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ON THE CATEGORIES Sp(X) AND Ban(X). II

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RÉSUMÉ. Soit X un espace localement compact séparé. On construit des foncteurs adjoints A et S entre la catégorie des espaces au-dessus de X et la catégorie des espaces fibrés de Banach au-dessus de X. En conséquence, pour un espace fibré de Banach quelconque $p: E \to X$, l'espace $\Gamma_0(p)$ des sections de E tendant vers zéro à l'infini se plonge dans $C_0(S(E))$. Comme application on donne une description de l'espace dual de $\Gamma_0(p)$ et une représentation intégrale des opérateurs sur $\Gamma_0(p)$. Ces résultats sont valables en particulier dans le cas des vecteurs fibrés et des champs continus d'espaces de Banach étudiés par Dixmier, Douady, Fell, Gelfand, Godement, etc.

1. INTRODUCTION.

In an earlier paper [14], we constructed a pair of adjoint functors A and S between the category Sp(X) of spaces over a space X and the category Ban(X) of Banach bundles over X. However, in that paper the condition was imposed on the objects of Ban(X) that their norm function be continuous or, equivalently as it turns out, that each object in Sp(X) have open projection. This suited our needs in [13], where we were motivated to provide a bundle theoretic proof of the continuity of a certain convolution product of functions encountered in constructing C*-algebras of topological groupoids (see also [12]). It also suits the need that arises in the study of induced representations of locally compact groups, such as that made by Fell in [2]. Nevertheless, there are situations in analysis where the appropriate condition on the norm function is that it be merely upper semi-continuous. This is the case, for example, in the representation theory of rings and algebras by sections [6] (see also [9, 10] and their bibliography for some recent developments) and in the study of Banach spaces in the category of sheaves on X, see [8, 11].

Our present aim is threefold. Firstly, it is desirable in applications to have the results of [14] available in the more general setting of Banach bundles with upper semi-continuous norm function. In particular, this is true if one attempts to do "functional analysis" in the context of sectional representation theory. In Section 2 we will make this extension, but we will only present details where [14] does not immediately generalize. The main technical change here is that the fibres of a closed map $q: Y \to X$ determine an upper semi-continuous decomposition of Y, and this fact substitutes for the earlier requirement of openness of q.

As a corollary, we obtain a necessary and sufficient condition for continuity of the norm function of a Banach bundle.

The counit of the adjunction established in Section 2 yields a yet more general form of Alaoglu's Theorem (*) than that given earlier in [14], see Corollary 1. This leads naturally to an embedding of spaces of selections (sections), of a Banach bundle $p: E \to X$, inside spaces of complex valued functions (continuous functions) on $S(E)$; on the space $\Gamma_0(p)$ of sections vanishing at infinity, this embedding is essentially that of Kitchen and Robbins [9], Appendix. Section 3 will be devoted to considering this embedding which, in principle, reduces the study of spaces of sections to the study of certain closed subspaces of function spaces, see Theorem 4.

The aforementioned embedding is well suited for purposes of integration theory and we use it, in Section 4, to represent the dual of $\Gamma_0(p)$ in terms of (equivalence classes of) measures on X and bounded selections of the dual "bundle" of E, following [15]. This result is employed, finally, to establish an integral representation of operators respectively weakly compact operators, respectively compact operators defined on $\Gamma_0(p)$.

It is worth noting two special cases in which all these results apply ; they are

(i) normed vector bundles,
(ii) the continuous fields of Banach spaces studied by Dixmier, Douady, Fell, Gelfand, Godement et al.

In fact, Theorem 2 provides, as is well-known, an immediate mechanism by which such fields can be viewed in terms of Banach bundles, as discussed here or, indeed, as in [14], since the norm function is continuous in these two cases.

