & \nearrow \Theta & X
 \end{array}$$

which satisfies the "equations" of a left Φ -action. (As usual, "equations" are to be understood with the help of the structural isomorphisms for

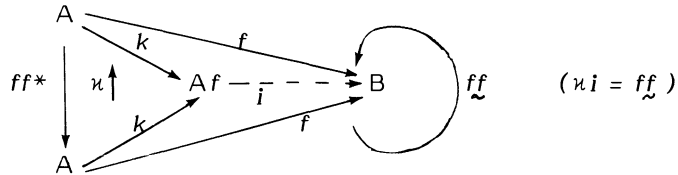
horizontal composition in M .) For opalgebras $[\frac{\Theta}{\omega}]$ and $[\frac{\Theta'}{\omega'}]$ with common domain, a *homomorphism of opalgebras* $[\frac{\Theta}{\omega}] \rightarrow [\frac{\Theta'}{\omega'}]$ is a transformation $\vartheta : \Theta \rightarrow \Theta'$ which is equivariant with respect to the Φ -actions. Write $M(A, \Phi)\text{-OPALG}(X)$ for the resulting category. A *Kleisli opalgebra* for (A, Φ) is an opalgebra $[\frac{K}{\kappa}]$, $K : A \rightarrow A_{\Phi}$, which is universal in that composing with it yields an equivalence of categories

$$M(A_{\Phi}, X) \sim M(A, \Phi)\text{-OPALG}(X).$$

Dually, one speaks of *algebras* $[\Omega : X \rightarrow A, \vartheta : \Omega\Phi \rightarrow \Omega]$ (right Φ -actions) and a universal such is called an *Eilenberg-Moore algebra*. The main reference for such matters is Street's paper [ST2].

Axiom 5. Every monad (A, Φ) in M has a representable Kleisli opalgebra $[\frac{K}{\kappa}]$, $k : A \rightarrow A_{\Phi}$. The equivalences $M(A_{\Phi}, X) \sim M(A, \Phi)\text{-OPALG}(X)$ respect representability and $[k^* : A_{\Phi} \rightarrow A, \kappa^*]$ is an Eilenberg-Moore algebra for (A, Φ) . $\diamond$

Let $f : A \rightarrow B$ be an arrow in K . We have a monad (A, ff^*) in M for which $[\frac{f}{ff^*}]$ is an opalgebra. (f is the counit for $f \dashv f^*$.) We will write Af for the Kleisli object A_{ff^*} . By Axiom 5 we have a "commutative" diagram



f is said to be a *surjection* if i in the diagram above is an equivalence.

Proposition 4. In the factorization $f \simeq ki$ above, k is a surjection and i is an inclusion.

Proof. Since $[\frac{K}{\kappa}]$ is a Kleisli opalgebra for ff^* , we have from [ST2] that ff^* is generated by the adjunction $k \dashv k^*$. That is, $kk^* \simeq ff^*$. Hence $Ak \sim Af$; so k is a surjection.

Explicitly, the isomorphism $kk^* \simeq ff^*$ is

$$k\tilde{i}k^* : kk^* \xrightarrow{\sim} kii^*k^*.$$

Since $[k^*, \kappa^*]$ is Eilenberg-Moore, we have $\tilde{k}i : k \xrightarrow{\sim} kii^*$. Since $[\frac{K}{\kappa}]$ is Kleisli we have $\tilde{i} : Af \xrightarrow{\sim} ii^*$ and i is an inclusion.

Proposition 5. If in a "commutative" diagram

$$\begin{array}{ccc} A & \xrightarrow{k} & B \\ f \downarrow & & \downarrow g \\ X & \xrightarrow{i} & Y \end{array}$$

in K , k is a surjection and i is an inclusion, then there exists an essentially unique arrow $d : B \rightarrow X$, in K , such that both triangles "commute".

Proof. $f \simeq fii^* \simeq kgi^*$. According to Axiom 5 the equivalence mediated by composition with k respects representability. Hence, there exists an essentially unique $d : B \rightarrow X$ in K and an isomorphism $\bar{\tau} : d \xrightarrow{\sim} gi^*$. So $kd \simeq f$. By adjointness we have $\tau : di \rightarrow g$. But

$$(k\tau : kdi \rightarrow kg) = (kdi \xrightarrow{\sim} fi \xrightarrow{\sim} kg)$$

where the isomorphisms are as above. Since k is Kleisli we have $\tau : di \xrightarrow{\sim} g$. Finally, such an isomorphism essentially determines d since $d \simeq dii^* \simeq gi^*$. $\diamond$

We will not assume for our axiomatic development that K admits the construction of all Eilenberg-Moore algebras; nevertheless, for those which exist we can establish two classical relationships between them and the Kleisli construction. Analogous synthetic results, in the context of Yoneda structures, were given first by Street and Walters [S&W]. (See especially Section 5 of that paper.)

Proposition 6. *If (X, t) is a monad in K for which an Eilenberg-Moore algebra $[u : X^t \rightarrow X, \alpha : ut \rightarrow u]$ exists and if the left adjoint of u , call it f , factors as $f \simeq ki$ with k a surjection and i an inclusion, then k is Kleisli for t .*

Proof. (We recall from [ST2] that universality for $[u, \alpha]$ guarantees the existence of $f \dashv u$ and, of course, that $t \simeq fu$.) If $f \simeq ki$, then $u \simeq i^*k^*$. Hence

$$t \simeq fu \simeq kii^*k^* \simeq kk^*.$$

Since $k : X \rightarrow Xk = X_{kk^*} \sim X_t$, the conclusion follows immediately. $\diamond$

In the opposite direction, recovering Eilenberg-Moore from Kleisli, we recall first the Yoneda structure version of Linton's Theorem [LIN]. If $k : X \rightarrow X_t$ is Kleisli for a monad (X, t) in K , the latter equipped with a Yoneda structure for which P has a left adjoint; then Eilenberg-Moore $u : X^t \rightarrow X$ is given by the following pullback in K :

(y is the Yoneda arrow)

$$\begin{array}{ccc}
 X^t & \xrightarrow{\quad} & P(X_t) \ (\sim (P X)_{(P_t)}) \\
 u \downarrow & & \downarrow P k \\
 X & \xrightarrow{\quad y \quad} & P X
 \end{array}$$

To state and prove this in the context of proarrows, observe first that in the examples for which it makes sense

$$\begin{array}{ccc}
 T & \xrightarrow{\bar{\phi}} & P Y \\
 v \downarrow & & \downarrow P f \\
 X & \xrightarrow{\quad y \quad} & P X
 \end{array}$$

"commutes" given $f : X \rightarrow Y$ in K , iff

$$\begin{array}{ccc}
 & T & \\
 v \swarrow & & \searrow \phi \\
 X & \xleftarrow{\quad f^* \quad} & Y
 \end{array}$$

"commutes" in M . Here we intend that $\phi : T \rightarrow Y$ in M correspond to the K -arrow $\bar{\phi} : T \rightarrow P Y$. (The arrow v in both diagrams is in K .) For want of a better word, call such a diagram a *kone* for f . (Our need for a terminology does not extend beyond the next proposition.) Clearly, the pullback of $P f$ along y corresponds to a universal kone.

Proposition 7. If $k : X \rightarrow X_t$ is Kleisli for (X, t) in K , then Eilenberg-Moore $u : X^t \rightarrow X$ is given by a universal kone

$$\begin{array}{ccc}
 & X^t & \\
 u \swarrow & & \searrow I \\
 X & \xleftarrow{\quad k^* \quad} & X_t
 \end{array}$$

Proof. k Kleisli for t in K yields k^* Eilenberg-Moore for t in M . Let K denote the vertex of a universal kone.

$$\begin{array}{c}
 \begin{array}{ccc}
 T \rightarrow K & K\text{-arrows} & \\
 \downarrow v & \searrow \phi & \\
 X & \xleftarrow{\quad k^* \quad} & X_t
 \end{array} \\
 \hline
 \begin{array}{ccc}
 T & \xrightarrow{\quad v \quad} & X \\
 \downarrow v & \searrow t & \\
 X & & X
 \end{array}
 \end{array}$$

kones

 t algebras in K
 $((\)_*)$ is locally fully faithful

$T \rightarrow X^t$ K -arrows.

◊

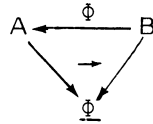
It is easy to identify I in the statement of proposition 7. u has a left adjoint, f , which factors as

$$f \simeq ki \quad \text{with} \quad k : X \rightarrow X_t \quad \text{and} \quad i : X_t \rightarrow X^t$$

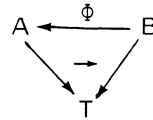
the comparison inclusion, $u \simeq f* \simeq i*k*$. Since also $u \simeq Ik*$ we have $I \sim i*$ by universality.

