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**STRONG INFINITESIMAL LINEARITY, WITH APPLICATIONS TO
STRONG DIFFERENCE AND AFFINE CONNECTIONS**

by A. KOCK and R. LAVENDHOMME

Résumé. Nous présentons ici diverses notions et constructions appartenant à la théorie axiomatique de la Géométrie Différentielle Synthétique ; nous montrons qu'elles fournissent des démonstrations simples et complètement indépendantes des coordonnées de certaines égalités pour les connexions affines, et qu'elles permettent de trouver un lien manquant pour relier ces connexions aux "sprays".

1. The strong notion of infinitesimal linearity.

We work over a field k of characteristic 0, for simplicity. Recall (from [4] Appendix A, say) that if R is a commutative k -algebra object in a category E with finite inverse limits, then there exists a functor

$$\text{Spec}_R : (\text{FP } k)^{\text{op}} \rightarrow E$$

(where $\text{FP } k$ is the category of finitely presented commutative k -algebras), which preserves finite inverse limits and takes $k[X]$ to R ; these two properties determine the functor up to an isomorphism. Since we shall keep R fixed, we write Spec for Spec_R .

Let $W \subset \text{FP } k$ be the full subcategory consisting of Weil algebras over k (cf. [4] I § 16, say). By an inverse limit of Weil Algebras, we understand an inverse limit diagram in $\text{FP } k$, consisting of Weil algebras ; or, equivalently, an inverse limit diagram in W , which is preserved by the inclusion $W \subset \text{FP } k$ (or equivalently, is preserved by the forgetful functor from W to Sets , or to $k\text{-Vect}$, the category of vector spaces over k).

The contravariant functor Spec takes finite colimit diagrams to limit diagrams, but not limits to colimits. However, it will be convenient to make the following

Definition 1.1. A diagram in E that appears as Spec ($= \text{Spec}_R$) applied to a finite inverse limit diagram of Weil algebras is called a *quasi-co-limit* in E (relative to R).

The motivation for this definition is that if E is Cartesian closed (which we henceforth assume) and if E, R satisfies the (Kock-Lawvere)

Axiom 1^W (cf. [4] I § 16), then R "perceives any quasi-colimit diagram to be an actual colimit diagram" ; more precisely

Proposition 1.2. *Suppose R satisfies Axiom 1^W. Then the contravariant functor $R^{(-)} : E \rightarrow E$ takes quasi-colimit diagrams into limit diagrams. And conversely, if a diagram of objects $\text{Spec}(W)$ ($W \in W$) is taken into a limit diagram by $R^{(-)}$, then the diagram is a quasi-colimit, provided, $\Gamma(R) = k$.*

(Here $\Gamma : E \rightarrow \text{Sets}$ denotes the global sections functor $\text{hom}_E(1, -)$.)

Proof. The composite (covariant) functor

$$W \subset \text{FP}k \xrightarrow{\text{Spec}} E \xrightarrow{R^{(-)}} E,$$

is, by Axiom 1^W, isomorphic to the functor $R_{\emptyset} -$; this is the functor which takes k^n , with a k -algebra structure with structure constants $\gamma_{\Sigma}^{i,j} \in k$, into $R^n \in E$, with k -algebra structure given by the same structure constants. This functor is easily seen to preserve those finite limits which are preserved by $W \subset \text{FP}k$; in fact, $R_{\emptyset} -$ can be defined as an *additive* functor from $k\text{-vect}$ (= finite dimensional vector spaces over k) to the category of R -modules in E , and it is exact since short sequences of vector spaces are split exact and thus remain exact upon application of any additive functor.

To prove the converse, consider the composite functor

$$W \xrightarrow{\text{Spec}} E \xrightarrow{\Gamma} \text{Sets} .$$

We have, for $W \in W$, by Axiom 1^W and definition of $R_{\emptyset} -$, that

$$(1.1) \quad \Gamma(R^{\text{Spec}(W)}) = \Gamma(R_{\emptyset}W) = W$$

(using $\Gamma(R) = k$). Now let D be a diagram in W such that $R^{\text{Spec} D}$ is a limit diagram. Then, since Γ preserves limits, $\Gamma(R^{\text{Spec} D})$ is a limit in Sets , and hence so is D itself, by (1.1). So D is a limit diagram of Weil algebras, thus $\text{Spec}(D)$ is a quasi-colimit. \diamond

Let us remark we do not have a converse Proposition : from "R⁽⁻⁾ converts quasi-colimits into limits", we cannot conclude "R satisfies Axiom 1^W". In fact, any reduced ring (= nilpotent free) in Sets will have the former property.