2. THE CATEGORIES *Sp*(X) AND *Ban*(X).

Let X denote a (fixed) locally compact Hausdorff space. Objects in the category $Sp(X)$, of spaces over X, are continuous surjective mappings $q: Y \to X$, where Y is a locally compact Hausdorff space ; such objects are also called fields of topological spaces in the literature. A morphism

$$\eta: (Y, q) \to (Y', q')$$

is a proper, fibre preserving map $\eta: Y \to Y'$. If, in addition, q is an open map, then $q: Y \to X$ is a space over X as defined in [14]. It is sometimes convenient to refer to such a map as an open space over X, and to keep the notation Sp(X), as in [14], for the full subcategory of $Sp(X)$ which they form. Further, we shall denote by $Spp(X)$ the full subcategory of $Sp(X)$ in which the projections q are proper maps, and

*) A similar result has been obtained by Mulvey and Pelletier [11] in the context of locales. However, their methods and objectives are very different from ours.

by Spp(X) the corresponding subcategory of Sp(X), as in [14] again.

An object in the category *Ban*(X), of Banach bundles over X, is a Banach bundle $p: E \to X$ as defined in [3, 7, 8, 9, 10], and we refer to [3] as a background source of information and terminology regarding Banach bundles. A morphism $\Psi: (E, p) \to (E', p')$ in *Ban*(X) is a continuous fibre preserving map $\Psi: E \to E'$ such that $\Psi_x = \Psi|_{E_x}$ is a linear contraction, that is to say $||\Psi_x|| \leq 1$, for all $x \in X$. We shall retain the notation Ban(X) of [14] for the full subcategory of *Ban*(X) formed by those Banach bundles whose norm function is continuous.

We proceed now to establish the functor $\bar{A}: Sp(X) \to Ban(X)$ following closely the development and notation of [14]. It will be convenient for the reader, however, and for use in later sections, to repeat the basic constructions made there. Thus, let $q: Y \to X$ be an object of $Sp(X)$, and let $p: A \to X$ be the corresponding map whose fibre A_x is the space $C_0(Y_x)$ of continuous complex valued functions defined on the fibre $q^{-1}(x)$ of q over x, and which vanish at infinity. Again each element φ of the space $k(Y)$, of continuous complex valued functions defined on Y with compact support, determines a selection $\tilde{\varphi}: X \to A$ defined by $\tilde{\varphi}(x) = \varphi|_{Y_x}$; we let Γ_c denote the vector space $\{\tilde{\varphi} \mid \varphi \in k(Y)\}$ of all such selections.

If $\tilde{\varphi} \in \Gamma_c$, then

$$||\tilde{\varphi}(x)|| = \sup_{y \in Y_x} |\varphi(y)| ,$$

and we denote by $||\tilde{\varphi}(.)||$ the resulting non-negative function defined on X. Our substitute for [13], Theorem 1 (ii), is the following result.

Theorem 1. *Let $q: Y \to X$ be an object of $Sp(X)$. Then for every $\varphi \in k(Y)$ the function $||\tilde{\varphi}(.)||$ is upper semi-continuous on X. Moreover, $||\tilde{\varphi}(.)||$ is continuous for every $\varphi \in k(Y)$ iff q is an open map.*

Proof. Let $\varphi \in k(Y)$ and let K denote the compact support of φ. Then $||\tilde{\varphi}(x)||$ has value zero if x belongs to the open set $X \backslash q(K)$, and has value $\sup_{y \in K_x} |\varphi(y)|$ otherwise, where $K_x = K \cap Y_x$. It therefore suffices to show that $||\tilde{\varphi}(.)||$ is upper semi-continuous on $q(K)$. To do this we can restrict q to K and consider $q: K \to q(K)$, since we are not supposing that q is open. It is no loss of generality, therefore, to assume that Y is compact and that K = Y. In this case q is closed and so the fibres $\{Y_x \mid x \in X\}$ of q form an upper semi-continuous decomposition D of Y.