3. ARTIN GLUEING AND LOCAL FINITE COLIMITS.

We assume that $(\)_* : K \rightarrow M$ satisfies Axioms 1 through 5. An arrow $A \leftarrow B : \phi$ in M (considered as a diagram in M) has a *collage* if there is a diagram



composing with which establishes an equivalence between $M(\underline{\phi}, T)$ and the category of diagrams



for any T . General collages were discussed by Street in [ST1] and the term "collage" was credited to Walters. Presumably, the word was chosen because collages, in the everyday sense of the word, are usually obtained by glueing.

Proposition 8. Every $A \leftarrow B : \phi$ in M has a collage.

Proof. Consider the following 2×2 matrix :

$$A \oplus B \xrightarrow{\begin{bmatrix} A & 0 \\ \phi & B \end{bmatrix}} A \oplus B$$

It is a monad via

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \xrightarrow{\begin{bmatrix} A & 0 \\ \sigma & B \end{bmatrix}} \begin{bmatrix} A & 0 \\ \phi & B \end{bmatrix} \xrightarrow{\begin{bmatrix} A & 0 \\ \gamma & B \end{bmatrix}} \begin{bmatrix} A & 0 \\ \phi + \phi & B \end{bmatrix}$$

where σ and γ , as displayed below, are the canonical transformations

$$0 \xrightarrow{\sigma} \phi \xleftarrow{\gamma} \phi + \phi$$

which uniquely exhibit ϕ as a monoid in the monoidal category $(M(B, A), +, 0)$.

An opalgebra for $\begin{bmatrix} A & 0 \\ \phi & B \end{bmatrix}$, with codomain T , amounts to a diagram of the kind preceeding the statement of this proposition. In particular the Kleisli opalgebra provides a collage for $A \leftarrow B : \phi$. $\diamond$

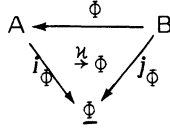
Corollary 9. *The injections into the collage for ϕ are representable and the equivalence mediated by the collage respects representability.*

Proof. By Axiom 5 the Kleisli arrow for $\begin{bmatrix} A & 0 \\ \phi & B \end{bmatrix}$ is representable ; call it

$$k : A \otimes B \longrightarrow A \otimes B \quad \begin{bmatrix} A & 0 \\ \phi & B \end{bmatrix} \sim \underline{\phi} .$$

By Axiom .4, k is necessarily of the form $k \simeq \begin{bmatrix} i \\ j \end{bmatrix}$, for representable i and j . The second statement is similarly immediate from those axioms. $\diamond$

We will denote the collage for ϕ by



and suppress the subscripts " ϕ " whenever possible.

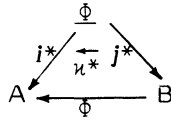
Corollary 10. *The injections i and j are inclusions, $\phi \simeq ji^*$ and $ij^* \simeq 0$.*

Proof. We have

$$\begin{bmatrix} A & 0 \\ \phi & B \end{bmatrix} \simeq kk^* \simeq \begin{bmatrix} i \\ j \end{bmatrix} \begin{bmatrix} i \\ j \end{bmatrix}^* \simeq \begin{bmatrix} i \\ j \end{bmatrix} [i^*, j^*] \simeq \begin{bmatrix} ii^* & ij^* \\ ji^* & jj^* \end{bmatrix} .$$

Now compare corresponding matrix entries. $\diamond$

Corollary 11. *The diagram*



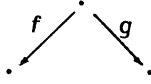
is also universal in M .

Proof. By Axiom 5, $[i^*, j^*] : \underline{\phi} \rightarrow A \otimes B$ is Eilenberg-Moore. $\diamond$

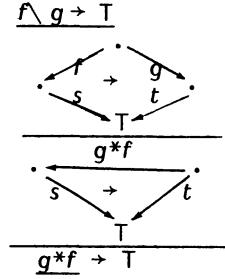
Remark 12. ϕ is the comma object ϕ/A . It is also the cocomma object $\phi \backslash B$. (Generally, cocomma squares are not instances of collages.) In particular, for all A , both $2.A$ and $\{2, A\}$ in M are given by $\underline{A}$, the collage object for the identity, $A \leftarrow A : A$. $\underline{A}$ is also $2.A$ in K . $\diamond$

Proposition 13. *K has all cocommas.*

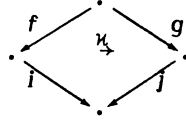
Proof. Given a span



$f \searrow g$ is given by $\underline{g^*f}$, for



Recall that a square



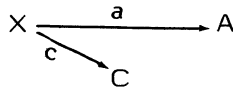
is said to be exact [GUI] (or satisfy the Beck-Chevalley condition) if $\kappa^* : g^*f \rightarrow ji^*$ is an isomorphism. $\diamond$

Corollary 14. *Cocomma squares in K are exact.*

Proof. Note the construction of $f \searrow g$ in Proposition 13. The isomorphism $g^*f \simeq ji^*$ is then seen to be an instance of that in Corollary 10. $\diamond$

It is quite helpful to know that every Φ in M factors as $\Phi \simeq gf^*$ for representables f and g . The next two propositions are included to illustrate this. The precise nature of the factorization is not required. As in [WD1] we use $\Psi \Rightarrow \Gamma$ to denote a right lifting of Γ through Ψ and $\Gamma \Leftarrow \Phi$ to denote a right extension of Γ along Φ .

Proposition 15. *M is biclosed iff for all*



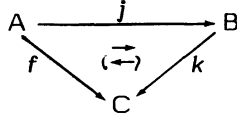
in K , $c \Leftarrow a$ and $a^* \Rightarrow c^*$ exist in M . (We may take a and c to be inclusions.)

Proof. $(cd^*) \Leftarrow (ab^*) \simeq b(c \Leftarrow a)d^*$ and $(ba^*) \Rightarrow (dc^*) \simeq d(a^* \Rightarrow c^*)b^*$. $\diamond$

Paré [PAR] first proved that a colimit is absolute iff it is preserved by the Yoneda embedding. Proposition 15 is an abstract version

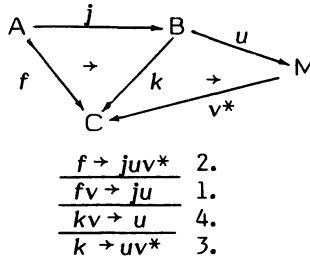
of that fact. It should be compared with the definition of "pointwise extension" given in [WD1].

Proposition 16.

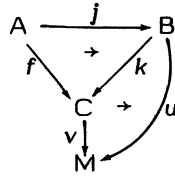


is an absolute left (resp. right) extension diagram in K iff $()_*$ (resp. $()^*$) applied to it yields a left extension (resp. lifting) diagram in M .

Proof. (Only if) Consider the following diagram in M :



(If) Consider the following diagram in K :

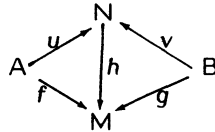


Now read the lines above as numbered. ◊

Provisionally, let $(\text{COSP}N K)(B, A)$ denote the category of co-spans from B to A in K . Thus objects are typified by

$$f : A \rightarrow M \leftarrow B : g$$

and morphisms by "commutative" diagrams



We have a functor from this category to $M(B, A)$, which assigns to a morphism h , as above,

$$vu^* \xrightarrow{\tilde{v}h u^*} vhh^*u^* \xrightarrow{\sim} gf^* .$$

$$\left[\begin{array}{c} f : A \rightarrow M \\ \sigma : \Phi f \rightarrow g \\ g : B \rightarrow M \end{array} \right] \xrightarrow{\left[\begin{array}{c} \varphi : f \rightarrow f' \\ \gamma : g \rightarrow g' \end{array} \right]} \left[\begin{array}{c} f' : A \rightarrow M \\ \sigma' : \Phi f' \rightarrow g' \\ g' : B \rightarrow M \end{array} \right]$$

where

$$\begin{array}{ccc} \Phi f & \xrightarrow{\Phi \varphi} & \Phi f' \\ \sigma \downarrow & & \downarrow \sigma' \\ g & \xrightarrow{\gamma} & g' \end{array} \quad \parallel$$

(As remarked in 12, Φ is a cocomma object in M .) Thus if $\tau : h \rightarrow h'$ is a transformation between arrows of cospans from $i : A \rightarrow \Phi \leftarrow B : j$ to $f : A \rightarrow M \leftarrow B : g$, then this explicit description shows that

$$(i \tau : f \rightarrow f) = (f : f \rightarrow f) \quad \text{and} \quad (j \tau : g \rightarrow g) = (g : g \rightarrow g)$$

imply that τ is essentially an identity. $\diamond$

If K has pushouts, then cospans can be composed and $COSPN K$ becomes a bicategory with bicategory valued homs. The functors

$$(COSPN K)(B, A) \rightarrow M(B, A)$$

then define a normal morphism of bicategories $COSPN K \rightarrow M$. In particular the functors $(COSPN K)(A, A) \rightarrow M(A, A)$ are normal monoidal functors between monoidal categories. The category of monoids in $(COSPN K)(A, A)$ is equivalent to A/K and the forgetful functor is given by

$$(f : A \rightarrow X) \mapsto (f : A \rightarrow X \leftarrow A : f).$$

Write $MND M(A)$ for the category of monoids in $M(A, A)$. (So $MND M(A)$ is the category of monads on A in M .) It follows that we have a "commutative" diagram

$$\begin{array}{ccc} A/K & \xrightarrow{\quad} & MND M(A) \\ \downarrow & & \downarrow \\ (COSPN K)(A, A) & \xrightarrow{\quad} & M(A, A) \end{array}$$

where $A/K \rightarrow MND M(A)$ is given by $f \mapsto ff^*$. Its left adjoint is the Kleisli construction, $\Phi \mapsto k : A \rightarrow A_\Phi$.