The following general notion of infinitesimal linearity is essentially the one first proposed by Bergeron [1], although we have simplified his formulation. It is a strengthening of the notion "infinitesimal linearity" first considered in [13] and [6]. It also implies the Wraith condition ("condition (8)" in [13], called Property W in [4], and thus also the Mi-

colinearity of [11]. And it implies the Symmetric Function Property (cf. [4]), the Iterated Tangent Bundle Property (cf. [2]), and many other properties, one of which will be essential in § 2.

Definition 1.3. An object $M \in E$ is called *infinitesimally linear* (in the strong sense) if the contravariant functor $M^{(-)} : E \rightarrow E$ takes quasi-colimit diagrams into limit diagrams. ("M perceives quasi-colimits as actual colimits".)

Thus, by Proposition 1.2, Axiom 1^W implies that R is infinitesimally linear in the strong sense. Also, the class L of objects in E which are infinitesimally linear in the strong sense is closed under all inverse limits, and if $X \in E$ and $M \in L$, then $M^X \in L$. In particular, any "affine scheme" $\text{Spec}(A)$ (with $A \in \text{FP}k$) will be in L . In any well-adapted model for synthetic differential geometry, any manifold will be in L . For R , this follows from Axiom 1^W and Proposition 1.2, and for other manifolds, the argument is then the same as the one given in [3], Theorem 4. Finally, the proof of [6] of "étale descent of infinitesimal linearity" goes through for the strong notion.

In general, L will not be closed under formation of subobjects in E . Even if E is a Grothendieck topos, we do not know whether $L \subset E$ is a reflexive subcategory.

2. The notion of strong difference.

As a first application, we shall introduce the notion of strong difference, which in classical differential geometry was considered by I. Kolar [9, 10], and E. White [14] (Theorem 2.7). All our considerations are in a topos E with a commutative k -algebra object R , which satisfies Axiom 1^W . We freely make use of the "set theoretic" way of speaking about E (see e.g. [4], Chapter 2, for a justification), and also of some of the standard notation of SDG. In particular, maps $D \rightarrow M$ are called *tangents*, and maps $D \times D \rightarrow M$ *2-tangents*, cf. [8]. (They correspond to the 2-sectors of [14].)

Let $(D \times D) \vee D \subset R^3$ denote the subobject

$$\{ (d_1, d_2, e) \mid d_i^2 = e^2 = d_i \cdot e = 0 \}$$

Then we have a commutative diagram

$$(2.1) \quad \begin{array}{ccc} D(2) & \longrightarrow & D \times D \\ \downarrow & & \downarrow \psi \\ D \times D & \xrightarrow{\varphi} & (D \times D) \vee D \end{array}$$

where $D(2)$ and DxD are the usual objects with this name (cf. [4] I § 6, say), and the un-named maps the inclusion ; φ and ψ are given by:

$$\varphi(d_1, d_2) = (d_1, d_2, 0), \quad \psi(d_1, d_2) = (d_1, d_2, d_1.d_2).$$

The diagram (2.1) is actually a quasi-pushout : it suffices by the second clause in Proposition 1.2 to test with R . By Axiom 1^W , maps

$$f : (DxD)VD \rightarrow R$$

are of the form

$$f(d_1, d_2, e) = a + b_1.d_1 + b_2.d_2 + b.e + c.d_1.d_2 .$$

Using also Axiom 1^W to get standard form for maps

$$DxD \rightarrow R \quad \text{and} \quad D(2) \rightarrow R,$$

the result follows trivially.

