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SHAPE THEORY IN A BICATEGORY

by Renato BETTI

INTRODUCTION

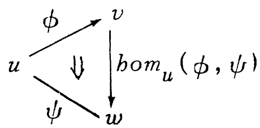
The purpose of this paper is to show that categorical shape theory may be considered in a general bicategory which admits the Kleisli construction of monads. It is a well known fact, explicitly remarked e.g. by Deleanu-Hilton [8], that a similar formulation of the shape category provides alternative proofs of many results. Bourn-Cordier [5] show that these results rely on a «bimodule calculus» and also that the inverse system approach (see for instance Mardešić-Segal [14]) can be dealt with in this general setting.

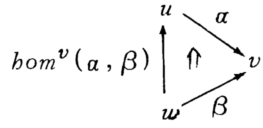
Here we want to stress this latter point of view and derive some consequences: known properties relative to shape invariant functors are obtained from formal properties of adjoint pairs and Kan extensions. It follows that some applications to module theory (Frei-Kleisli [10, 11], Kleisli [12]) become particular cases of properties of general category theory.

Moreover a new approach to «Čech-condition» is introduced. Shape categories are characterized in terms of indexed limits and the Čech condition turns out to be sufficient to present each object as a canonical limit.

1. THE BICATEGORICAL SETTING.

Let us consider a bicategory B such that each hom-category $B(u, v)$ is small-complete and cocomplete, and such that colimits are preserved by composition. Suppose moreover that B is biclosed, i.e., it admits right Kan extensions $hom_u(\phi, \psi)$ and right liftings $hom^v(\alpha, \beta)$ of pairs of 2-cells as in the following diagrams





$$\frac{\theta \circ \phi \dashrightarrow \psi}{\theta \longrightarrow hom_u(\phi, \psi)} \qquad \frac{\alpha \circ \tau \longrightarrow \beta}{\tau \longrightarrow hom^v(\alpha, \beta)}$$

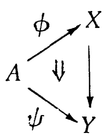
When B is as above, also the bicategory B-*mod* of small categ-egories based on B with bimodules as arrows satisfies the same properties (see Betti [2], Betti, Carboni, Street & Walters [3]).

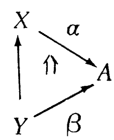
In B-*mod* the right Kan extension $hom_A(\phi, \psi)$ is explicitly given by:

$$hom_A(\phi, \psi)(y, x) = \int^a hom_{ea}(\phi(x, a), \psi(y, a))$$

where the symbol «e» denotes the underlying object for categories based on a bicategory. Analogously:

$$hom^A(\alpha, \beta)(x, y) = \int^a hom^{ea}(\beta(y, a), \alpha(x, a))$$





A particular case considered in the following is when $B = V$ is a one-object symmetric bicategory, i.e. a symmetric, closed category. In this case the bimodule calculus coincides with that explicitly given by Bourn-Cordier [5] and first established by Bénabou [1].

We claim that the general properties of shape categories depend essentially on the following two bicategorical lemmata.

LEMMA 1. *If ϕ admits a left adjoint ϕ', then*

$$hom_u(\phi, \psi) \simeq \psi \circ \phi'.$$

If ϕ admits a right adjoint ϕ'', then there exists the left Kan extension

$$Lan_\phi \psi \simeq \psi \circ \phi''.$$

PROOF. $\phi' \dashv \phi$ gives the following bijective correspondances:

$$\frac{\dfrac{\theta\circ\phi\to\psi}{\theta\circ\phi\circ\phi'\longrightarrow\psi\circ\phi'}}{\theta\longrightarrow\psi\circ\phi'}\ ,$$

$\phi \dashv \phi''$ gives

$$\frac{\dfrac{\psi\longrightarrow\theta\circ\phi}{\psi\circ\phi''\longrightarrow\theta\circ\phi\circ\phi''}}{\psi\circ\phi''\longrightarrow\theta}$$ □

LEMMA 2. *An arrow* α *preserves right Kan extensions iff it admits a left adjoint* β .

