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ON THE GROTHENDIECK TOPOLOGIES IN THE TOPOSES OF PRESHEAVES

by V. PATRYSHEV

Résumé. On discute les conditions sous lesquelles les topologies de Grothendieck dans un topos de préfaisceaux sont déterminées par les ensembles d'objets de la catégorie de base. Le livre de P. T. Johnstone "Topos Theory" contient toutes les définitions et résultats nécessaires.

0. Introduction.

When our program to build the Grothendeicek topologies over a category was ready, we first obtained all the topologies for the categories as simple as **2** and **4**. They were 4 and 16. It became at once clear that for any $\ulcorner n \urcorner$ the result should be 2^n. And for the finite trees it is still valid. The question now arises : what are the conditions for such a relation to hold?

1. The basic relation.

Let us consider a bounded geometric morphism f between two toposes, $f : \mathbf{E} \to \mathbf{F}$. Each Grothendieck topology j in $\mathbf{E}$ ($j \in GT(\mathbf{E})$) gives a topology $f_* j$ in $\mathbf{F}$: $sh_{f_* j}(\mathbf{F})$ is the image of $sh_j(\mathbf{E}) \to \mathbf{E} \to \mathbf{F}$.

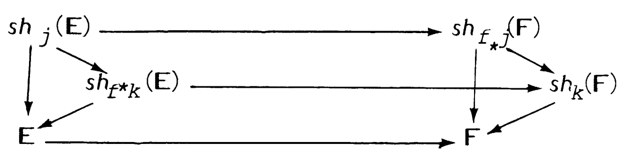
Conversely, any topology $k \in GT(\mathbf{F})$ gives a topology f^*k in $\mathbf{E}$, $sh_{f^*k}(\mathbf{E})$ being the pullback of $sh_k(\mathbf{E})$ against k.

These f_* and f^* are lattice morphisms between $GT(\mathbf{E})$ and $GT(\mathbf{F})$.

1.1. Proposition. *The morphisms f_* and f^* constructed above are adjoint,*

$$f^* \dashv f_* : GT(\mathbf{E}) \to GT(\mathbf{F}).$$

This is easily seen from the following diagram :



◊

Let **C** be a category in **E**. For the geometric morphism

$$c : E/C_0 \to E^{C^{op}},$$

take

$$tops\ (D_0 \rightarrowtail C_0) = c_*(j^o_{D_0}) \in GT(E^{C^{op}}),$$

where $j^o_{D_0}$ is the open topology in E/C_0 corresponding to D_0, and it gives the lattice morphism

$$tops : Sub(C_0)^{op} \to GT(E^{C^{op}}).$$

On the other hand, for a topology $j \in GT(E^{C^{op}})$, $c^*(j)$ is the smallest topology k in E/C_0 such that $dns_0 : 1_{C_0} \rightarrowtail J_0$ is k-dense. Take $spot(j)$ to be the largest subobject of C_0, $D_0 \rightarrowtail C_0$, for which

$$\begin{array}{ccc} D_0 & \rightarrowtail & J_0 \\ \downarrow & & \downarrow p_0 \\ D_0 & \rightarrowtail & C_0 \end{array}$$

is a pullback. Or, in other words,

$$spot(j) = \{c \mid (\omega \in J_0 \wedge p_0\omega = c) \Rightarrow \omega = true(c)\}.$$

The topology $tops\ (D_0)$ may also be expressed as

$$tops(D_0)_0 = \{\omega \mid (p_0\omega \in D_0) \Rightarrow (\omega = true(p_0))\}.$$

2. When $(Sub\ C_0)^{op}$ is a reflexive sublattice of $GT(E^{C^{op}})$.

For a subobject $D_0 \rightarrowtail C_0$ the construction

$$D_0^k = \{c \mid \exists < f, g > ((m(f, g) = e(c) \wedge (d_1 f \in D_0))\} \rightarrowtail C_0$$

is known as its *Karoubian closure*.