Consider also the map

$$(2.2) \quad D \xrightarrow{\varepsilon} (DxD)VD \quad \text{given by} \quad \varepsilon(d) = (0, 0, d).$$

We consider in what follows an object M which is infinitesimally linear in the strong sense. It therefore perceives (2.1) as a pushout. Thus if $\tau_1, \tau_2 : DxD \rightarrow M$ are two 2-tangents which coincide on $D(2) \subset DxD$, we get a unique

$$I : (DxD)VD \rightarrow M \quad \text{with} \quad I \circ \varphi = \tau_1 \quad \text{and} \quad I \circ \psi = \tau_2 .$$

Definition 2.1. The *strong difference* of τ_1 and τ_2 denoted $\tau_2 \dot{-} \tau_1$ is the tangent vector $I \circ \varepsilon : D \rightarrow M$.

Just as easily as for (2.1), one sees that the following diagram (2.2) is a quasi-pushout

$$(2.3) \quad \begin{array}{ccc} 1 & \xrightarrow{\quad} & D \\ \downarrow & & \downarrow \varepsilon \\ DxD & \xrightarrow{\quad \varphi \quad} & (DxD)VD \end{array}$$

(with φ and ε as above, and the un-named maps being given by 0). Thus, if $\tau : DxD \rightarrow M$ and $t : D \rightarrow M$ have $\tau(0, 0) = t(0)$, there exists a unique

$$u : (DxD)VD \rightarrow M \quad \text{with} \quad u \circ \varphi = \tau \quad \text{and} \quad u \circ \varepsilon = t.$$

Definition 2.2. The (*translation -*) *sum* of t and τ , denoted $t \dot{+} \tau$, is the 2-tangent $u \circ \psi : DxD \rightarrow M$.

To state succinctly the algebraic structure provided by $\dot{+}$ and $\dot{-}$, we remind the reader that if $(V, +, 0)$ is an abelian group and A is a non-empty (better "inhabited") set, then to give A the structure of a *translation space*, or *affine space*, or *torsor*, over V means to give maps

$$V \times A \xrightarrow{\dot{+}} A, \quad A \times A \xrightarrow{\dot{-}} V$$

satisfying equations (a)-(d) below, for any $\tau_i \in A$, $t_j \in V$. Any fixed $\tau_0 \in A$ gives rise to an identification of V with A , via $t \mapsto t \dot{+} \tau_0$, in such a way that $\dot{+}$ becomes $+$ and $\dot{-}$ becomes $-$. It follows that when testing a meaningful equation involving $\dot{+}$, $\dot{-}$, $+$ and $-$, it suffices to replace $\dot{+}$ by $+$ and $\dot{-}$ by $-$, and test whether the result is an identity of the theory of abelian groups.

We can now state

Proposition 2.3. *The fibres of the restriction morphisms*

$$(M^{D \times D})_x \rightarrow (M^{D(2)})_x$$

(for $x \in M$ arbitrary) have a natural structure of translation space over $(M^D)_x$.

Proof. It suffices to verify the identities

- (a) $(\tau_2 \dot{-} \tau_1) \dot{+} \tau_1 = \tau_2$,
- (b) $(t \dot{+} \tau) \dot{-} \tau = t$,
- (c) $0 \dot{+} \tau = \tau$,
- (d) $(t_1 + t_2) \dot{+} \tau = t_1 \dot{+} (t_2 \dot{+} \tau)$.

The three first follow easily from the definitions of $\dot{-}$ and $\dot{+}$. The fourth is proved using the easily verified fact that the following diagram is a quasi-pushout

$$(2.4) \quad \begin{array}{ccc} D \times D & \xrightarrow{\varphi} & (D \times D)VD \\ \psi \downarrow & & \downarrow \zeta_2 \\ (D \times D)VD & \xrightarrow{\zeta_1} & (D \times D)VD(2), \end{array}$$

where

$$(D \times D)VD(2) = \{ \{ (d_1, d_2, e_1, e_2) \mid d_1^2 = e_1^2 = d_1 \cdot e_1 = e_1 \cdot e_2 = 0 \} \},$$

and

$$\zeta_1(d_1, d_2, e) = (d_1, d_2, e, 0), \quad \zeta_2(d_1, d_2, e) = (d_1, d_2, d_1 \cdot d_2, e).$$