Fix $x \in X$ and let $\varepsilon > 0$. For each $y \in Y_x$ there is a neighborhood U_y of y in Y such that

$$|\varphi(y') - \varphi(y)| < \varepsilon \quad \text{for all} \quad y' \in U_y .$$

Hence, $\bigcup_{y \in Y_x} U_y$ is a neighborhood of Y_x in Y. By the nature of D, there is a neighborhood V of x in X such that

$$Y_x \subset q^{-1}(V) \subset \bigcup_{y \in Y_x} U_y \,.$$

Now let $x' \in V$; then there is an element β of $Y_{x'}$ such that

$$\|\tilde{\varphi}(x')\| = |\varphi(\beta)| \,.$$

Choose $y \in Y_x$ such that $\beta \in U_y$, then $|\varphi(\beta) - \varphi(y)| < \varepsilon$. Hence,

$$|\varphi(\beta)| < \varepsilon + |\varphi(y)| \leqq \varepsilon + |\varphi(\alpha)| \,,$$

where $\alpha \in Y_x$ is such that $\|\tilde{\varphi}(x)\| = |\varphi(\alpha)|$ Thus,

$$\|\tilde{\varphi}(x')\| < \varepsilon + \|\tilde{\varphi}(x)\| \qquad \text{for all } x' \in V,$$

and so $\|\tilde{\varphi}(.)\|$ is upper semi-continuous on X, as required.

Remark. The reason why this argument is not symmetrical in α and β, and hence that $\|\tilde{\varphi}(.)\|$ is not continuous in general, is that the neighborhoods U_y which contain α may not meet $Y_{x'}$.

That $\|\tilde{\varphi}(.)\|$ is continuous if q is open follows from [13], Theorem 1. The converse follows from [5], Theorem 3, with minor modifications to allow for non compactness of X. *) ◊

By applying Hofmann's basic existence Theorem [7], Theorem 3.6, to the restrictions $\tilde{\varphi}|V$ of elements $\tilde{\varphi}$ of Γ_c to the open sets V in X, we obtain the following theorem.

Theorem 2. *Let $q : Y \to X$ be an object of $Sp(X)$. Then there is a unique coarsest topology on A such that :*

(i) $p : A \to X$ is a Banach bundle.
(ii) All the elements $\tilde{\varphi}$ of Γ_c are sections of p.

If, further, q is in fact an object of $Spp(X)$, then there is only one topology on A satisfying (i) and (ii). ** ◊

In case that $q: Y \to X$ is a proper map the fibres Y_x are compact spaces, and so

$$\{\tilde{\varphi}(x) \mid \tilde{\varphi} \in \Gamma_c\} = A_x \qquad \text{for each} \quad x \in X.$$

It is from this observation that the second conclusion of Theorem 2 follows.

*) In preparing [13], we overlooked the existence of [5] which contains some results similar to those of [13]. This reference indicates possible applications of our results to approximation theory.

**) There is an alternative way of obtaining the second conclusion, at least if X is compact, by viewing C(Y) as a C(X)-module by means of q , see [9], §3 Example 2.

Theorem 2 determines the object function of $\bar{A}$, and on morphisms $\eta : (Y, q) \to (Y', q')$ in $Sp(X)$, $\bar{A}$ is defined by

$$(\bar{A}\,\eta)(f) = f \circ \eta_x \qquad \text{for } f \in \bar{A}(Y')_x \text{ and } x \in X,$$

where η_x denotes $\eta | Y_x$.

Since X is locally compact, any object in $Ban(X)$ is full in the sense that a section passes through any given point of the total space [3], and so Proposition 1 of [14] and its proof immediately generalise to $Ban(X)$, and we obtain :

Proposition 1. $\bar{A}$ *is a contravariant functor from* $Sp(X)$ *to* $Ban(X)$. ◊

Finally, we note that only the slightest change needs to be made to the proof of [14], Proposition 3, viz. replacement of $U(\mathfrak{F}, V, \varepsilon/3)$ as defined there by $U(\mathfrak{F}, X, \varepsilon/3)$, to obtain a valid conclusion here, as follows.