2.1. Proposition. *For any topology in* $E^{C^{op}}$ *its spot is Karoubian-closed.*

Proof. Let

$$m(f, g) = e(c), \quad \text{and} \quad d_1 f = d_0 g \in spot(j),$$

and let ω of type J_0 be such that $p_0(\omega) = c$, then

$$p_0(g(\omega)) = d_0 g \in spot(j), \quad \text{and} \quad g(\omega) = true(p_0 g(\omega)).$$

So

$$\omega = fg(\omega) = f(true(p_0 g(\omega))) = true(p_0 fg(\omega)) = true(c),$$

that is $c \in spot(c)$. ◊

2.2. Proposition. *The lattice of Karoubian-closed subobjects of* C_0 *is a* *reflexive sublattice of* $GT(E^{C^{op}})$.

Proof. It is enough to show that for any $D_0 \rightarrowtail C_0$, $spot(tops(D_0))$ is its Karoubian closure. Let $c \in spot(tops(D_0))$. Then for $\omega \in tops(D_0)$,

$$p_0(\omega) = c \quad \text{iff} \quad \omega = true(c).$$

Take the free presheaf $R(c)$. $Z \rightarrowtail R(c)$ is $tops(D_0)$-dense iff Z_0 equals to $R(c)_0$ over D_0. But then the subpresheaf

$$Z = \{ f \mid d_1 f = c \wedge \exists <g, h> (f = m(h, g) \wedge d_0 h \in D_0)\}$$

(which is the image of

$$C_1 \underset{C_0}{\times} D_0 \underset{C_0}{\times} C_1 \underset{C_0}{\times} U \rightarrowtail C_1 \underset{C_0}{\times} C_1 \underset{C_0}{\times} U \xrightarrow{m \times U} C_1 \underset{C_0}{\times} U)$$

is dense in $R(c)$. It follows that for ω classifying Z in $R(c)$, $\omega(c) = true(c)$, and $Z = R(c)$ at c, $e(c) \in Z_0$, and we have

$$E \models (\exists <g, h> (e(c) = m(h, g) \wedge d_0 h \in D_0)). \qquad \Diamond$$

2.3. Definition. A category C in E is *pseudoantisymmetric* if

$$E \models ((d_0 f = d_1 g \wedge d_1 f = d_0 g) \Rightarrow (d_0 f = d_1 f \wedge h = f^{1})),$$

or, in the diagram

$$\begin{array}{ccc} Aut(C) & \rightarrowtail & C \\ \downarrow & & \downarrow (d_0, d_1) \\ C & \xrightarrow{(d_0, d_1)} & C_0 \times C_0 \end{array}$$

is a pullback.

2.4. Proposition. *For a pseudoantisymmetric* C *each* $D_0 \rightarrowtail C_0$ *is Karoubian-closed, and the lattice* $(Sub\ C_0)^{op}$ *is a reflective sublattice in* $GT(E^{C^{op}})$. $\Diamond$

3. When the topologies in $E^{C^{op}}$ are determined by subobjects of C_0, or *spot* becomes iso.

For this property to hold, we obviously need still another restriction to be put upon the category **C**. Note first that any finite segment of N^{op} possesses the property, while the whole N^{op} does not. The obstacle is its infiniteness : take the topology $\neg\neg$ in E^N - its *spot* is void. So we need some sort of boundedness, to be more precise, left-boundedness. And the topos E/C_0 should be Boolean.

3.1. Definition. A pseudoantisymmetric category $\mathbf{C}$ in a topos $\mathbf{E}$ is *fairly-ordered* if for any $D_0 \rightarrowtail C_0$ there is some $p: W \rightarrowtail D_0$ such that

$$\mathbf{E} \models ((d_0 f \in D_0 \ \wedge \ d_1 f \in W) \Rightarrow d_0 f = d_1 f)$$

and $\pi_2 : W \times D_0 \to D_0$ is epi.

Expressing this in Kripke-Joyal, we have

$$\mathbf{E} \models (\exists p(p \in q) \Rightarrow (\exists p(p \in q \wedge \ \forall p'((p' \in q \ \wedge p' \leqq p) \Rightarrow p' = p))))$$

for q of the type Ω^{C_0}, where $p' \leqq q$ means

$$\exists f(d_0 f = p' \wedge \ d_1 f = p).$$

Any well-ordered poset is fairly-ordered. Finite antisymmetric categories in Ens are also fairly-ordered.