We have an initial object in K , so the temporary assumption of pushouts gives us coequalizers. The left adjoint to $A/K \rightarrow (COSPN K)(A, A)$ is

$$(f : A \rightarrow M \leftarrow A : g) \mapsto (A \rightarrow \text{coeq}(f, g)).$$

If K has countable sums locally, then $\Phi \mapsto \sum_{n \in \mathbb{N}} \Phi^n$ gives us a free monad construction. (Note that the full force of "locally" is needed for

this. Note too that the proof of proposition 3 is easily modified to handle the statement of that proposition with the words "finite" deleted. In short a global assumption about countable sums could be made.) Putting all these adjoints together we get,

$$\begin{array}{ccc} \mathcal{A}/\mathcal{K} & \xrightleftharpoons[\perp]{} & \mathcal{M}ND \mathcal{M}(A) \\ \updownarrow \dashv & & \updownarrow \dashv \\ (\mathcal{C}OSPN \mathcal{K})(A, A) & \xrightleftharpoons[\perp]{} & \mathcal{M}(A, A) \end{array}$$

providing a relationship between dual structure-semantics (the top row adjunction) and dual comprehension.

We turn now to local finite colimits. A preliminary lemma will be useful. Consider the following square in $\mathcal{M}$:

$$\begin{array}{ccc} A & \xleftarrow{\Phi} & B \\ \Gamma \downarrow & \searrow \varepsilon & \downarrow \Delta \\ A' & \xleftarrow{\Phi'} & B' \end{array}$$

Pasting

$$\begin{array}{ccc} A' & \xleftarrow{\Phi'} & B' \\ i' \swarrow & \searrow \kappa' & \swarrow j' \\ & \Phi' & \end{array}$$

to the square yields, via universality of Φ , an arrow $\Sigma: \Phi \rightarrow \Phi'$ in $\mathcal{M}$. We can write

$$\Sigma = \left[\begin{array}{c} \Gamma i' \\ (\varepsilon i')(\Delta \kappa') \\ \Delta j' \end{array} \right]$$

Similarly, we can replace ε in the square by ε^{-1} , paste

$$\begin{array}{ccc} & \Phi & \\ i^* \swarrow & \searrow \kappa^* & \swarrow j^* \\ A & \xleftarrow{\Phi} & B \end{array}$$

to the result and obtain, via universality of Φ' (Corollary 11), an arrow

$$\Pi = [j^* \Delta, (j^* \varepsilon^{-1})(\kappa^* \Gamma), i^* \Gamma] : \Phi \rightarrow \Phi'.$$

Lemma 19. $\Sigma \approx \Pi$.

Proof. $\Sigma \approx [\Sigma j^*, \Sigma \kappa^*, \Sigma i^*]$. A direct calculation yields

$$\Sigma \approx \left[\begin{array}{ccc} A \xrightarrow{0} B' & 0_{A A'} \rightarrow \Gamma & A \xrightarrow{\Gamma} A' \\ 0_{B B'} \rightarrow \Delta & & \Phi \Gamma \xrightarrow{\varepsilon} \Delta \Phi \\ B \xrightarrow{\Delta} B' & \Delta \Phi \rightarrow \Delta \Phi & B \xrightarrow{\Delta \Phi} A' \end{array} \right]$$

Similarly we have

$$\Pi \simeq \begin{bmatrix} i \Pi \\ \kappa \Pi \\ j \Pi \end{bmatrix} \simeq \begin{bmatrix} A \xrightarrow{0} B' & 0_{A A'} \rightarrow \Gamma & A \xrightarrow{\Gamma} A' \\ 0_{B B'} \rightarrow \Delta & \Delta \Phi \xrightarrow{\sim} \Phi \Gamma & \Phi \Gamma \rightarrow \Phi \Gamma \\ B \rightarrow B' & & B \xrightarrow{\Phi \Gamma} A' \end{bmatrix}$$

Comparing entries shows $\Sigma = \Pi$.

◊

We will write $\underline{\Gamma, \Delta} (= \underline{\Gamma, \epsilon, \Delta})$ for the "common value" of Σ and Π above.

Proposition 20. *M has finite colimits locally.*

Proof. In virtue of Proposition 1 (or Propositions 1 and 3) it suffices to show that *M* has local pushouts (or local coequalizers). So let

$$\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$$

be a span in $M(B, A)$. We have $A \leftarrow A : A+A$ (i.e., $A+A$ is the sum in $M(A, A)$ of the identity with itself). From

$$\begin{array}{ccc} A & \xleftarrow{A+A} & A \\ & \searrow \gamma & \swarrow A \\ & A & \end{array}$$

where $\gamma : A+A \rightarrow A$ is the codiagonal in $M(A, A)$, we get a codiagonal arrow, $A \leftarrow \underline{A+A} : c$, in *K*. Using Corollary 11 we define

$$\underline{A+A} \leftarrow B : [\Gamma, \sigma+\tau, \Phi+\Psi]$$

as in the following diagram :

$$\begin{array}{ccc} & B & \\ \Phi+\Psi \swarrow & & \searrow \Gamma \\ A & \xleftarrow{A+A} & A \end{array} \quad (\Gamma(A+A) \simeq \Gamma+\Gamma \text{ by Proposition 3})$$

We show that the composite

$$A \xleftarrow{c} \underline{A+A} \xleftarrow{[\Gamma, \sigma+\tau, \Phi+\Psi]} B$$

is the required pushout.

Let Δ be an arbitrary object of $M(B, A)$. Then

$$\begin{array}{c} \frac{[\Gamma, \sigma+\tau, \Phi+\Psi]c \rightarrow \Delta : B \rightarrow A}{[\Gamma, \sigma+\tau, \Phi+\Psi] \rightarrow \Delta c^* : B \rightarrow \underline{A+A}} \quad (c^* = [A, \gamma, A]) \\ \hline [\Gamma, \sigma+\tau, \Phi+\Psi] \rightarrow [\Delta, \Delta \gamma \Delta] : B \rightarrow \underline{A+A} \\ \hline \Gamma \xrightarrow{\xi} \Delta, \Phi+\Psi \xrightarrow{\eta} \Delta : B \rightarrow A \end{array}$$

$$\begin{array}{ccc}
 & \text{such that} & \\
 \Gamma + \Gamma' & \xrightarrow{\zeta + \zeta} & \Delta + \Delta \\
 \sigma + \tau \downarrow & & \downarrow \Delta \gamma \\
 \Phi + \Psi & \xrightarrow{(\xi) \quad \eta} & \Delta \\
 \hline
 \Phi \xrightarrow{\xi} \Delta, \Psi \xrightarrow{\eta} \Delta : B \rightarrow A & & \\
 & \text{such that} & \\
 \Gamma & \xrightarrow{\tau} & \Psi \\
 \sigma \downarrow & & \downarrow \eta \\
 \Phi & \xrightarrow{\xi} & \Delta
 \end{array}$$

Hence $[\Gamma, \sigma + \tau, \Phi + \Psi]c$ is the pushout in $M(B, A)$. To see that it is preserved by precomposition with all $\Delta : B' \rightarrow B$, note that

$$\Delta[\Gamma, \sigma + \tau, \Phi + \Psi] \simeq [\Delta\Gamma, \Delta\sigma + \Delta\tau, \Delta\Phi + \Delta\Psi]$$

(where we again use Proposition 3). For preservation by postcomposition, consider $\Delta : A \rightarrow A'$ and

$$\begin{array}{ccc}
 A & \xleftarrow{A+A} & A \\
 \Delta \downarrow & & \downarrow \Delta \\
 A' & \xleftarrow{A'+A'} & A'
 \end{array}$$