One then considers $u : (D \times D)VD \rightarrow M$ characterized by

$$u(0, 0, e) = t_2(e) \quad \text{and} \quad u(d_1, d_2, 0) = \tau(d_1, d_2),$$

and next $v : (D \times D)VD \rightarrow M$ characterized by

$$v(0, 0, e) = t_1(e) \quad \text{and} \quad v(d_1, d_2, 0) = u(d_1, d_2, d_1 \cdot d_2),$$

and finally $w : D \times D \times D(2) \rightarrow M$ characterized by

$$w \circ \zeta_1 = u \quad \text{and} \quad w \circ \zeta_2 = v.$$

One sees easily that

$$(t_1 \dot{+} (t_2 \dot{+} \tau))(d_1, d_2) = v(d_1, d_2, d_1 \cdot d_2).$$

So it just remains to be verified that one also has

$$((t_1 + t_2) \dot{+} \tau)(d_1, d_2) = v(d_1, d_2, d_1 \cdot d_2).$$

To do this, let us put

$$z(d_1, d_2, e) = w(d_1, d_2, e, e).$$

We have

$$z(d_1, d_2, 0) = w(d_1, d_2, 0, 0) = u(d_1, d_2, 0) = \tau(d_1, d_2);$$

and, because

$$w(0, 0, e, 0) = u(0, 0, e) = t_2(e)$$

and

$$w(0, 0, 0, e) = v(0, 0, e) = t_1(e),$$

we have

$$w(0, 0, e, e) = (t_1 + t_2)(e).$$

Thus

$$\begin{aligned} ((t_1 + t_2) \dot{+} \tau)(d_1, d_2) &= z(\psi(d_1, d_2)) = z(d_1, d_2, d_1 \cdot d_2) \\ &= w(d_1, d_2, d_1 \cdot d_2, d_1 \cdot d_2) = v(d_1, d_2, d_1 \cdot d_2), \end{aligned}$$

which is the desired result. \diamond

We note that we have on $M^{D \times D}$ two laws for multiplying by a scalar $\lambda \in R$, defined by

$$(\lambda \cdot_1 \tau)(d_1, d_2) = \tau(\lambda \cdot d_1, d_2) \quad \text{and} \quad (\lambda \cdot_2 \tau)(d_1, d_2) = \tau(d_1, \lambda \cdot d_2).$$

One easily verifies the following proposition :

Proposition 2.4. *If τ_1 and τ_2 are 2-tangents which coincide on $D(2) \subset D \times D$ then*

$$\lambda \cdot (\tau_2 \dot{+} \tau_1) = \lambda \cdot_1 \tau_2 \dot{+} \lambda \cdot_1 \tau_1 = \lambda \cdot_2 \tau_2 \dot{+} \lambda \cdot_2 \tau_1.$$

One also has, for τ a 2-tangent and t a tangent

$$\lambda \cdot_1 (t \dot{+} \tau) = \lambda \cdot t \dot{+} \lambda \cdot_1 \tau, \quad \lambda \cdot_2 (t \dot{+} \tau) = \lambda \cdot t + \lambda \cdot_2 \tau. \quad \diamond$$

Remark. If one is only interested in the "algebra of 2-tangents" on M , it is not necessary to require M to be infinitesimally linear in the strong sense. It suffices in fact that M is infinitesimally linear in the usual sense, and satisfies condition W . Let us just prove this for the case of strong difference. Let τ_1 and τ_2 be 2-tangents which coincide on $D(2) \subset DxD$, and denote their (common) restrictions to the two axes by X and Y . By cartesian adjointness

$$M^{DxD} \simeq (M^D)^D,$$

one may view τ_1 and τ_2 as tangent vectors on M^D at Y . Because M , and thus M^D , is infinitesimally linear in the usual sense, one may consider the difference $\sigma = \tau_2 - \tau_1$ in the R -module $T_Y(M^D)$. This difference $\sigma : D \rightarrow M^D$ takes in fact its values in the space $T_X M \subset M^D$. Now this R -module V is Euclidean (cf. [11], i.e. $V^D \simeq V \times V$ canonically), because M has the property W . Hence there exists a unique tangent vector $t \in T_X M$ such that, for any $d \in D$, $\sigma(d) = Y + d.t$. One then put

$$t := \tau_2 \dot{-} \tau_1.$$

(This way of defining $\dot{-}$ is a direct paraphrasing of Kolař's definition, [9] § 4.)