3.2. Proposition. *Let* $\mathbf{C}$ *be a fairly-ordered category in a topos* $\mathbf{E}$*, and let* $\mathbf{E}/C_0$ *be Boolean. Then (tops, spot) is an isomorphism between*

$$(Sub\ C_0)^{op} \quad and \quad GT(\mathbf{E}^{\mathbf{C}^{op}}).$$

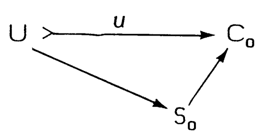
Proof.

3.2.1. In any case we have $tops(spot(j)) \geqq j$, and for our $\mathbf{C}$,

$$spot \circ tops = 1.$$

Let j be a topology in $\mathbf{E}^{\mathbf{C}^{op}}$, and let j' denote $tops(spot(j))$. The corresponding subobjects of $\Omega_{\mathbf{E}^{\mathbf{C}^{op}}}$ (further denoted by Ω) are J and J'. The inclusion $J' \leqq J$ is to be shown.

3.2.2. Let $\rho_0 : R_0 \rightarrowtail C_0$ (corresponding to the full subcategory $\rho : R \rightarrowtail C$) be the largest subobject of C_0 such that $\rho^* J' = \rho^* J$ in $\mathbf{E}^{\mathbf{R}^{op}}$. Take $\sigma_0 : S_0 \rightarrowtail C_0$ (corresponding to $\sigma : S \rightarrowtail \mathbf{C}$) to be a complement of R_0 in C_0. By 3.1 there is



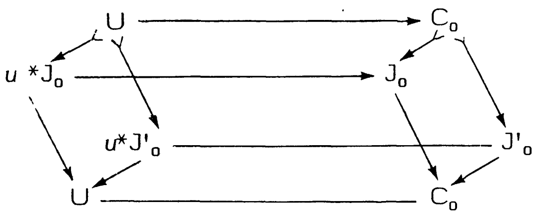
such that

$$((d_0 f \in S_0 \wedge \ d_1 f \in U) \Rightarrow d_0 f = d_1 f)$$

and $\pi_2 : U \times S_0 \to S_0$ is epi. Let us prove now that $u^* J'_0 \leqq u^* J_0$.

3.2.3. Look at the commutative diagram (a) hereafter. Suppose for some $t : T \to U$, $t^* u^* J_0 = 1_T$. Then

$$T \leqq spot(j), \qquad J' \leqq tops(T),$$



and

$$t^*u^*J'_0 = t^*u^*tops(T) = 1_T = t^*u^*J_0 .$$

So $T \leqq R_0 \cap S_0 = 0$.

Now for D_0 = complement of

$$u^*(C_0 \rightarrowtail \xrightarrow{dns_0} J_0)$$

in u^*J_0 the composition

$$D_0 \longrightarrow u^*J_0 \longrightarrow U$$

is epi. The composition

$$D_0 \longrightarrow u^*J_0 \longrightarrow U$$

gives a J-dense subpresheaf $Z \rightarrowtail R(D_0)$ with

$$Z_0 \cap (D_0 \rightarrowtail \xrightarrow{(e,\ 1)} C_1 \underset{C_0}{\times} D_0) = 0.$$

3.2.4. Here we will see that

$$Z_0 \rightarrowtail C_1 \underset{C_0}{\times} D_0 \xrightarrow{d_0 \pi_1} C_0$$

factors through $R_0 \rightarrowtail C_0$. Really, take $T = \sigma_0^*(Z_0)$, then

$$T \longrightarrow S_0 \times Z_0 \longrightarrow S_0 \times D_0 \longrightarrow S_0 \times U \longrightarrow C_0 \times C_0$$

factors through C_1 , it will be denoted $t\colon T \to C_1$. Since $d_0 t \in S_0$ and $d_1 t \in U$, we have $d_0 t = d_1 t$, and $T \to C_0 \times C_0$ factors through the diagonal, making the square

$$\begin{array}{ccc} T & \longrightarrow & D_0 \\ \downarrow & & \downarrow \\ Z_0 & \longrightarrow & C_1 \underset{C_0}{\times} D_0 \end{array}$$

commute. Then

$$T = T \cap (D_0 \longrightarrow C_1 \underset{C_0}{\times} D_0) = 0.$$

3.2.5. For $V_0 = D_0 \times u^*J'_0$ we have (as well as for D_0) a j-dense subpresheaf $Z \rightarrowtail R(V_0)$ (classified by

$$C_1 \times_{C_0} V_0 \longrightarrow D_0 \longrightarrow u^*J_0 \longrightarrow \Omega_0\ ) \qquad ,$$

and Z_0 is also over R_0.