Define $\underline{\Delta}, \underline{\Delta} : A+A \rightarrow A'+A'$ as after Lemma 19. From the description of $\underline{\Delta}, \underline{\Delta}$ as an arrow into a comma object we get

$$[\Gamma, \sigma + \tau, \Phi + \Psi] \underline{\Delta}, \underline{\Delta} \simeq [\Gamma\Delta, \sigma\Delta + \tau\Delta, \Phi\Delta + \Psi\Delta]$$

The proof is completed by showing that

$$\begin{array}{ccc}
 A+A & \xrightarrow{\underline{\Delta}, \underline{\Delta}} & A'+A' \\
 c \downarrow & & \downarrow c' \\
 A & \xrightarrow{\Delta} & A'
 \end{array}$$

"commutes", where c' is the corresponding codiagonal for A' . This follows from the description of $\underline{\Delta}, \underline{\Delta}$ as an arrow out of a cocomma object. Explicitly :

$$\underline{\Delta}, \underline{\Delta} c' \simeq \underline{\Delta}, \underline{\Delta} \begin{bmatrix} A' \\ \gamma' \\ A \end{bmatrix} \simeq \begin{bmatrix} \Delta \\ \Delta\gamma' \\ \Delta \end{bmatrix} \simeq \begin{bmatrix} \Delta \\ \gamma\Delta \\ \Delta \end{bmatrix} \simeq c\Delta.$$

We have chosen the above construction for pushouts in $M(B, A)$ because it invites comparison with the construction for sums given in Proposition 3. The matrix calculus permitted by the latter immediately gives another more symmetric sum construction. Indeed, if $\Phi, \Psi : B \rightarrow A$,

then $\Phi + \Psi$ is given by the composite

$$B \xrightarrow{[\Phi, B]} A \oplus B \xrightarrow{\begin{bmatrix} A \\ \Psi \end{bmatrix}} A.$$

It is natural to seek such a description for the pushout of $\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$ in $M(B, A)$. Experimentation with the examples, particularly

$$(\)_* : TOP \rightarrow TOPLEX^{CO}$$

suggests that such is provided by

$$B \xrightarrow{[B, \sigma, \Phi]} \underline{\Gamma} \xrightarrow{\begin{bmatrix} A \\ \tau \\ \Psi \end{bmatrix}} A.$$

This is a correct general construction. We postpone a proof until Section 5 (Corollary 31).

It should be clear from the proof of Proposition 20 that a slight modification of the construction gives local coequalizers.

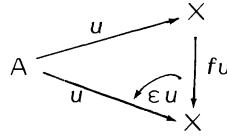
Proposition 21. For $\sigma, \tau : \Gamma \rightrightarrows \Phi$ in $M(B, A)$, the coequalizer of σ and τ is given by the following composite :

$$A \xleftarrow{c} \underline{A+A} \xleftarrow{[\Gamma, \begin{pmatrix} \sigma \\ \tau \end{pmatrix}, \Phi]} B. \quad \diamond$$

4. SURJECTIONS REVISITED.

Our results in Section 2 suggest that surjections-inclusions constitute a bicategorical E-M factorization system for K , when $(\)_* : K \rightarrow \mathcal{M}$ is proarrow equipment satisfying Axioms 1 through 5. However, we should have each kind of arrow closed under composition. For inclusions this is trivial but a moment's reflection shows that for surjections the situation is more subtle. After all, PTT functors in CAT are not composition-closed. To show that surjections are in fact closed under composition we find it convenient to establish first a general monadicity theorem for arbitrary bicategories, Proposition 22. The experienced reader will find nothing particularly new in either the statement or the proof of Proposition 22, however, we need precisely the version there and a published account of it does not seem to have been given. It is reminiscent of some of the results in W. Butler's unpublished, but well-circulated, tripleability theorems. We respectfully suggest that Proposition 22 be called the FTT (formal tripleability Theorem). Temporarily, K is just an arbitrary category.

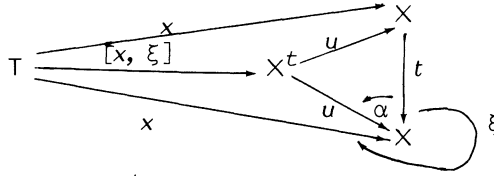
Consider an adjunction $f : X \xrightleftharpoons{u} A : u$ in K , with counit ϵ . Consider the following diagram



It is an fu algebra. A right adjoint u , as above, is said to be *monadic* if $[u, \epsilon u]$ is an Eilenberg-Moore algebra. For a monad (X, t, η, μ) with Eilenberg-Moore algebra

$$[u : X^t \rightarrow X, \alpha : ut \rightarrow u],$$

arrows into X^t may be denoted by pairs as in :



In particular, the identity on X^t is $[u, \alpha]$ and $[t, \mu] = f \dashv u$. The unit for the adjunction is η . The counit

$$uf = u[t, \mu] = [ut, u\mu] \rightarrow [u, \alpha] = X^t$$

is α regarded as a homomorphism. Given an adjunction $f : X \rightleftarrows A : u$ and an fu algebra $[x : T \rightarrow X, \xi : xfu \rightarrow x]$, note that we have a parallel pair, $\xi f, x f \epsilon : x f u f \rightrightarrows x f$, in $K(T, A)$. Recall that a colimit (or limit) in $K(B, A)$ is said to exist *pointwise* if it is preserved by $K(b, A)$ for all $b : T \rightarrow B$ (for all T).

Proposition 22. For an adjunction $f : X \rightleftarrows A : u$, with counit ϵ , in an arbitrary bicategory K , u is monadic iff :

(i) for every algebra $[x, \xi]$, the coequalizer of ξf and $x f \epsilon$ exists and is preserved by composition with u ,

(ii) for the algebra $[u, \epsilon u]$, the coequalizer of $\epsilon u f$ and $u f \epsilon$ exists pointwise,

$$(iii) \quad u f u f \xrightarrow[\quad u f \epsilon]{\quad \epsilon u f} u f \xrightarrow{\quad \epsilon} A.$$

is a coequalizer.

Proof. Assume u is monadic. Write $t = fu$ and write

$$[u : X^t \rightarrow X, \alpha : ut \rightarrow u] \quad \text{for} \quad [u : A \rightarrow X, \epsilon u : u f u \rightarrow u].$$

Let $[x, \xi] : T \rightarrow X^t$ be given and compose it with the commuting fork displayed in (iii) above to obtain, in algebra notation, the commuting fork

$$[x t t, x t \mu] \xrightarrow[\quad x \mu]{\quad \xi t} [x t, x \mu] \xrightarrow{\quad \xi} [x, \xi].$$

For any $[b, \beta] : T \rightarrow X^t$ and commuting fork

$$[xtt, xt\mu] \xrightarrow[\underline{x\mu}]{\xi t} [xt, x\mu] \xrightarrow{\vartheta} [b, \beta],$$

it is easy to verify that $(x\eta)\vartheta : [x, \xi] \rightarrow [b, \beta]$ is the unique homomorphism that commutes with ξ and ϑ . This verifies the existence of the coequalizer required in (i). Composing it with u yields

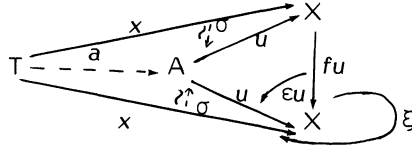
$$xtt \xrightarrow[\underline{x\mu}]{\xi t} xt \xrightarrow{\xi} x,$$

which is absolutely a coequalizer. Now set

$$[x, \xi] = [u, \alpha] (= X^t = A).$$

This verifies (iii). It also verifies (ii) after one notes how the coequalizer required in (i) was constructed.

Conversely, assume (i), (ii and (iii). Consider



where $[x, \xi]$ is an fu algebra. We have to show that there is an essentially unique a and, for such, a unique isomorphism σ as shown, satisfying $(\sigma fu)(a \in u)\sigma^{-1} = \xi$.

Assume that a solution a, σ exists. Consider

$$\begin{array}{ccc} xfu & \xrightarrow[\underline{xf\epsilon}]{\xi f} & xf \\ \sigma fu \downarrow \wr & & \downarrow \wr \sigma f \\ au & \xrightarrow[\underline{auf\epsilon}]{a \in uf} & auf \xrightarrow{a \in} a \end{array}$$

By (iii) and (ii) the bottom row is a coequalizer. The bottom square commutes "naturally". The top square is seen to commute by applying f to the assumed equation. Thus a is essentially uniquely determined as a coequalizer of ξf and $xf\epsilon$. Moreover, for the coequalizer transformation $xf \rightarrow a$, $\sigma : x \rightarrow au$ is the correspondent via the adjointness $f \dashv u$. So both aspects of uniqueness have been verified.

