CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

MARIA CRISTINA PEDICCHIO
FABIO ROSSI

Monoidal closed structures for topological spaces : counter-example to a question of Booth and Tillotson

Cahiers de topologie et géométrie différentielle catégoriques, tome 24, n° 4 (1983), p. 371-376
<http://www.numdam.org/item?id=CTGDC_1983__24_4_371_0>

CAHIERS DE TOPOLOGIE *Vol. XXIV-4 (1983)*
ET GÉOMÉTRIE DIFFÉRENTIELLE

MONOIDAL CLOSED STRUCTURES FOR TOPOLOGICAL SPACES: COUNTER-EXAMPLE TO A QUESTION OF BOOTH AND TILLOTSON

by Maria Cristina PEDICCHIO and Fabio ROSSI *

INTRODUCTION.

In recent years, the problem of studying monoidal closed structures [2] on the category *Top* of topological spaces and continuous maps has interested many authors (see, for instance, Wyler [6], Wilker [5], Greve [3], and Booth & Tillotson [1]). In particular, the method of Booth & Tillotson is to topologize, by an arbitrary class $\mathfrak{A}$ of spaces, both the set of continuous maps and the set-product of topological spaces (*$\mathfrak{A}$-open* topology and *$\mathfrak{A}$-product*, respectively). If $\mathfrak{A}$ satisfies suitable conditions, then the $\mathfrak{A}$-open topology and the $\mathfrak{A}$-product are related by an exponential law and determine a monoidal closed structure on *Top* ([1], Theorem 2.6, page 42).

In the list of their examples, the authors study the case, important for sequential spaces, of $\mathfrak{A} = \{N_\infty\}$, where N_∞ is the one-point compactification of the discrete space N of natural numbers. This class does not satisfy the hypothesis of Theorem 2.6 of [1].

One question they raise is to determine whether this $\mathfrak{A}$-structure is monoidal closed on *Top* ([1], Section 6 (i), page 46).

The aim of our note is to answer this question negatively, proving, by an example, that the considered structure is not monoidal closed.

To obtain this result we will use some properties, due to Isbell [4], that we recall in Section 1.

1. NOTATION. X, Y, Z will always denote topological spaces and $hom(X, Y)$ will indicate the set of all continuous functions (maps) from X to Y. 1 will stand for a singleton space, and 2 for a Sierpinski space, i.e. a space consisting of two points 0, 1 with $\emptyset$, $\{1\}$, $\{0, 1\}$ as its

* Work partially supported by the Italian CNR.

open sets.

Let us first summarize some notions and results stated in [1] and [4].

Let $\mathfrak{A}$ be an arbitrary class of topological spaces.

DEFINITION 1.1 [1]. If $A \in \mathfrak{A}$, $f: A \to Y$ is a map and U is an open set in Z, then $W(f, U)$ will denote the following subset of $hom(Y, Z)$:

$$W(f, U) = \{g: Y \to Z \mid gf(A) \subseteq U\}.$$

By requiring the family of all these sets to be an open subbase, we introduce a topology on $hom(Y, Z)$, called the *$\mathfrak{A}$-open topology.*

The corresponding function space will be denoted by $M_{\mathfrak{A}}(Y, Z)$; $M_{\mathfrak{A}}(-,-)$ is a bifunctor from $Top^{op} \times Top$ to Top.

DEFINITION 1.2 [1]. We define the *$\mathfrak{A}$-product* $X \times_{\mathfrak{A}} Y$ of X and Y as the set $X \times Y$ with the final topology with respect to all incoming maps of the forms :

$$1_X \times f: X \times A \to X \times Y, \quad f \in hom(A, Y), \quad A \in \mathfrak{A},$$
$$i: \{x\} \times Y \to X \times Y, \quad x \in X.$$

It is easy to see that $- \times_{\mathfrak{A}} -$ is a bifunctor from $Top \times Top$ to Top.

DEFINITION 1.3 [1]. The class $\mathfrak{A}$ of spaces will be said to be a *regular class* if for each $A \in \mathfrak{A}$, every point of A has a fundamental system of neighborhoods which are closed in A and are epimorphic images of spaces in $\mathfrak{A}$.

PROPOSITION 1.4 [1]. *If $\mathfrak{A}$ is a regular class of compact spaces, then there is an adjunction*

$$< - \times_{\mathfrak{A}} Y, M_{\mathfrak{A}}(Y, -), \theta >: Top \to Top, \quad \textit{for any } Y \in Top.$$

Furthermore the exponential law

$$\theta: M_{\mathfrak{A}}(X, M_{\mathfrak{A}}(Y, Z)) \to M_{\mathfrak{A}}(X \times_{\mathfrak{A}} Y, Z),$$

$$\theta(f') = f, \quad f(x, y) = f'(x)(y), \quad x \in X, \ y \in Y,$$

is a continuous bijection.

DEFINITION 1.5 [4]. A topology on the set $Op\,Y$ of open sets of a topological space Y is a *topological topology* iff it makes finite intersection and arbitrary union continuous operations.