The monomorphism $u^*J'_0 \rightarrowtail \Omega_0$ gives a j'-dense subobject $T \rightarrowtail R(u^*J'_0)$ Its j-closure is the pullback

$$\begin{array}{ccc} \overline{T} & \longrightarrow & J_0 \\ \downarrow & & \downarrow \\ C_1 \times_{C_0} u^*J'_0 & \longrightarrow & \Omega_0 \end{array}$$

In E/R_0 this looks like

$$\begin{array}{ccc} R_0 \times_{C_0} \overline{T}_0 & \longrightarrow & R_0 \times_{C_0} J_0 \\ \downarrow & & \downarrow \\ R_0 \times_{C_0} C_1 \times_{C_0} u^*J'_0 & \longrightarrow & R_0 \times_{C_0} \Omega_0 \end{array}$$

But we have

$$R_0 \times_{C_0} J_0 = R_0 \times_{C_0} J'_0, \qquad \text{so} \qquad R_0 \times_{C_0} \overline{T}_0 = \overline{R_0 \times_{C_0} T_0}$$

is also the j'-closure of T, that is, it equals to $R_0 \times_{C_0} C_1 \times_{C_0} u^*J'_0$. Now, since Z_0 is over R_0,

$$\begin{array}{ccc} Z_0 & \longrightarrow & \overline{T}_0 \\ \downarrow & & \downarrow \\ C_1 \times_{C_0} V & \longrightarrow & C_1 \times_{C_0} u^*J'_0 \end{array}$$

commutes, and we have

$$\begin{array}{ccc} Z & \longrightarrow & \overline{T} \\ \downarrow & & \downarrow \\ R(V) & \longrightarrow & R(u^*J'_0) \end{array}$$

with $Z \rightarrowtail R(V)$ j-dense and $\overline{T}$ closed, thus T is also j-dense in $R(u^*J'_0)$. But therefore

$$u^*J' = u^*J, \qquad \text{then} \qquad U \leqq S_0 \cap R_0 = 0, \qquad \text{and} \qquad S_0 = 0.$$

We have

$$R_0 = C_0, \qquad \text{and} \qquad tops(spot(j)) = j. \qquad \Diamond$$

The question naturally arises : what properties are necessary for a category **C** and topos E/C_0 to give the isomorphism between topologies and subobjects? The properties are just the same as in 3.2. The propositions below prove this. The following lemma will be an example.

3.3. Lemma. *If a category* C *in a topos* E *is a monoid, and (tops, spot) is an isomorphism, then* C *is a group(oid).*

Proof. For $D_0 \rightarrowtail C_0$ the full subcategory $D \rightarrowtail C$ is in fact a subobject of 1 in $E^{C^{op}}$. But then any topology in $E^{C^{op}}$ is open, $E^{C^{op}}$ is Boolean, and is thus a group. ◊

This shows how one could prove in an arbitrary category with *spot* iso all the endomorphisms are invertible.

But first of all we should prove the following

3.4. Lemma. *If for a category* C *in a topos* E, *(tops, spot) is an isomorphism, then the topos* E/C_0 *is Boolean.*

Proof. We have the reflection

$$GT(E/C_0) \longrightarrow GT(E^{C^{op}}),$$

the isomorphism *tops* factors through it, thus any topology in E/C_0 will come from $Sub\, C_0$, and is consequently open. E/C_0 is Boolean. ◊

Next we shall successively obtain the properties of C for

$$spot = tops^{-1} .$$

3.5. Lemma. *Let (tops, spot) be an iso for* C *in a topos* E, *and let* $f : U \to C$ *be such that* $d_1 f = d_0 f$. *Then f is invertible.*