PROOF. If $\alpha \dashv \beta$, then :

$$\frac{\theta\longrightarrow\alpha\circ hom_u(\phi,\psi)}{\beta\circ\theta\longrightarrow hom_u(\phi,\psi)}\quad\text{(adjunction } \alpha\dashv\beta)$$

$$\frac{\beta\circ\theta\longrightarrow hom_u(\phi,\psi)}{\beta\circ\theta\circ\phi\longrightarrow\psi}\quad\text{(right Kan extension)}$$

$$\frac{\beta\circ\theta\circ\phi\longrightarrow\psi}{\theta\circ\phi\longrightarrow\alpha\circ\psi}\quad\text{(adjunction } \alpha\dashv\beta)$$

$$\frac{\theta\circ\phi\longrightarrow\alpha\circ\psi}{\theta\longrightarrow hom_u(\phi,\alpha\circ\psi)}\quad\text{(right Kan extension)}\ .$$

Conversely, if α preserves right Kan extensions, take $\beta = hom_v(\alpha, 1)$. □

Dual statements hold true for the right and left liftings.

DEFINITION (Street [15]). Let $\phi : v \to v$ be a monad in B. The *Kleisli object of* ϕ is an object k of B endowed with a ϕ-algebra $d: v \to k$ such that, for each object x, the map induced by the composition with d:

$$B(k,x)\longrightarrow \phi\text{-}alg(v,x)$$

is an isomorphism.

When this is the case, d has a right adjoint d^*, the monad $d^*\circ d$ is isomorphic to ϕ and the object k satisfies the classical universal property of Kleisli algebras. Technically the Kleisli object is a lax colimit, or a «collage» with a more recent terminology (Street [16]).

It is easy to check that in B-*mod* any monad $\phi : A \nrightarrow A$ has a Kleisli object K, which can be described as the category with the same objects of A, the same underlying, and

$$K(a,b) = \phi(a,b)$$

(see also Thiebaud [17]).

2. SHAPE OBJECTS AND SHAPE INVARIANT ARROWS.

Let $K: A \to T$ be an arrow which admits a right adjoint K^*. From the axiomatic approach to shape categories of Bourn - Cordier [5] we assume the following

DEFINITION. The *shape of* K is the Kleisli object S_K of the monad $hom_A(K,K)$. Let us denote by $D: T \to S_K$ the canonical arrow of Kleisli objects.

In B-*mod*, K is a functor $A \to T$, considered as the bimodule: $K_*: A \nrightarrow T$. $K_*(x,a) = T(x, Ka)$ admits the right adjoint

$$K^*(a,x) = T(Ka, x).$$

The above definition thus amounts to the classical one for shape categories: S_K has the same objects as T, the same underlying, and

$$S_K(x,y) \approx hom_A(K_*(y,-), K_*(x,-)) \approx hom_A(K_*, K_*)(y,x).$$

In this case the canonical arrow of Kleisli objects is provided by the functor $D: T \to S_K$ which is the identity on objects and is defined on arrows as follows: for any ordered pair (x,y), the arrow

$$T(x,y) \longrightarrow S_K(x,y) \approx hom_A(K_*, K_*)(y,x)$$

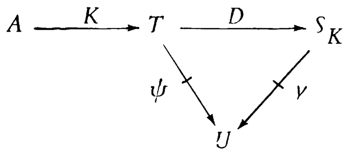
is given by the morphism of bimodules $1_T \to hom_A(K_*, K_*)$ corresponding to $1: K_* \to K_*$.

It is easy to check that in general we have

$$(*) \qquad hom_A(K_*, K_*) = D^* \circ D.$$

Dual definitions can be given for the *coshape object of* K; it is the Kleisli object of the monad $hom^T(K^*, K^*)$.