Kolař [10], and White [14], Definition 2.17, noted how strong difference may be used to express the Lie bracket of two vector fields. We shall state and prove this result in our context. We remind the reader (confer e. g. [4], I § 9) that the Lie bracket $[X, Y]$ of two vector fields X and Y on M is characterized by

$$(2.5) \quad [X, Y]_X(d_1, d_2) = (Y_{\dot{-}d_2} \circ X_{\dot{-}d_1} \circ Y_{d_2} \circ X_{d_1})(x)$$

($x \in M$). We define also, for $x \in M$, a 2-tangent $(Y.X)_X$ at x by

$$(Y.X)_X(d_1, d_2) := Y(X(x, d_1), d_2) = (Y_{d_2} \circ X_{d_1})(x).$$

For convenience, denote the vector field $\dot{-}X$ by \bar{X} , and let $S : DxD \rightarrow DxD$ denote the twist map $(d_1, d_2) \mapsto (d_2, d_1)$.

Proposition 2.5. We have

$$(2.6) \quad [X, Y]_X = (\bar{X}.Y)_X \circ S \dot{-} (Y.\bar{X})_X.$$

Proof. First note that the two terms on the right are 2-tangents with some restriction to $D(2)$ because \bar{X}_{d_1} and Y_{d_2} commute for $(d_1, d_2) \in D(2)$ (generalize Exercise I.9.1 in [4]). Construct $l : (DxD)VD \rightarrow M$ by

$$l(d_1, d_2, e) = (Y_{d_2} \circ [X, Y]_e \circ X_{\dot{-}d_1})(x).$$

We have

$$l(\varphi(d_1, d_2)) = l(d_1, d_2, 0) = (Y.\bar{X})_X(d_1, d_2)$$

and

$$I(\psi(d_1, d_2)) = I(d_1, d_2, d_1 \cdot d_2) = (Y_{d_2} \circ [X, Y]_{d_1, d_2} \circ \bar{X}_{d_1})(x)$$

but since \bar{X}_{d_1} cancels X_{d_1} and Y_{d_2} cancels Y_{-d_2} in (2.5), this equals $(\bar{X} \cdot Y)_x \circ S(d_1, d_2)$, so that

$$((\bar{X} \cdot Y)_x \circ S)(d_1, d_2) = (Y \cdot X)_x(d_1, d_2) = I(0, 0, d_1 \cdot d_2) = [X, Y]_{d_1, d_2}(x). \quad \diamond$$

3. Applications to the theory of affine connections.

Let M , as before, be an object which is infinitesimally linear in the strong sense. As in [7], with the simplification of [5], an (affine) connection on M is a map

$$M^D \times_M M^D \xrightarrow{\nabla} M^D \times D$$

satisfying the following equational conditions (with t_1 and t_2 tangents at the same point of M , thus $(t_1, t_2) \in M^D \times_M M^D$):

$$(1) \quad \nabla(t_1, t_2)(d_1, 0) = t_1(d_1); \quad \nabla(t_1, t_2)(0, d_2) = t_2(d_2),$$

$$(2) \quad \begin{aligned} \nabla(\alpha \cdot t_1, t_1)(d_1, d_2) &= \nabla(t_1, t_2)(\alpha \cdot d_1, d_2), \\ \nabla(t_1, \alpha \cdot t_2)(d_1, d_2) &= \nabla(t_1, t_2)(d_1, \alpha \cdot d_2) \end{aligned}$$

for all $d_i \in D$, $\alpha \in R$. The first condition says that ∇ is a splitting of the evident map

$$K : M^D \times D \rightarrow M^D \times_M M^D$$

which sends a 2-tangent τ to $(\tau(., 0), \tau(0, .))$. The second condition is (in view of [4] Proposition I.10.2, say) a linearity condition.