For existence use (i) and define $\beta : xf \rightarrow a$ to be a coequalizer of ξf and $xf\epsilon$. Also by (i), $\beta u : xfu \rightarrow au$ is a coequalizer of ξfu and $xf\epsilon u$. However, ξ is a coequalizer (absolutely) of ξfu and $xf\epsilon u$ so we have an isomorphism

$$\begin{array}{ccc} xfu & \xrightarrow{\xi} & x \\ & \searrow \beta u & \downarrow \wr \sigma \\ & & au \end{array}$$

The coequalizer $\xi: xfu \rightarrow x$ is further preserved by applying the left adjoint f . Hence we have

$$\begin{array}{ccc}
 xfu & \xrightarrow{\xi f} & xf \\
 \downarrow x\tau \varepsilon & \searrow \beta u f & \downarrow \sigma f \\
 xf & \xrightarrow{\beta} & a \\
 & & \downarrow a\varepsilon \\
 & & a
 \end{array}
 \quad \begin{array}{c}
 \beta \\
 \text{curved arrow}
 \end{array}$$

where the rightmost region commutes by uniqueness of transformations out of the coequalizer xf . Hence

$$(\sigma fu)(a\varepsilon u) = \beta u = (\xi)(\sigma).$$

◇

Proposition 22 is easily dualized. For an adjunction $f: X \xrightarrow{f} A: u$ say that f is *opmonadic* if $[\frac{f}{f\varepsilon}]$ is a Kleisli opalgebra for fu . Note that for an fu opalgebra $[\frac{x}{\xi}]$, $x: X \rightarrow T$, we have a parallel pair

$$\varepsilon ux, u\xi: ufu \rightrightarrows ux \quad \text{in } K(A, T).$$

Proposition 22^{OP}. For an adjunction $f: X \xrightarrow{f} A: u$, with counit ε , in an arbitrary bicategory K ; f is opmonadic iff:

(i^{OP}) for every opalgebra $[\frac{x}{\xi}]$, the coequalizer of εux and $u\xi$ exists and is preserved by (pre)composition with f ,

(ii^{OP}) for the opalgebra $[\frac{f}{f\varepsilon}]$, the coequalizer of εuf and $uf\varepsilon$ exists absolutely,

$$(iii^{OP}) \quad ufuf \xrightarrow[\quad uf\varepsilon]{\quad \varepsilon uf} uf \longrightarrow A$$

is a coequalizer.

◇

Corollary 23. For an adjunction $\Phi: X \xrightarrow{\Phi} A: \Phi^*$, with counit $\tilde{\Phi}$, in a bicategory M with local coequalizers, the following are equivalent:

- (i) Φ^* is monadic,
- (ii) Φ^* reflects isomorphisms in M ,

$$(iii) \quad \begin{array}{ccc} \Phi^* \Phi \Phi^* \Phi & \xrightarrow[\Phi^* \Phi \tilde{\Phi}]{\Phi \Phi^* \Phi} & \Phi^* \Phi \xrightarrow{\tilde{\Phi}} A \end{array}$$

is a coequalizer,

- (iv) Φ opreflects isomorphisms in M ,
- (v) Φ is opmonadic.

Proof. Conditions (i) and (ii) of proposition 22 and (i^{OP}) and (ii^{OP}) of Proposition 22^{OP} are automatically satisfied in a bicategory with local coequalizers, so the equivalence of (i), (iii) and (v) above is immediate.

In any bicategory, Φ^* monadic implies Φ^* reflects isomorphisms. In the present situation we have

$$\begin{array}{ccc} \Phi^* \Phi \Phi^* \Phi & \xrightarrow[\Phi^* \Phi \Phi]{\Phi \Phi^* \Phi} & \Phi^* \Phi \\ & \searrow \Phi & \downarrow \varphi \\ & & A \end{array},$$

where the top fork is a coequalizer and the bottom fork, and hence the triangle, commute. Applying Φ^* makes both forks coequalizers and hence $\varphi \Phi^*$ is an isomorphism. Therefore, if Φ^* reflects isomorphisms, condition (iii) above is satisfied.

By " Φ opreflects isomorphisms" we mean "for all composable transformations τ , $\Phi \tau$ an isomorphism implies τ an isomorphism". The equivalence of (iv) with the other statements is just dual to that in the paragraph above. $\diamond$

Corollary 24. For any arrow $f : A \rightarrow B$ in a bicategory K , with proarrow equipment $()_* : K \rightarrow M$ satisfying Axioms 1 through 5, the following are equivalent :

- (i) f^* reflects isomorphisms in M ,
- (ii) f is a surjection,
- (iii) f opreflects isomorphisms in M .

$\diamond$

Z If f^* above is also in K and reflects isomorphisms in K , it does not follow that f^* reflects isomorphisms in M .

Corollary 25. In any bicategory K , with proarrow equipment $()_* : K \rightarrow M$ satisfying Axioms 1 through 5, surjections are closed under composition.

Proof. Conditions (i) and (iii) above are clearly stable under composition. $\diamond$

Corollary 26. If fk is a surjection, then so is k . $\diamond$

If Φ and Ψ are monads on X in M and $\vartheta : \Phi \rightarrow \Psi$ is a morphism of monads, then for general reasons we have an arrow $X_{\vartheta} : X_{\Phi} \rightarrow X_{\Psi}$ as below :

$$\begin{array}{ccc} X & & \\ \Phi \downarrow & \nearrow k_{\Psi} & \\ X & \xrightarrow{k_{\Phi}} & X_{\Phi} \\ & \searrow k_{\Psi} & \downarrow X_{\vartheta} \\ X & & X_{\Psi} \end{array} \quad \begin{array}{c} \text{curly arrow from } X_{\Phi} \text{ to } X_{\Psi} \\ \text{labelled } (\vartheta k_{\Psi}) \eta_{\Psi} \end{array}$$

For any transformation $\tau : \Phi \rightarrow \Psi : B \rightarrow A$ (Φ and Ψ arbitrary) we have, in Proposition 16, implicitly mentioned an arrow $\underline{\tau} : \underline{\Phi} \rightarrow \underline{\Psi}$ which is a morphism of cospans from B to A . It is a special case of X_{ϑ} . Indeed

$$\begin{bmatrix} A & 0 \\ \Phi & B \end{bmatrix} \xrightarrow{\begin{bmatrix} A & 0 \\ \tau & B \end{bmatrix}} \begin{bmatrix} A & 0 \\ \Psi & B \end{bmatrix}$$

is a morphism of monads. Thus an immediate consequence of Corollary 26 is :

Corollary 27. *The arrows $\times_{\Phi}$ and $\sqsubseteq$ are surjections.* $\diamond$

5. MATRIX ARITHMETIC EXTENDED.

Consider monads (A, Φ) and (B, Ψ) in M . An arrow $A_{\Phi} \rightarrow X$ "is" a Φ -opalgebra

$$\left[\begin{smallmatrix} \Gamma \\ \gamma \end{smallmatrix} \right] \quad (\Gamma : A \rightarrow X, \gamma : \Phi \Gamma \rightarrow \Gamma)$$

or *left Φ -module*. An arrow $X \rightarrow B_{\Psi}$ "is" a Ψ -algebra

$$[\Delta, \delta] \quad (\Delta : X \rightarrow B, \delta : \Delta \Psi \rightarrow \Delta)$$

or *right Ψ -module*. The composite

$$\left[\begin{smallmatrix} \Gamma \\ \gamma \end{smallmatrix} \right] [\Delta, \delta] : A_{\Phi} \rightarrow B_{\Psi}$$

"is" a *left Φ , right Ψ -module*. (The actions associate.) Indeed, any arrow $A_{\Phi} \rightarrow B_{\Psi}$ "is" an arrow $A \rightarrow B$ together with a pair of associating actions. Somewhat more interesting is a composite of the forme $Y \rightarrow A_{\Phi} \rightarrow X$.

Proposition 28. *If $[\Delta, \delta] : Y \rightarrow A_{\Phi}$ and $\left[\begin{smallmatrix} \Gamma \\ \gamma \end{smallmatrix} \right] ; A_{\Phi} \rightarrow X$ then $[\Delta, \delta] \left[\begin{smallmatrix} \Gamma \\ \gamma \end{smallmatrix} \right] \simeq \Delta_{\Phi} \Gamma$ where*

$$\Delta \Phi \Gamma \xrightarrow[\Delta \gamma]{\delta \Gamma} \Delta \Gamma \longrightarrow \Delta_{\Phi} \Gamma$$

is a coequalizer in $M(Y, X)$.