Proposition 2. *For any object* $q : Y \to X$ *of* $Sp(X)$, *the evaluation map* $\rho : (f, y) \mapsto f(y)$ *is continuous on* $A(Y) \times_X Y$. ◊

We turn next to the construction of the functor $\bar{S} : Ban(X) \to Sp(X)$. Thus, let $p : E \to X$ be an object of $Ban(X)$ and let $p^* : E^* \to X$ be the associated map on the disjoint union E^* of the topological duals E_x^* of E_x. For each section σ of E, we let F_σ denote the mapping defined on E^* by

$$F_\sigma(f) = f(\sigma(p^*(f)))$$

and we give E^* the weak topology generated by the collection Ω of all the F_σ's together with p^*. Finally, let $\bar{S}(E)$ denote the set of all $f \in E^*$ with $\|f\| \leq 1$, and let q denote the restriction of p^* to $\bar{S}(E)$.

In [15], §3, we extended the appropriate results of [14], namely Propositions 4, 6 and 7, to our present level of generality. Often in proving such results one encounters the function $\|\sigma(.)\|$, where σ is a section, and one knows that $\|\sigma(x)\| < \varepsilon$ - say. Previously we used continuity to produce a neighborhood V of x in X such that

$$\|\sigma(y)\| < \varepsilon \qquad \text{for all } y \in V.$$

But, of course, upper semi-continuity also provides such a V. In any event we have the following result.

Proposition 3. $q : \bar{S}(E) \to X$ *is an object of* $Spp(X)$. ◊

This proposition yields the object function of $\bar{S}$, which on morphisms is defined as follows. Let $\Psi : (E, p) \to (E', p')$ in $Ban(X)$ and let Ψ^* denote the conjugate of Ψ, then we define $\bar{S}(\Psi)$ to be $\Psi^*|_{\bar{S}(E')}$.

Proposition 4. *$\bar{S}$ is a contravariant functor from Ban(X) to Sp (X) (in fact to Spp(X)).* ◊

The main result of this section is the following theorem in which we adopt now, and henceforth, the notational convention of [14], Theorem 3.

Theorem 3. *The functor A^{op} : Spp(X) $\to$ Ban$(X)^{op}$ is left adjoint to S : Ban$(X)^{op}$ $\to$ Spp(X) via an adjunction whose counit ε has components ε_E^{op} , E ϵ Ban(X) , where ε_E: E $\to$A(S(E)) is defined by*

$$\varepsilon_E(s)(f) = f(s) \quad \text{for} \quad f \in S(E) \quad \text{and} \quad s \in E.$$

Proof. The proof follows the same steps as that of [14], Theorem 3, except that in Step 1 we use the generalized form of Proposition 1 of [14], and in Step 2 we use Proposition 2 in place of [14], Proposition 3. ◊

Noting the uniqueness assertion of Theorem 2 relative to an object of *Spp* (X) and the proof of the corollary to [14], Theorem 3, we obtain :

Corollary 1. *The map ε_E is an isometric isomorphism of E, E ϵ Ban(X) , onto a Banach subbundle of A(S(E)) , which need not be a closed subset of A(S(E)).* ◊

Corollary 2. *Let p : E $\to$ X be an object of Ban(X). In order that the norm function of E be continuous, it is necessary and sufficient that the map q : S(E) $\to$ X be open.* ◊

Remark. It is natural to enquire as to the validity of the results of this section over spaces X more general than we are currently considering, and we will briefly investigate what this entails.