We now define the notion of covariant derivation :

Definition 3.1. Let X and Y be vector fields on M . The *covariant derivative of Y with respect to X* , denoted $\nabla_X Y$, is the vector field given by

$$(\nabla_X Y)_x = (Y \cdot X)_x \doteq \nabla(X_x, Y_x).$$

This expression for $\nabla_X Y$ in terms of strong difference is without doubt known by Kolar and White, but we could not find a reference. In the generality we work in here, it is easily seen to coincide with the definition in [7]. As a first application, we shall prove the Koszul law, which in [7] could only be proved when $TM \rightarrow M$ was a locally trivial bundle (which for the application to classical geometry would restrict its use to smooth (singularity free) M).

Proposition 3.2 (Koszul's law). Let X and Y be vector fields on M , and $f : M \rightarrow \mathbb{R}$ a function. Then

$$\nabla_X(f \cdot Y) - f \cdot \nabla_X Y = X(f) \cdot Y.$$

Proof. Let us calculate the left hand side at a point $x \in M$. We get by definition that it equals

$$\begin{aligned} & ((fY \cdot X)_x \doteq \nabla(X_x, f(x) \cdot Y_x)) - f(x) \cdot ((Y \cdot X)_x \doteq \nabla(X_x, Y_x)) \\ &= ((fY \cdot X)_x \doteq \nabla(X_x, f(x) \cdot Y_x)) - (f(x) \cdot (Y \cdot X)_x \doteq \nabla(X_x, f(x) \cdot Y_x)) \end{aligned}$$

by Proposition 2.4. From the general principle for calculating in translation spaces (as stated before Proposition 2.3), we may cancel the second and fourth term, to get

$$(3.1) \quad (f \cdot Y \cdot X)_x \doteq f(x) \cdot (Y \cdot X)_x,$$

provided this latter expression makes sense. But the two 2-tangents that occur here are given by

$$(3.2) \quad (d_1, d_2) \mapsto Y(X(x, d_1), f(X(x, d_1)) \cdot d_2)$$

and

$$(3.3) \quad (d_1, d_2) \mapsto Y(X(x, d_1), f(x) \cdot d_2)$$

respectively, which clearly agree when $d_1 = 0$ or $d_2 = 0$. Thus the strong difference (3.1) does make sense. It is calculated by considering the map $l : (DxD)VD \rightarrow M$ given by

$$l(d_1, d_2, e) := Y(X(x, d_1), X(f)(x) \cdot e + f(x) \cdot d_2),$$

since putting $e = 0$ here yields (3.3), and putting $e = d_1 \cdot d_2$ yields, by definition of the directional derivative $X(f)$,

$$Y(X(x, d_1), (X(f)(x) \cdot d_1 + f(x)) \cdot d_2) = Y(X(x, d_1), f(X(x, d_1)) \cdot d_2)$$

which is (3.2). Thus, the difference (3.1) itself is $l(0, 0, -)$ which is clearly $X(f) \cdot Y(x)$, as desired. \diamond

As a second application we shall define the connection-map associated to the connection ∇ . If $\tau : DxD \rightarrow M$ is a 2-tangent at $x \in M$, let t_1 and t_2 be the two tangent vectors at x obtained by restriction of τ along the two axes. The strong difference between τ and $\nabla(t_1, t_2)$ is another tangent at x which we denote $C(\tau)$. Thus we get a map

$$C : M^{D \times D} \rightarrow M^D, \quad C(\tau) = \tau \doteq \nabla(t_1, t_2).$$

For instance, the definition of $\nabla_X Y$ can be written $\nabla_X Y = C(Y \cdot X)$. One defines then the *torsion* associated to ∇ by

$$T(X, Y) = C(Y.X) - C(Y.X \circ S)$$

(where S is the twist map, as above), or substituting also the definition of C in terms of ∇ :

$$(3.4) \quad T(X, Y) = (Y.X \nabla(X, Y)) - (Y.X \circ S \nabla(Y, X)).$$

The proof of the following classical identity will now be almost pure "translation-space" calculations :