Proof. Write $k : A \rightarrow A_{\Phi}$ for the Kleisli opalgebra for Φ . By the FTT we have a coequalizer, composition stable by Proposition 20,

$$k * k k * k \xrightarrow[k * k k]{k k * k} k * k \xrightarrow{k} A_{\Phi}.$$

Applying $[\Delta, \delta] - \left[\begin{smallmatrix} \Gamma \\ \gamma \end{smallmatrix} \right]$ to this diagram (i.e. precomposing with $[\Delta, \delta]$ and postcomposing $\left[\begin{smallmatrix} \Gamma \\ \gamma \end{smallmatrix} \right]$) and noting the remarks about such composites in Section 4 (preceding Proposition 22) we obtain

$$\Delta \Phi \Gamma \xrightarrow[\Delta \delta]{\delta \Gamma} \Delta \Gamma \longrightarrow [\Gamma, \gamma]_{\delta}^{\Delta}$$

a coequalizer. $\diamond$

So in general, composites $A_{\Phi} \rightarrow B_{\Psi} \rightarrow C_{\Gamma}$ are "tensor products" of modules. Note that the full force of local coequalizer is required to

equip such a composite with the requisite left Φ , right Γ -structure.

The title of this section is admittedly overly ambitious. The point is to encourage comparison of composites such as

$$(\Upsilon \xrightarrow{[\Phi, \Psi]} A \otimes B \xrightarrow{[\begin{smallmatrix} \Gamma \\ \Delta \end{smallmatrix}]} X) \simeq (\Upsilon \xrightarrow{\Phi \Gamma + \Psi \Delta} X)$$

with composites such as in Proposition 28. In each case the intermediate object is simultaneously a "limit" and a "colimit" and the result is a local colimit. A simple combination of these cases is given in Proposition 30.

Lemma 29. For $A \leftarrow B : \Gamma$ in $M(B, A)$, the following diagram is a push-out in $M(\underline{\Gamma}, \underline{\Gamma})$:

$$\begin{array}{ccc} & j^* \Gamma i & \\ \kappa^* i \swarrow & & \searrow j^* \kappa \\ i^* i & & j^* j \\ \downarrow \sim & & \downarrow \sim \\ & \underline{\Gamma} & \end{array}$$

where, as usual, we have

$$\begin{array}{ccc} & \underline{\Gamma} & \\ j^* \swarrow & & \searrow j^* \\ A & & B \\ \downarrow i & \xrightarrow{\kappa} \Gamma & \downarrow i \\ & \underline{\Gamma} & \end{array}$$

exhibiting $\underline{\Gamma}$ as both a "limit" and a "colimit".

Proof. (Sketch) Since $[\begin{smallmatrix} i \\ j \end{smallmatrix}] : A \otimes B \rightarrow \underline{\Gamma}$ is Kleisli (for the monad $\begin{bmatrix} A & 0 \\ \Gamma & B \end{bmatrix}$) we have a local coequalizer

$$[i^*, j^*][\begin{smallmatrix} i \\ j \end{smallmatrix}] \rightrightarrows [i^*, j^*][\begin{smallmatrix} i \\ j \end{smallmatrix}] \longrightarrow \underline{\Gamma}.$$

Expanding and noting that $ij^* \simeq 0$, $ji^* \simeq \Gamma$ this becomes

$$i^* i i^* + j^* \Gamma i + j^* j j^* \rightrightarrows i^* i + j^* j \longrightarrow \underline{\Gamma}.$$

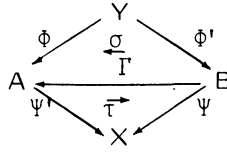
The diagram restricted to $j^* \Gamma i$ then yields the required pushout. $\diamond$

Proposition 30. For $A \leftarrow B : \Gamma$, consider the following composite :

$$\Upsilon \xrightarrow{[\Phi', \sigma, \Phi]} \underline{\Gamma} \xrightarrow{\begin{bmatrix} \Psi' \\ \tau \\ \Psi \end{bmatrix}} X.$$

It is given by $\Phi \Psi' +_{\Phi' \Gamma \Psi} \Phi \Psi$, the pushout of $\sigma \Psi' : \Phi \Psi' + \Phi' \Gamma \Psi' \rightarrow \Phi' \Psi : \Phi' \tau$ in $M(B, A)$.

Proof. The given data is



Apply the technique in the proof of proposition 28 using Lemma 29. $\diamond$

Corollary 31. For $\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$ in $M(B, A)$, the pushout is given by

$$B \xrightarrow{[B, \sigma, \Phi]} \underline{\Gamma} \xrightarrow{\begin{bmatrix} A \\ \tau \\ \Psi \end{bmatrix}} A \quad (\simeq B \xrightarrow{[B, \tau, \Psi]} \underline{\Gamma} \xrightarrow{\begin{bmatrix} A \\ \sigma \\ \Phi \end{bmatrix}} A).$$

Proof. Take

$$(\Phi' : Y \rightarrow B) = (B : B \rightarrow B) \quad \text{and} \quad (\Psi' : A \rightarrow X) = (A : A \rightarrow A)$$

in Proposition 30. $\diamond$

On the other hand, any arrow $\underline{\Phi} \rightarrow \underline{\Psi}$ is a "matrix" consisting of four arrows and four transformations as in the proof of Lemma 19. A good notation would be helpful. We conclude this section with some simple results, stated without proofs.

Proposition 32. For $A \leftarrow B : \Phi$

$$i^* \simeq \begin{bmatrix} A \\ \Phi \end{bmatrix}, \quad j^* \simeq \begin{bmatrix} 0 \\ \sigma \\ B \end{bmatrix}, \quad j^* \Phi \simeq \begin{bmatrix} 0 \\ \sigma \\ \Phi \end{bmatrix} \xrightarrow{\kappa^* = \begin{bmatrix} \sigma \\ \Phi \end{bmatrix}} \begin{bmatrix} A \\ \Phi \end{bmatrix}$$

and

$$i \simeq [0, \sigma, A], \quad j \simeq [B, \Phi, \Phi], \quad \Phi i \simeq [0, \sigma, \Phi] \xrightarrow{\kappa = [\sigma, \Phi]} [B, \Phi, \Phi]$$

where all σ 's are transformations out of 0's. $\diamond$

Corollary 33. For the codiagonal $c : A \rightarrow A$ we have $i \dashv c \dashv j$. $\diamond$

Proposition 34. For $\tau : \Gamma \rightarrow \Psi : B \rightarrow A$,

$$\underline{\tau} = \begin{bmatrix} i_{\Psi} \\ (\tau i_{\Psi}) \kappa_{\Psi} \\ j_{\Psi} \end{bmatrix} : \underline{\Gamma} \rightarrow \underline{\Psi}$$

satisfies

$$\underline{\tau} i_{\Psi}^* = \begin{bmatrix} -A \\ \sigma \\ \Psi \end{bmatrix} \quad \text{and} \quad j_{\Psi} \underline{\tau}^* = [B, \sigma, \Psi]$$

where the σ 's are transformations out of 0's. $\diamond$

Corollary 35. For $\sigma : \Phi \leftarrow \Gamma \rightarrow \Psi : \tau$ in $M(B, A)$ the pushout is given by the following composite :

$$\begin{array}{ccccc} A & & \underline{\Gamma} & & B \\ i_{\Phi}^* \swarrow & & \searrow \sigma & & \swarrow \tau^* \\ \underline{\Phi} & & & & \underline{\Psi} \\ & & & & j_{\Psi} \end{array} \quad (\simeq j_{\Phi} \sigma^* \underline{\tau} i_{\Phi}^*).$$

Write Σ for the pushout above. By Proposition 17 we have :

Proposition 36.

$$\begin{array}{ccc} & \Gamma & \\ \sigma \swarrow & & \searrow \tau \\ \underline{\Phi} & & \underline{\Psi} \\ & \Sigma & \end{array}$$

is a "pushout" in $(\text{COSPN } K)(B, A)$ and hence also a "pushout" in K . $\diamond$

Lemma 37. Let K be an arbitrary bicategory. Let (A, t, η, μ) and (A, t', η', μ') be monads in K such that

$$\begin{array}{ccc} A & \xrightarrow{\eta} & t \\ \eta' \downarrow & & \downarrow \eta't \\ t' & \xrightarrow{t'\eta} & t't \end{array}$$

is a local pushout. Then $(\begin{smallmatrix} \eta t' \\ t \eta' \end{smallmatrix}) = \delta : t't \rightarrow tt'$ is a distributive law with the property that an opalgebra for the composite monad $t \delta t'$ "is" an arrow $h : A \rightarrow X$ together with a t -opalgebra structure and a t' -opalgebra structure. (I.e., the usual distributivity requirement for such structures is automatically satisfied in this situation.) If Kleisli objects exist in K , the following square is a "pushout" :

$$\begin{array}{ccc} & A & \\ \swarrow & & \searrow \\ A_t & & A_{t'} \\ \searrow & & \swarrow \\ & A_{t \delta t'} & \end{array}$$

$\diamond$

By Corollary 27, $\underline{\sigma}$ and $\underline{\tau}$ of Proposition 36 are universal opalgebras for the monads $\underline{\sigma}\underline{\sigma}^*$ and $\underline{\tau}\underline{\tau}^*$. It can be shown that

$$\begin{array}{ccc} \Gamma & \xrightarrow{\tilde{\underline{\sigma}}} & \underline{\sigma}\underline{\sigma}^* \\ \tilde{\underline{\tau}} \downarrow & & \downarrow \\ \underline{\tau}\underline{\tau}^* & \longrightarrow & \underline{\tau}\underline{\tau}^* \underline{\sigma}\underline{\sigma}^* \quad (\simeq \underline{\sigma}\underline{\sigma}^* \underline{\tau}\underline{\tau}^*) \end{array}$$

is a pushout in $M(\underline{\Gamma}, \underline{\Gamma})$, hence Lemma 37 explains the existence of the global "pushout" $\underline{\Sigma}$. It is a Kleisli object for a composite monad.