DEFINITION (Frei [9], Deleanu - Hilton [8]). An arrow $\psi: T \to U$ *is shape* *invariant* if there exists $\gamma : S_K \to U$ such that $\psi = \gamma \circ D$:



THEOREM (Frei [9], Deleanu-Hilton [8]). *Right Kan extensions are shape invariant.*

PROOF. Suppose $\psi = hom_A(K, \alpha)$; take $\gamma = \psi \circ D^*$. The proof now comes from a calculation just involving the universal property of right Kan extensions and the essential feature $(*)$ of S_K. □

THEOREM (Frei-Kleisli [10, 11]). *Let $\psi : T \to U$ be a shape-invariant arrow. If ψ preserves the right Kan extension $hom_A(K, K)$, then it is a right Kan extension along K.*

PROOF. Suppose $\psi \simeq \gamma \circ D$; take

$$\alpha = \gamma \circ D \circ K \simeq hom_{S_K}(K^* \circ D^*, \gamma).$$

We have to show

$$\psi \simeq hom_A(K, \alpha), \text{ i.e. } \gamma \circ D \simeq hom_A(K, \gamma \circ D \circ K).$$

The bijective correspondance

$$\frac{\beta \longrightarrow \gamma \circ D}{\beta \circ K \longrightarrow \gamma \circ D \circ K}$$

is obvious in one direction ; in the other one it is obtained as follows :

$$\frac{\beta \circ K \longrightarrow \gamma \circ D \circ K}{\beta \longrightarrow hom_A(K, \gamma \circ D \circ K)} \quad \text{(right Kan extension)}$$

$$\frac{\beta \longrightarrow hom_A(K, \gamma \circ D \circ K)}{\beta \longrightarrow \gamma \circ D \circ hom_A(K, K)} \quad (\psi \text{ preserves } hom_A(K, K))$$

$$\frac{\beta \longrightarrow \gamma \circ D \circ hom_A(K, K)}{\beta \longrightarrow \gamma \circ D \circ D^* \circ D.} \quad (\text{by } (*))$$

□

COROLLARY. *If $\psi : T \to U$ is shape-invariant and admits a left adjoint, then it is a right Kan extension along K.*

Deleanu-Hilton [8] and Frei [9] calculate the shape category of a functor having a left adjoint. Applications to this case are also given in Cordier-Porter [6]. It is easy to show that the main feature of S_K, in this case, depends only on the adjunction.

Suppose that $K: A \to T$ has a left adjoint $L: T \to A$. We have:

$$L \dashv L^* \simeq K \dashv K^* .$$

A direct consequence of Lemma 1 is thus: $hom_A(K, K) \simeq K \circ L$, i.e. (Deleanu-Hilton [8], Theorem 4.3) S_K is the Kleisli object of the monad $K \circ L$. Moreover, in *B-mod* the bijection $S_K(x, y) \simeq A(Lx, Ly)$ proved in Deleanu-Hilton [8], is now reduced to a simple calculation (again Lemma 1):

$$S_K(x, y) \simeq hom_A(K_*(y, \cdot), K_*(x, \cdot)) \simeq hom_A(K_*(y, \cdot), L^*(x, \cdot))$$
$$\simeq L^*(x, \cdot) \circ L_*(y, \cdot) \simeq A(Lx, Ly).$$

3. APPLICATIONS TO MODULE THEORY.

We want now to recover some applications of Kleisli [12], when B is the category *Ab-mod* of categories based on the closed category of abelian groups, with bimodules as morphisms.

Let A, T be rings with unit elements, i.e. one-object categories. Let $K: A \to T$ be a ring-homomorphism, i.e. a functor. Then the shape category S_K is the endomorphism ring $End_A T$ of T considered as a left A-module. The functor $D: T \to S_K$ is given on arrows by

$$x \mapsto \text{left multiplication by } x: T \to T.$$

A bimodule $T \nrightarrow 1$ (1 denotes the trivial one-object category) is just a left T-module.

The module $\psi: T \nrightarrow 1$ is shape invariant when it can be extended to an $End_A T$-module. ψ is a right Kan extension along K when it is of the form $Hom_A(T, \alpha)$, and it is a left Kan extension when it has the form $T \otimes_A \gamma$.

Recall from Lawvere [13] that a module $\psi: T \nrightarrow 1$ has a left adjoint exactly when it is a finitely generated projective module. The previous corollary thus applies directly to such modules.