Proposition 3.3. *Let ∇ be a connection with torsion T . Then for any pair of vector fields X, Y we have*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Proof. The left hand side is spelled out in (3.4). The right hand side is, by Definition 3.1:

$$(Y.X \nabla(X, Y)) - (X.Y \nabla(Y, X)) - [X, Y].$$

The first parenthesis here also occurs in (3.4), so it suffices to prove

$$-((Y.X) \circ S \nabla(Y, X)) + (X.Y \nabla(Y, X)) = -[X, Y].$$

The left hand side equals $-((Y.X) \circ S \nabla(X, Y))$ because this makes sense, so that we may calculate as in an abelian group. This in turn equals $-[X, Y]$, by Proposition 2.5, which is $-[X, Y]$. \diamond

4. The ray property, and sprays.

We finish by another instance of strong infinitesimal linearity. In [5] (inspired by [2]), the first author proved a synthetic form of the Ambrose-Palais-Singer Theorem, utilizing a "ray property of order 2 relative to R^n ". We recall the definition :

The object M satisfies the *ray property* of order r , relative to R^n , if M "perceives the following diagram as a coequalizer" :

$$(4.1) \quad D_r \times R \times R^n \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} D_r \times R^n \xrightarrow{w} D_r(n),$$

where

$$u(\delta, \lambda, \underline{x}) = (\delta \cdot \lambda, \underline{x}), \quad v(\delta, \lambda, \underline{x}) = (\delta, \lambda \cdot \underline{x}), \quad \text{and} \quad w(\delta, \underline{x}) = \delta \cdot \underline{x}.$$

(Precisely : The contravariant functor $M^{(-)} : E \rightarrow E$ converts (4.1) into an equalizer.)

Proposition 4.1. *If M is infinitesimally linear in the strong sense, M satisfies the ray property of order r relative to R^n (for any r, n).*

Proof. We do the case $r = 2$ only. We first note that the diagram

$$(4.2) \quad D_2 \times D_2 \times D_2(n) \begin{array}{c} \xrightarrow{u'} \\ \xrightarrow{v'} \end{array} D_2 \times D_2(n) \xrightarrow{w'} D_2(n)$$

is a quasi-coequalizer (with u', v', w' defined with the same formulas as u, v and w above). In fact, consider

$$(4.3) \quad k[\underline{X}, T, S] \begin{array}{c} \xleftarrow{u''} \\ \xleftarrow{v''} \end{array} k[\underline{X}, T] \xleftarrow{w''} k[\underline{X}]$$

where $\underline{X} = (X_1, \dots, X_n)$ and where the k -algebra homomorphisms are given by formulas similar to u, v and w :

$$w''(X_i) = X_i \cdot T, \quad v''(X_i) = X_i \cdot T, \quad v''(T) = S, \\ u''(X_i) = X_i, \quad u''(T) = T \cdot S.$$

This is an equalizer diagram in the category of k -algebras, as can be seen by a consideration of total degrees. Dividing out by relevant ideals in (4.3) gives a diagram of Weil algebras whose Spec is (4.2), and this diagram of Weil algebras is an equalizer (this can be seen by obvious canonical vector space splitting of the dividing-out maps). Consequently (4.2) is a quasi-coequalizer. Thus M perceives it as a coequalizer. Consider now the diagram (straight arrows)

$$\begin{array}{ccccc} D_2 \times D_2 \times D_2(n) & \begin{array}{c} \xrightarrow{u'} \\ \xrightarrow{v'} \end{array} & D_2 \times D_2(n) & \xrightarrow{w'} & D_2(n) \\ \downarrow & & \downarrow i & \nearrow w & \downarrow f' \\ D_2 \times R \times R & \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} & D_2 \times R & \xrightarrow{f} & M \end{array}$$

where $f \circ u = f \circ v$. Since M perceives the upper row as a coequalizer, we get a unique f' (dotted arrow) with $f' \circ w' = f \circ i$. It remains to be proved that $f = f' \circ w$. This results easily from the fact that M perceives the multiplication $D_2 \times D_2 \rightarrow D_2$ to be surjective (specialize the quasi-coequalizer (4.2) to $n = 1$). So we test $f = f' \circ w$ on an element of form $(\delta_1, \delta_2, \underline{x})$ ($\delta_i \in D_2, \underline{x} \in R$). We have