6. THE EXAMPLES.

Proposition 38. $()_* : \text{CAT} \rightarrow \text{PROF}$ satisfies Axiom 4.

Proof. CAT duality, $()^{\text{op}} : \text{CAT}^{\text{co}} \xrightarrow{\sim} \text{CAT}$ extends to PROF , as in

commutative diagram below.

$$\begin{array}{ccc}
 & \xleftarrow{(\)^{op}} & \\
 CAT^{co} & \xrightleftharpoons[(\)^{op}]{\approx} & CAT \\
 (\)_* \downarrow & & \downarrow (\)_* \\
 PROF^{op} & \xrightleftharpoons[(\)^{op}]{\approx} & PROF
 \end{array}$$

In particular, any colimit diagram in $PROF$ consisting entirely of profunctors of the form f_* becomes a limit diagram in $PROF$ when each f_* is replaced by f^* . So we have only to show that $(\)_*$ preserves the finite sums of CAT .

$$\begin{array}{c}
 \frac{0 \rightarrow A \text{ in } PROF}{A^{op} \times 0 \rightarrow SET \text{ functors}} \\
 \frac{}{0 \rightarrow SET \text{ functors}} \\
 \hline
 1 \\
 \frac{B + C \rightarrow A \text{ in } PROF}{A^{op} \times (B + C) \rightarrow SET \text{ functors}} \\
 \frac{(A^{op} \times B) + (A^{op} \times C) \rightarrow SET \text{ functors}}{A^{op} \times B \rightarrow SET, A^{op} \times C \rightarrow SET \text{ functors}} \\
 \hline
 B \rightarrow A, C \rightarrow A \text{ in } PROF.
 \end{array}$$

◇

Proposition 39. $(\)_* : CAT \rightarrow PROF$ satisfies Axiom 5.

Proof. Let (A, Φ, η, μ) be a monad in $PROF$. A_Φ is the category whose objects are those of A with $A_\Phi(a, b) = \Phi(a, b)$. Composition is given by $\mu : \Phi\Phi \rightarrow \Phi$ and a (bijective Φ on objects) functor $k : A \rightarrow A_\Phi$ is obtained from $\eta : A \rightarrow \Phi$. The opalgebra structure, $\kappa : \Phi k \rightarrow k$ is constructed from μ . Explicitly :

$$\begin{aligned}
 \Phi k_*(a, c) &\simeq \int^b \Phi(b, c) \times k_*(a, b) \simeq \int^b \Phi(b, c) \times A_\Phi(a, bk) \\
 &\simeq \int^b \Phi(b, c) \times \Phi(a, b) \simeq \Phi\Phi(a, c) \xrightarrow{\mu} \Phi(a, c) \simeq A_\Phi(a, ck) \simeq k_*(a, c).
 \end{aligned}$$

It is easy to check that this opalgebra, $[k_\kappa]$, is universal amongst opalgebras $[f_\varphi]$ with f in CAT . It continues to be universal if f is a functor $A \rightarrow SET^{X^{op}}$ ($SET \nsubseteq CAT$) but the latter is a profunctor $A \rightarrow X$. So A_Φ is Kleisli in $PROF$ and representable opalgebras give rise to representables out of A_Φ .

Finally, note that $[k_\kappa]$ Kleisli for (A, Φ) implies

$$\begin{array}{ccc}
 A^{op} & & \\
 \downarrow \uparrow \kappa^{op} & \searrow k^{op} \circ \kappa_* & \\
 A^{op} & \xrightarrow{k^{op} \circ \kappa_*} & A^{op} = (A_\Phi)^{op}
 \end{array}$$

is a Kleisli diagram for the monad $(A^{\text{op}}, \phi^{\text{op}})$. Applying $()^{\text{op}}$ to this diagram yields an Eilenberg-Moore diagram for the monad (A, Φ) . (Recall the duality diagram of the proof of Proposition 38 and note that $k^{\text{op}} \circ \phi^{\text{op}} = k \circ \phi$.) $\diamond$

The preceding two propositions can of course be established without the use of functors that suggest a change of universe. The proofs are tedious but suitably modified they allow CAT to be replaced by V-CAT, S-ind CAT, cat(S), etc.

Proposition 40. $()_* : \text{TOP} \rightarrow \text{TOPLEX}^{\text{co}}$ satisfies Axiom 4.

Proposition 41. $()_* : \text{TOP} \rightarrow \text{TOPLEX}^{\text{co}}$ satisfies Axiom 5.

Proof. A monad in $\text{TOPLEX}^{\text{co}}$ is a cotriple, on a topos, whose underlying functor is left exact. If (E, g) is such it is well-known that the category of Eilenberg-Moore coalgebras, E_g , is again a topos and that the cofree functor $k_* : E \rightarrow E_g$ is the direct image part of a geometric morphism. Necessarily, k_* is an opcoalgebra for g , via $\delta : g \rightarrow gg$ since $ek_* = (eg, e\delta)$. Write $\kappa : k_* \rightarrow gk_*$. Let $\left[\begin{smallmatrix} h \\ \omega \end{smallmatrix} \right]$ be any opcoalgebra with $h : E \rightarrow F$ in TOPLEX . For all (e, α) in E_g , we have

$$(e, \alpha) \xrightarrow{\alpha} (eg, e\delta) \xrightarrow[e\delta]{\alpha g} (egg, eg\delta)$$

an equalizer in E_g . So a left exact functor $\bar{h} : E_g \rightarrow F$ satisfying $k_* \bar{h} = h$ and $\kappa \bar{h} = \omega$ must be given by

$$(e, \alpha) \bar{h} \longrightarrow eh \xrightarrow[e\omega]{\alpha h} egh,$$

an equalizer in F . Hence there exists an essentially unique such $\bar{h}$. $\bar{h}$ has a left exact left adjoint (thus making it an arrow in TOP) iff h does. This verifies all the Kleisli aspects of Axiom 5.

Finally, the forgetful functor $k^* : E_g \rightarrow E$ exhibits E_g as an Eilenberg-Moore object in $\text{TOPLEX}^{\text{co}}$. $\diamond$

It is clear that Propositions 40 and 41 apply equally well to

$$()_* : \text{ABEL} \rightarrow \text{ABELLEX}^{\text{co}} \quad \text{and} \quad ()_* : \text{GEOM} \rightarrow \text{LEX}^{\text{co}},$$

where ABEL is abelian categories and geometric morphisms (see [RW3]) and GEOM is left exact categories and geometric morphisms.

THE AXIOMS.

We begin with a notion of morphism of proarrow equipments. Only Axioms 1, 2 and 3 are assumed. A *strong morphism* from $()_* : K \rightarrow M$ to

$(\)_+ : L \rightarrow N$ is a pair of homomorphisms $V : K \rightarrow L$, $U : M \rightarrow N$ such that

$$\begin{array}{ccc} K & \xrightarrow{V} & L \\ (\)_* \downarrow & & \downarrow (\)_+ \\ M & \xrightarrow{U} & N \end{array}$$

commutes up to (specified) equivalence. (The equivalence should really be regarded as part of the data.) We write U for both U and V above and say that $U : M \rightarrow N$ preserves representability. Since homomorphisms preserve adjunctions we have also that

$$\begin{array}{ccc} K^{\text{coop}} & \xrightarrow{U^{\text{coop}}} & L^{\text{coop}} \\ (\)_* \downarrow & & \downarrow (\)_+ \\ M & \xrightarrow{U} & N \end{array}$$

"commutes". We will write $U : (\)_* \rightarrow (\)_+$. An example is provided in the duality diagram of Proposition 38. (If $(\)_* : K \rightarrow M$ is proarrow equipment then so is $(\)_* : K^{\text{co}} \rightarrow M^{\text{op}}$.)