Proof. Let U_{ni} be the complement of $f^*(Iso\ (C))$, D_0 be the image of

$$U_{ni} \xrightarrow{fni} C \xrightarrow{d_0} C_0,$$

and D the corresponding full subcategory in C. It is clear that *fni* is not invertible anywhere in D. The complement of the image of $D_1 \times_{D_0} U_{ni} \to D_1$ (considered as the subpresheaf of $R(D_0)$ in $E^{D^{op}}$) is empty, so this image is $\neg\neg$-dense. Then $\neg\neg$ is not true anywhere in $E^{D^{op}}$,

$$tops(spot(\neg\neg)) = (\Omega \longrightarrow 1 \xrightarrow{true} \Omega)$$

in $E^{D^{op}}$, $sh_{\neg\neg}(E^{D^{op}})$ is degenerate, and $E^{D^{op}}$ is also degenerate. $D_0 = 0$, and f is invertible in C. ◊

3.6. Lemma. *Let (tops, spot) be an iso for* C *in a topos* E. *Let f be an invertible* U*-element of* C_1, $d_0 f = x$, $d_1 f = y$. *Then* $x = y$.

Proof. Consider two topologies :

$$j_x = tops(x), \quad \text{and} \quad j_y = tops(y).$$

The monomorphism $u: A \rightarrowtail B$ in $E^{C^{op}}$ is

j_x-dense iff $x^*A_0 = x^*B_0$,
j_y-dense iff $y^*A_0 = y^*B_0$.

The two conditions are equivalent because $f: x \to y$ is invertible. ◊

The next proposition is merely the corollary of the two lemmas above.

3.7. Proposition. *Let (tops, spot) be an iso for* C *in a topos* E *. Then* C *is pseudoantisymmetric.* ◊

3.8. Proposition. *Let (tops, spot) be an iso for* C *in a topos* E*. Then* C *is fairly-ordered.*

Proof. Let $I \rightarrowtail C_0$ be the *spot* of double negation (*spot* $(\neg\neg)$), and $f: U \to C_1$ be such that $d_1 f$ is in I. Take the presheaf R(I) and its subpresheaf Z - the image of

$$C_1 \underset{C_0}{\times} U \xrightarrow{C_1 \times f} C_1 \underset{C_0}{\times} C_1 \underset{C_0}{\times} I \xrightarrow{m} C_1 \underset{C_0}{\times} I.$$

The complement of the image is empty, then Z is $\neg\neg$-dense, which means equal to R(I) over I, so f is an isomorphism.

Note that the same holds for an arbitrary full subcategory $D \rightarrowtail C$, and the property of the *tops* being iso is hereditary with respect to full subcategories. ◊

3.9. Remark. The "I" used in 3.8 is actually "the set of initial points" of C. To be more precise,

$$I = \Pi_{d_1} [(d_0, d_1)^* \Delta_{C_0}].$$

Really, the full subcategory $I \rightarrowtail C$ is almost discrete - it has only automorphisms. It is the subobject of 1 in $E^{C^{op}}$, and the corresponding *tops* (I) is open. $sh_{j_I^o} \to E^{C^{op}}$ is logical, and $\neg\neg = 1$ in $sh_{j_I^o}$. So $sh_{j_I^o} \to E^{C^{op}}$ factors through $sh_{\neg\neg}(E^{C^{op}})$. ◊

4. Conclusion.

The situation when any topology in $E^{C^{op}}$ is determined by the subobjects of C_0 seems to be most pleasant from the "materialistic" point of view. That is, if we think of C as a generalized time (internal time for a physical body), the isomorphism

$$(Sub\ C_0) \longrightarrow GT(E^{C^{op}})$$

means that any event, which develops in time, can be discerned by marking time, and ghosts cannot appear. The conditions upon our time are simple : it ought to be discrete, irreversible, and bounded at one side. But groups of automorphisms (space symmetry) are permitted at any moment.

Or, to be less philosophical, let C be a computational scheme of some process in a computer. It is non-linear in the case of multiprocessing. And the property of $tops = spot^{-1}$ gives the possibility to spot a deviation of the realization at the points of the scheme - the quality the structured programs are expected to possess.

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