1) Firstly, one needs to know that any object of *Ban*(X) is full to ensure that Ω separates the points of E*. The best general hypothesis which will guarantee this is complete regularity of X (see [3], Corollary 2.10). However, to apply Theorem 2 one needs to know that the set of values $\tilde{\varphi}(x)$, with $\varphi \in k(Y)$ say, is dense in $C_0(Y_x)$ for any space Y over X. This necessitates an application of the Tietze extension Theorem and one therefore needs to know that Y, and hence X, is normal. Under this hypothesis, the functor A can be constructed.

2) Now one is faced with showing that S(E) is normal, and the obvious approach, which we used in [14], is to embed S(E) in the product of the normal space X with the (normal) product of the images of the F_σ with σ , say, running over the set of all bounded sections of E. But products and subspaces of normal spaces need not be normal, and this strategy, whilst showing complete regularity of S(E), seems unlikely to succeed in demonstrating normality.

For these reasons it is not clear that S can be so constructed, and in view of this local compactness is probably the best general hypothesis to make on X. It is worth noting, however, that the theory developed in [11] makes no restriction on X at all.

3. SPACES OF SECTIONS AND THE EMBEDDING OF $\Gamma_0(p)$ IN $C_0(S(E))$.

Let $p: E \to X$ be an object of *Ban*(X). A selection σ of E will be said to vanish at infinity if for each $\varepsilon > 0$ there is a compact set $K \subset X$ such that

$$\|\sigma(x)\| < \varepsilon \quad \text{for all } x \in X \setminus K.$$

We denote by $\Sigma(p)$ the set of all bounded selections of E endowed with the uniform norm :

$$\|\sigma\| = \sup_{x \in X} \|\sigma(x)\|$$

By $\Pi(p)$, $\Gamma_0(p)$ and $\Gamma_C(p)$ we denote, respectively, the subspaces of $\Sigma(p)$ consisting of all bounded sections, sections vanishing at infinity and sections with compact support. Then $\Sigma(p)$ is a Banach space [3], 1.12, $\Pi(p)$ and $\Gamma_0(p)$ are closed subspaces of $\Sigma(p)$ and $\Gamma_0(p)$ is the uniform norm completion of $\Gamma_C(p)$, [15], Proposition 4. If $q: Y \to X$ is an object of *Spp*(X), we denote by B(Y) the Banach space of all bounded complex valued functions on Y whose restrictions to Y_x are continuous for each $x \in X$, endowed with the uniform norm. Finally, we denote by C(Y) the subspace of the continuous functions in B(Y), and retain the notation $C_0(Y)$ and $k(Y)$ with the meaning already established.

Consider an object $q: Y \to X$ of *Spp* (X) and its image $\overline{p}: A(Y) \to X$ under the functor A. An element φ of B(Y) determines a selection $\tilde{\varphi}$ of A(Y) as in §2, that is

$$\tilde{\varphi}(x) = \varphi|_{Y_x}$$

and a simple computation shows that $\|\tilde{\varphi}\| = \|\varphi\|$. Hence, $\tilde{\varphi} \in \Sigma(\overline{p})$. In the opposite direction we have :

Proposition 5. *Any element σ of $\Sigma(\overline{p})$ determines uniquely an element φ of* B(Y) *such that $\tilde{\varphi} = \sigma$. Moreover, φ is continuous iff $\tilde{\varphi}$ is a section.*

Proof. Define φ on Y by

$$\varphi(y) = \sigma(q(y))(y).$$

Then

$$\varphi|_{Y_x} = \sigma(x),$$

and is therefore continuous, $\tilde{\varphi} = \sigma$ and φ is bounded since, in fact, we have $\|\varphi\| = \|\sigma\|$; hence $\varphi \in B(Y)$.

Suppose φ is continuous and $x \in X$. Let U be a compact neighborhood of x in X and choose

$$\vartheta \in k(Y) \quad \text{with} \quad \vartheta = 1 \text{ on } q^{-1}(U).$$

Then $\vartheta\varphi \in k(Y)$ and so $\widetilde{\vartheta\varphi}$ is continuous by Theorem 2. Since $\widetilde{\vartheta\varphi} = \tilde{\varphi}$ on U, it follows that $\tilde{\varphi}$ is continuous at x as required.