$$f(\delta_1, \delta_2, \underline{x}) = f(u(\delta_1, \delta_2, \underline{x})) = f(v(\delta_1, \delta_2, \underline{x})) = f(\delta_1, \delta_2, \underline{x}) \\ = f(i(\delta_1, \delta_2, \underline{x})) = f'(w'(\delta_1, \delta_2, \underline{x})) = f'(\delta_1, \delta_2, \underline{x}) = f'(w(\delta_1, \delta_2, \underline{x})).$$

proving the Proposition. ◊

We present now some considerations from [5] on sprays. According to Smale (cf. [12], Definition 6), a *spray* on M may be defined as a map

$$TM = M^D \xrightarrow{\sigma} M^{D_2} = T_2M,$$

which splits the restriction map arising from $D \rightarrow D_2$, and commutes with fibrewise multiplication by scalars. Thus, for $t \in T_x M$, $\delta \in D_2$, we have $\sigma(t)(\delta) \in M$ defined, and

$$(4.4) \quad \sigma(t)(d) = t(d) \quad \forall d \in D,$$

$$(4.5) \quad \sigma(\lambda.t)(\delta) = \sigma(t)(\lambda.\delta) \quad \forall \lambda \in R, \quad \delta \in D_2.$$

Assume that M is infinitesimally linear in the strong sense, and that $T_x M \simeq R^n$ for each $x \in M$. By Proposition 4.1, M has the ray property with respect to R^n , hence with respect to $T_x M$ (the subobject $D_2(T_x M)$ may be defined as the image of $D_2(n)$ under one, or any, isomorphism $T_x M \simeq R^n$). Using (4.1) with $r = 2$ and with $D_2(T_x M)$ instead of $D_r(n)$ the map $\hat{\sigma} : D_2 \times T_x M \rightarrow M$ given by

$$\hat{\sigma}(\delta, t) := \sigma(t)(\delta)$$

coequalizes u and v by (4.5), so we get a map

$$e = e_x : D_2(T_x M) \rightarrow M \quad \text{with} \quad e \circ w = \hat{\sigma};$$

and which satisfies

$$(4.6) \quad e(d.t) = t(d) \quad \forall d \in D, \quad t \in T_x M,$$

by (4.4). The collection of the e_x 's define a map e from a certain subset $D_2(TM)$ to M , and this map is the 2-jet of the exponential map of the spray σ .

A symmetric (= torsion free) connection ∇ may now be associated to σ by putting (for t_i tangents at x)

$$(t_1, t_2)(d_1, d_2) := e_x(d_1.t_1 + d_2.t_2),$$

noting that $d_i \in D$ implies that the vector $d_1.t_1 + d_2.t_2$ is actually in $D_2(T_x M)$.

In the first author's proof of the Ambrose-Palais-Singer Theorem in [5], the ray property (of order 2) for M (with respect to $T_x M$) was the only property which was not (by then) subsumed under strong infinitesimal linearity. By the considerations above (based on Proposition 4.1), we can now sharpen the formulation of the Ambrose-Palais-Singer Theorem of loc. cit. into

Theorem 4.2. *Let M be infinitesimally linear in the strong sense, with $T_x M \simeq R^n \quad \forall x \in M$. Then there are natural bijective correspondences between the following three kind of data :*

- (i) sprays $\sigma : M^D \rightarrow M^{D_2}$,
- (ii) symmetric connections $\nabla : M^D \times_M M^D \rightarrow M^{D \times D}$,
- (iii) "partial exponential maps", i.e. maps $e : D_2(M^D) \rightarrow M$ which sat-

isfy (4.6).

(The remaining correspondence, from (ii) to (i), is simply that

$$\sigma(t)(d_1 + d_2) := \nabla(t, t)(d_1, d_2),$$

(cf. [5]).

In particular, applying (i) \leftrightarrow (ii) in a fully well-adapted model for synthetic differential geometry yields the classical Ambrose-Palais-Singer Theorem for ordinary smooth manifolds. The present proof thus completes the proofs given in [5] and in the preliminary version of [2].

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