Now the (dual of) Theorem 2.2 of Frei-Kleisli [11] can be reformulated and proved as follows:

THEOREM. *Let $K: A \to T$ be a ring homomorphism. If T, considered as a A-module $\tau: A \,+\!\!\!\to\, 1$ has a left adjoint, then every shape invariant T-module is a right Kan extension along K.*

PROOF. More generally, suppose A and T are categories (enriched in a bicategory) and K is a functor such that, for each object x, the bimodule

$$K_*(x,\cdot): A \,+\!\!\!\to\, \hat{ex}$$

has a left adjoint a_x ($\hat{ex}$ denotes the trivial one-object category with underlying ex). By the previous theorem, it is enough to show that any ψ : $T \,+\!\!\!\to\, \hat{ex}$ preserves $hom_A(K_*, K_*)$:

$$hom_A(K_*, K_*)(x,y) = hom_A(K_*(y,\cdot), K_*(x,\cdot)) \simeq K_*(x,\cdot) \circ a_y .$$

So:

$$(\psi \circ hom_A(K_*, K_*))(x) \simeq \int_y \psi(y) \circ hom_{ey}(K_*, K_*)(y,x)$$
$$\simeq \int_y (\psi(y) \circ K_*(y,\cdot) \circ a_x)$$

and

$$hom_A(K_*, \psi \circ K_*)(x) \simeq hom_A(K_*(x,\cdot), \psi \circ K_*) \simeq (\psi \circ K)_* \circ a_x$$
$$\simeq \int_y (\psi(y) \circ K_*(y,\cdot)) \circ a_x . \qquad \square$$

4. THE ČECH CONDITION.

DEFINITION. $K: A \to T$ *is shape adequate* if $hom_A(K,K) \circ K \simeq K$.

Bourn-Cordier [5] show that, in B-*mod*, K is shape adequate iff

$$T(x, Ka) \simeq S_K(x, Ka),$$

i.e. when D is fully-faithful on pairs (x, Ka). Frei [9] points out that this condition (called condition C in [9], the terminology «shape-adequate» can be found in Tholen [18]) is the most general sufficient one for $D \circ K$ to be codense.

It is known (Frei [9], Deleanu-Hilton [7]) that when K is shape adequate, each S_K-object x admits a limit presentation, namely:

$$x = \varprojlim D \circ K \circ d_x$$

from the comma category

$$(x\downarrow K) \xrightarrow[d_x]{} A \xrightarrow[K]{} T \xrightarrow[D]{} S_K .$$

This property can now be formulated as follows :

THEOREM. *Each object x of S_K is the limit of $D \circ K$ indexed by the bimodule $K_*(x, -) : A \nrightarrow \hat{e}x$.*

PROOF. From Borceux-Kelly [4], recall that the limit $\{F, \phi\}$ of $F: A \to X$, indexed by the bimodule $\phi: A \nrightarrow \hat{u}$ (when it exists) is an object representing the right Kan extension $hom_A(\phi, F_*)$. Such an object is characterized by a family of isomorphisms

$$X(y, \{F, \phi\}) \simeq hom_A(\phi, F_*(y, -))$$

for each object y. To prove the theorem it is thus enough to verify

$$S_K(y, x) \simeq hom_A(K_*(x, -), (D \circ K)_*(y, -)),$$

and

$$(D \circ K)_*(y, -) \simeq S_K(y, K-) \simeq T(y, K-)$$

holds true because K is shape adequate. □

More generally one could ask for limits indexed by suitable bimodules.

DEFINITION. Let Ω be a family of bimodules $\phi: A \nrightarrow \hat{u}$. $K: A \to T$ satisfies the *Čech condition* with respect to Ω if for each T-object x there exist a_x in Ω and a 2-cell $a_x \to K_*(x, -)$ such that the induced 2-cell

$$hom_A(K_*(x, -), K_*) \longrightarrow hom_A(a_x, K_*)$$

is an isomorphism.

THEOREM. *If $K: A \to T$ is shape adequate and satisfies the Čech condition with respect to Ω then each object of S_K is a limit indexed in Ω.*

PROOF. We have $x \simeq \{D \circ K, a_x\}$, because

$$S_K(y, x) \simeq hom_A(K_*(x, -), K_*(y, -)) \simeq hom_A(a_x, K_*(y, -))$$
$$\simeq hom_A(a_x, (D \circ K)_*(y, -)). \qquad \square$$

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