Finally, if $\tilde{\varphi}$ is continuous, then the expression

$$\varphi(y) = \tilde{\varphi}(q(y))(y)$$

can be written as a composite of mappings involving the evaluation map of Proposition 2. It follows that φ is continuous. ◊

Now, it is clear that the mapping $\varphi \mapsto \tilde{\varphi}$ is linear and therefore an isometric isomorphism, and we obtain :

Proposition 6. *Under the identification* $\varphi \mapsto \tilde{\varphi}$ *we have :*

1. $\Sigma(\bar{p}) = \{\tilde{\varphi} \mid \varphi \in B(Y)\}$.
2. $\Gamma(\bar{p}) = \{\tilde{\varphi} \mid \varphi \in C(Y)\}$.
3. $\Gamma_0(\bar{p}) = \{\tilde{\varphi} \mid \varphi \in C_0(Y)\}$.
4. $\Gamma_c(\bar{p}) = \Gamma_c = \{\tilde{\varphi} \mid \varphi \in k(Y)\}$.

If $p : E \to X$ *is any object of* $Ban(X)$ *and* $Y = S(E)$, *then the mapping* T *defined by* $T(\sigma) = \varepsilon_E \sigma$, *where* ε_E *is defined as in Theorem 3, is an isometric isomorphism of* $M(p)$ *onto a closed subspace of* $M(\bar{p})$, *where* M *denotes any of the four spaces* Σ, Γ, Γ_0 *or* Γ_c.

Proof. The assertions 1 and 2 have already been established. For 3, if $\|\tilde{\varphi}(x)\| < \varepsilon$ outside the compact set K, then $|\varphi(y)| < \varepsilon$ outside the compact set $q^{-1}(K)$, and the converse is similar. So too is the proof of 4.

As far as T is concerned, if $\sigma \in \Sigma(\bar{p})$, then

$$\|\varepsilon_E \sigma(x)\| = \|\sigma(x)\| \qquad \text{for each} \quad x \in X.$$

From this it follows that

$$\|\varepsilon_E \sigma\| = \|\sigma\|$$

and also that $T : M(p) \to M(\bar{p})$ in each of the four cases listed above. That T is linear is clear and therefore T is an isometric isomorphism as required.

The closure assertion follows from completeness in cases 1, 2 and 3, and it only remains to show that $T(\Gamma_c(p))$ is closed in $\Gamma_c(\bar{p})$. So suppose that $\sigma_n \in \Gamma_c(p)$ for each natural number n and that $T(\sigma_n) \to w$ in $\Gamma_c(\bar{p})$; then w has compact support. Since T is an isometry and $\{T(\sigma_n)\}$ is a Cauchy sequence, it follows that $\{\sigma_n\}$ is a Cauchy sequence and, hence, that $\sigma_n \to \sigma$ in $\Gamma(p)$ for some section σ. Therefore, $T(\sigma_n) \to T(\sigma)$ and we conclude that $T(\sigma) = w$. Whence σ has compact support or, in other words, $\sigma \in \Gamma_c(p)$. Therefore $w \in T(\Gamma_c(p))$ as required. ◊

By composing the map T with the inverse of $\sim$ we obtain the map

$$F : \sigma \mapsto F_\sigma , \quad \text{where} \quad F_\sigma(f) = f(\sigma(q(f)))$$

for any selection σ.

The main result of this section is the following theorem.

Theorem 4. *Let $p : E \to X$ be any object of $Ban(X)$. Then the mapping F is an isometric isomorphism of $M(p)$ onto a closed subspace of $F(S(E))$ in case $M = \Sigma, \Gamma, \Gamma_0$ or Γ_c and F is, respectively,* B, C, C_0 or k. ◊

Remark. This theorem has, by means of Proposition 5, an obvious analogue for spaces of unbounded selections or sections respectively unbounded functions, equipped with, say, the uniform-on-compacta topology or for that matter any other function space topology. We will not trouble to formulate these results.