Now suppose that $U : M \rightarrow N$ has a left adjoint $F : N \rightarrow M$. (I.e., we have equivalences of hom categories, $M(NF, M) \sim N(N, MU)$, "natural" in M and N .) If F is also a strong homomorphism of proarrow equipments then the components of the unit, $Nn : N \rightarrow NFU$ and the counit $Me : MUF \rightarrow M$ are representables. Their right adjoints

$$Nn^+ : NFU \rightarrow N \quad \text{and} \quad Me^* : M \rightarrow MUF$$

tend to give $U : M \xrightarrow{\perp} N : F$ in addition to the assumed $F \dashv U$.

Indeed, the triangle identities hold. However, while n and e are necessarily "natural", in N and M respectively, n^+ and e^* need not be. A very special instance of this is provided by Counterexample 2. This leads us to say that $U : (\)_* \rightarrow (\)_+$ has a left adjoint if $U : M \rightarrow N$ has a left adjoint which preserves $+$ representables and the right adjoints of the unit and counit are "natural". Applying $(\)^{\text{coop}}$ to an adjunction reverses the sense of it so in this situation we get a pair of "commutative" diagrams :

$$\begin{array}{ccc} K & \xleftarrow{F} & L \\ \downarrow (\)_* & \xrightarrow{\perp} U & \downarrow (\)_+ \\ M & \xleftarrow{F} & N \end{array} \quad \begin{array}{ccc} K^{\text{coop}} & \xrightarrow{U^{\text{coop}}} & L^{\text{coop}} \\ (\)_* \downarrow & \xleftarrow{F \perp} & \downarrow (\)_+ \\ M & \xrightarrow{\perp} U & N \end{array}$$

Write I for the bicategory with one object, 0 , and $I(0, 0) = 1$. (It is the terminal bicategory!) $I \rightarrow I$ is proarrow equipment. If $(\)_* : K \rightarrow M$ is proarrow equipment then so is $(\)_* \times (\)_* : K \times K \rightarrow M \times M$. Clearly,

$! : M \rightarrow I$ and $\Delta : M \rightarrow M \times M$ define strong homomorphisms of proarrow equipments, $! : ()_* \rightarrow I$ and $\Delta : ()_* \rightarrow ()_* \times ()_*$.

Proposition 42. *Axiom 4 is equivalent to $! : ()_* \rightarrow I$ and $\Delta : ()_* \rightarrow ()_* \times ()_*$ having left adjoints.* $\diamond$

We will use the machinery of this section to give an analogous formulation of Axiom 5. The situation is considerably more complex. In [T2] Street defined, for a bicategory K , a bicategory $MND(K)$ whose objects are monads (X, t, η, μ) in K . There is a diagonal homomorphism $I : K \rightarrow MND(K)$, a right adjoint for which is equivalent to the existence of Eilenberg-Moore objects in K . However, for any K , I has a left adjoint. It is a forgetful homomorphism. Existence of Kleisli objects in K is equivalent to the existence of a left adjoint to a similar diagonal, $K \rightarrow MND(K^{opp})$ (which always has a forgetful right adjoint).

Henceforth, assume that M has local coequalizers. $MOD(M)$ is defined to be the bicategory whose objects are monads $(A, \Phi, \eta, \mu) = (A, \Phi)$ in M , whose arrows $(A, \Phi) \rightarrow (B, \Psi)$ are left Φ , right Ψ modules and whose transformations are equivariant transformations between modules. More explicitly, an arrow $(A, \Phi) \rightarrow (B, \Psi)$ is a triple (γ, Γ, δ) , where $\Gamma : A \rightarrow B$ is an arrow in M and $\gamma : \Phi \Gamma \rightarrow \Gamma \Psi : \delta$ are associating actions. Modules are composed using the coequalizer formula of Proposition 28. We have a diagonal $\Delta : M \rightarrow MOD(M)$ given by $X \rightarrow (X, 1_X)$.

Lemma 43. Δ has a right adjoint iff M has Eilenberg-Moore objects. Δ has a left adjoint iff M has Kleisli objects. $\diamond$

For proarrow equipment $()_* : K \rightarrow M$ we define a bicategory $HOM()_*$ as follows: The objects of $HOM()_*$ are those of $MOD(M)$, monads (A, Φ) in M . An arrow $(A, \Phi) \rightarrow (B, \Psi)$ is a pair (f, φ) where $f : A \rightarrow B$ is in K and

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Phi \downarrow & \nearrow \varphi & \downarrow \Psi \\ A & \xrightarrow{f} & B \end{array}$$

is a transformation in M satisfying

$$\begin{array}{ccc} f & \xrightarrow{\eta f} & \Phi f \\ & \searrow f \eta & \downarrow \varphi \\ & & f \Psi \end{array} \quad \text{and} \quad \begin{array}{ccc} & \nearrow \Phi \varphi & \Phi f \Psi \xrightarrow{\varphi \Psi} f \Psi \Psi \\ \Phi \Phi f & & \downarrow f \mu \\ & \searrow \mu f & \Phi f \xrightarrow{\varphi} f \Psi \end{array}$$

For $(f, \phi), (g, \gamma) : (A, \Phi) \rightarrow (B, \Psi)$, a transformation $(f, \phi) \rightarrow (g, \gamma)$ is a transformation $\tau : f \rightarrow g\Psi$ satisfying

$$\begin{array}{ccccc} & & f\Psi & \xrightarrow{\tau\Psi} & g\Psi\Psi \\ & \nearrow \phi & & & \searrow g\mu \\ \hat{\Psi} f & & & & g\Psi \\ & \searrow \Phi\tau & \Phi g\Psi & \xrightarrow{\gamma\Psi} & g\Psi\Psi \\ & & & & \nearrow g\mu \end{array}$$

Arrows in $HOM(\ast)$ are composed by pasting squares. If also

$$\sigma : (g, \gamma) \rightarrow (h, \vartheta) : (A, \Phi) \rightarrow (B, \Psi),$$

the composite $\tau.\sigma$ is given by

$$f \xrightarrow{\tau} g\Psi \xrightarrow{\sigma\Psi} h\Psi\Psi \xrightarrow{h\mu} h\Psi.$$

Example 44. Let $(\)_+ : SET \rightarrow MAT$ denote the proarrow equipment obtained by restricting $(\)_\ast : CAT \rightarrow PROF$ to the discrete objects of CAT . (So MAT is equivalent as a bicategory to the bicategory of spans, $SPN(SET)$.) Then $HOM(+)$ is equivalent as a bicategory to CAT . $\diamond$

For $(f, \phi) : (A, \Phi) \rightarrow (B, \Psi)$ as above $f\Psi$ becomes a Φ, Ψ module, free on the right via $f\mu : f\Psi\Psi \rightarrow f\Psi$ with left Φ action given by

$$\Phi f\Psi \xrightarrow{\phi\Psi} f\Psi\Psi \xrightarrow{f\mu} f\Psi.$$

A transformation $\tau : (f, \phi) \rightarrow (g, \gamma)$ gives rise to an equivariant transformation $f\Psi \rightarrow g\Psi$ via

$$f\Psi \xrightarrow{\tau\Psi} g\Psi\Psi \xrightarrow{g\mu} g\Psi.$$

The assignments define a homomorphism $MON(\ast) : HOM(\ast) \rightarrow MOD(M)$. Every transformation $f\Psi \rightarrow g\Psi$ in $MOD(M)$ is obtained as above from a unique transformation $(f, \phi) \rightarrow (g, \gamma)$ in $HOM(\ast)$. (Indeed, it is obtained by composing $f\eta : f \rightarrow f\Psi$ and $f\Psi \rightarrow g\Psi$.) Furthermore, $\Psi f^\ast$ is Ψ, Φ module from (B, Ψ) to (A, Φ) in $MOD(M)$. The right Φ action is constructed using $\phi^\ast : f^\ast\Phi \rightarrow \Psi f^\ast$. We have $f\Psi \dashv \Psi f^\ast$ in $MOD(M)$. Hence :

Lemma 45. $MON(\ast) : HOM(\ast) \rightarrow MOD(M)$ is proarrow equipment. $\diamond$

These and related matters will be dealt with in further detail in a forthcoming paper by Rosebrugh and Wood [RW4].

Example 46. For $(\)_+ : SET \rightarrow MAT$ in Example 44,

$$MON(\ast) : HOM(\ast) \rightarrow MOD(MAT)$$

is equivalent to $(\)_\ast : CAT \rightarrow PROF$. $\diamond$

The homomorphism $\Delta: M \rightarrow MOD(M)$ defines a strong homomorphism of proarrow equipments, $\Delta: ()_* \rightarrow MON()_*$.

Proposition 47. *Axiom 5 is equivalent to $\Delta: ()_* \rightarrow MON()_*$ having a left adjoint.* $\diamond$

Artin glueing can be described globally in this manner too. The bicategories required for the description are related to the bicategory of glueing data in [NIE] in much the same way that HOM and MOD are related to MND .

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