Before moving on to the main applications of this result, we will pause briefly to consider its relevance to compactness of sets of sections and derive a version of Ascoli's Theorem.

Suppose X is compact and note then that Y is compact also for any object $q : Y \to X$ of $Spp(X)$. Let $p : E \to X$ be any Banach bundle over X, and suppose G is a subset of $\Gamma(p)$. In order to give criteria for compactness of G, it is natural to attempt to define "equicontinuity" for elements of G. But this seems difficult to do (even for product-bundles) if only for the reason that "$\sigma(x) - \sigma(x')$" makes no sense if $x \neq x'$. One solution is simply to require equicontinuity of G to mean equicontinuity of the set

$$F(G) = \{ F_\sigma \mid \sigma \in G \}$$

of functions. It is then a simple application of Ascoli's Theorem to obtain the following result, in which half of the conclusion, at least, has intuitive content.

Theorem 5. *A closed subset G of $\Gamma(p)$ is compact iff G is equicontinuous and, for each $x \in X$, the set*

$$G(x) = \{ \sigma(x) \mid \sigma \in G \}$$

is compact in E_x. ◊

4. THE DUAL OF $\Gamma_0(p)$ AND OPERATORS ON $\Gamma_0(p)$.

Unless stated to the contrary, X will throughout this section be supposed additionally to satisfy the strong lifting hypothesis of [15], and $p: E \to X$ will denote an object of *Ban*(X). In [15] we proved the following theorem.

Theorem 6. *Given a bounded linear mapping $\varphi : \Gamma_0(p) \to C$, there exists a family of linear functionals*

$$\eta_{\varphi,x} = \eta_x \quad \textit{with} \quad \eta_x \in E_x^* \quad \textit{for all} \quad x \in X,$$

and a bounded positive Radon measure $\mu_\varphi = \mu$ on X such that

1. $||\eta_x|| \leq 1$ *for all* $x \in X$.
2. $\varphi(\sigma) = \int_X \eta_x(\sigma(x))\, d\mu(x)$ *for all* $\sigma \in \Gamma_0(p)$.

If, further, $p : E \to X$ has continuous norm and is separable, then the η_x are μ-almost everywhere uniquely determined. ◊

This theorem was first established in slightly less general form by Gierz and Keimel in [3]. However, our proof is simpler and constitutes an application of the embedding F : φ can be regarded as a functional on the subspace $F(\Gamma_0(p))$ of $C_0(S(E))$, and can therefore be extended by the Hahn-Banach Theorem to all of $C_0(S(E))$ to give correspondingly a bounded Radon measure m on S(E). The result then follows by disintegrating m relative to $\mu = q(m)$, where q is the projection of S(E).

Our aim here is to use Theorem 6 to describe the dual of $\Gamma_0(p)$ and to obtain an integral representation of operators on $\Gamma_0(p)$. Before doing this we extend slightly Proposition 2 of [15]. This result demonstrates the complete equivalence of the problem of disintegrating measures with that of representing elements of $\Gamma_0(p)^*$ as in Theorem 6.

Proposition 7. *Let X be any locally compact Hausdorff space. Then the following two statements are equivalent :*

1. For all objects $p : E \to X$ of Ban (X), every functional φ on $\Gamma_0(p)$ has an integral representation as in Theorem 6.

2. For all objects $q : Y \to X$ of Spp (X) and all bounded positive Radon measures m on Y, m has a disintegration

$$m = \int_X \lambda_x\, d\mu(x), \qquad \textit{where} \quad \mu = q(m)$$

and for μ almost all x, $||\lambda_x|| = 1$ and the support of λ_x is contained in $q^{-1}(x)$. ◊

Further developments along these lines in the direction of disintegration theory have been made by Gierz in [4].

With the notation of Theorem 6 we write

$$\varphi = \int_X \eta_x \, d\mu(x)$$

if φ has the representation given there, and we denote by $<\sigma, \eta>$ the function defined by

$$<\sigma, \eta>(x) = \eta_x(\sigma(x)).$$

Now let Π' denote the set of all pairs (μ, η) such that $<\sigma, \eta>$ is μ-integrable for all $\sigma \in \Gamma_0(p)$, where μ is a bounded positive Radon measure on X and $\eta : X \to E^*$ is a bounded selection of E^* with

$$\|\eta\| = \sup_{x \in X} \|\eta_x\| \leq 1.$$

We identify (μ', η') and (μ, η) if

$$\int <\sigma, \eta> d\mu = \int <\sigma, \eta'> d\mu'$$

for all $\sigma \in \Gamma_0(p)$; and let $[\mu, \eta]$ denote the equivalence class of (μ, η) and Π denote the set of all such equivalence classes. Theorem 6 shows that there is a bijective mapping $\Lambda : \Pi \to \Gamma_0(p)^*$ defined by

$$\Lambda([\mu, \eta]) = \int_X \eta_x \, d\mu(x).$$

By means of Λ, $\Gamma_0(p)^*$ can be identified with Π, so that Λ carries the vector space structure and norm of $\Gamma_0(p)^*$ onto Π. When this is done, the vector space operations on Π can, in fact, be expressed quite naturally in terms of equivalence classes, as follows. Firstly, for scalar multiplication we have

$$\alpha[\mu, \eta] = [\mu, \alpha\eta]$$

for each scalar α. And secondly, for addition, if E and E' belong to Π, then by [15], Proposition 3, there exist representatives (μ, η) and (μ', η') of E and E' such that

$$E + E' = [\mu, \eta] + [\mu', \eta'] = [\mu + \mu', \eta'']$$

for some η''.

Having thus described the dual of $\Gamma_0(p)$, we proceed now to the main result of this section, namely an integral representation of operators defined on $\Gamma_0(p)$. This is a consequence of a theorem of Bartle, see [1], VI.7.1, and [16], 18.8.

Theorem 7. *Suppose X has the strong lifting property, Z is a locally compact Hausdorff space and $p : E \to X$ is an object of Ban(X). If $\Phi : \Gamma_0(p) \to C_0(Z)$ is a bounded linear operator, then there is a bounded weak* continuous map :*

$$\lambda : Z \to \Gamma_0(p)^*, \qquad \lambda(z) = [\mu^z, \eta^z],$$

such that

1. $$(\Phi\sigma)(z) = \int_X \langle \sigma , \eta^z \rangle \, d\mu^z$$
for all $\sigma \in \Gamma_0(p)$ *and* $z \in Z$.

2. $$\|\Phi\| = \sup_{z \in Z} \|\lambda(z)\|$$

3. The function $\lambda(.)$ *vanishes at infinity in the weak* sense, that is for each* $\sigma \in \Gamma_0(p)$ *the scalar function* $\lambda(.)(\sigma)$ *vanishes at infinity.*

Conversely, if $\lambda: Z \to \Gamma_0(p)^*$ *is a bounded weak* continuous map satisfying 3, then the expression 1 determines a bounded linear operator* $\Phi : \Gamma_0(p) \to C_0(Z)$ *whose norm is given by 2.*

Furthermore, Φ *is weakly compact iff* λ *is weakly continuous, and is compact iff* λ *is strongly continuous.* ◊

Remark. In the case of an operator $\Phi : \Gamma_0(p) \to \Gamma_0(p')$, Φ can be composed with the embedding F of Theorem 4 to obtain

$$F\Phi : \Gamma_0(p) \to C_0(S(E'))$$

to which Theorem 7 can be applied. In this sense we obtain a representation of operators between spaces of sections.

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