PROPOSITION 1.6 [4]. *If F and G are adjoint endofunctors of Top $F \dashv G$, then $G(2) = Y^*$ is a topological topology on the set $Op\,Y$, where $Y = F(1)$.*

Conversely, if Y is an arbitrary space and Y^* is a topological topology on $Op\,Y$, then there is (up to isomorphism) precisely one pair of adjoint endofunctors

$$F \dashv G : Top \to Top \quad \text{such that} \quad F(1) = Y \text{ and } G(2) = Y^*.$$

Moreover $G(Z)$ is $hom(Y, Z)$ with the initial topology with respect to all the functions of the form

$$\lambda_U : Hom(Y, Z) \to Y^*, \qquad U \in Op\,Z,$$

where $\lambda_U(f) = f^{-1}(U)$ for each $f \in hom(Y, Z)$.

Now, we are able to prove the following proposition.

PROPOSITION 1.7. *If $\mathfrak{A}$ is a regular class of compact spaces, then*

a) For any $Y \in Top$, $M_{\mathfrak{A}}(Y, 2)$ is a topological topology Y^* on $Op\,Y$.

b) $M_{\mathfrak{A}}(Y, Z)$ is the set $hom(Y, Z)$ with the initial topology with respect to all the functions

$$\lambda_U : hom(Y, Z) \to Y^*, \quad U \in Op\,Z,$$

where $\lambda_U(f) = f^{-1}(U)$, $f \in hom(Y, Z)$.

c) Let $W_0 = \{U_\alpha\}$ be a family of open sets of Z and let $\langle Y, Z\rangle_{W_0}$ denote the initial topology on $Hom(Y, Z)$ with respect to all the functions λ_{U_α}, $U_\alpha \in W_0$ (see b). If W_1 is the set of all finite intersections of U_α and W_2 is the set of all arbitrary unions of U_α, then the initial topologies $\langle Y, Z\rangle_{W_0}$, $\langle Y, Z\rangle_{W_1}$ and $\langle Y, Z\rangle_{W_2}$ coincide.

d) We topologize the set $Op(X \times_{\mathfrak{A}} Y)$, the subbasic open sets being all the sets of the form

$$\hat{W}(g, g') = \{U,\ U \in Op(X \times_{\mathfrak{A}} Y) \mid U \supseteq g(A) \times g'(A')\}$$

where $g \in hom(A, X)$, $g' \in hom(A', Y)$ *and* $A, A' \in \mathfrak{A}$.

We denote this topology by $<X \times_{\mathfrak{A}} Y>$. *For any* $X, Y \in Top$, *we have*

$$(X \times_{\mathfrak{A}} Y)^* \leqslant <X \times_{\mathfrak{A}} Y>.$$

If the class $\mathfrak{A}$ *determines a monoidal closed structure on Top, then, necessarily*

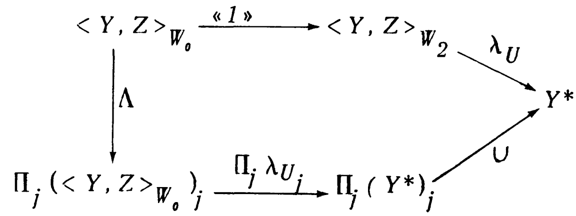
$$(X \times_{\mathfrak{A}} Y)^* = <X \times_{\mathfrak{A}} Y>.$$

PROOF. a and b follow from 1.4 and 1.6.

c) It is obvious that

$$< Y, Z >_{W_2} \geqslant < Y, Z >_{W_0}.$$

Conversely, let $U \in W_2$, then $U = \bigcup_j U_j$, $U_j \in W_0$. It follows that



is commutative, where

$$(< Y, Z >_{W_0})_j = < Y, Z >_{W_0}, \quad (Y^*)_j = Y^* \quad \text{for any } j,$$

Δ is the canonical diagonal map and «1» the identity function of $hom(Y, Z)$. Since Y^* is a topological topology, the union operation $\cup$ is continuous, then $\lambda_U \cdot$«1» is also continuous for each $U \in W_2$. It follows, from the universal property of initial topology, that «1» is continuous.

A similar proof applies to the family W_1.

d) From 1.3 the exponential law is a continuous bijection

$$M_{\mathfrak{A}}(X, Y^*) = M_{\mathfrak{A}}(X, M_{\mathfrak{A}}(Y, 2)) \xrightarrow{\theta} M_{\mathfrak{A}}(X \times_{\mathfrak{A}} Y, 2) = (X \times_{\mathfrak{A}} Y)^*.$$

From b and c, $M_{\mathfrak{A}}(X, Y^*)$ has the initial topology with respect to all the maps

$$\lambda_V : M_{\mathfrak{A}}(X, Y^*) \to X^*$$

where V is a subbasic open set of Y^*. It is easy to see that $M_{\mathfrak{A}}(X, Y^*)$

and $<X \times_{\mathfrak{A}} Y>$ are homeomorphic. We obtain the thesis from the following commutative diagram :

$$\begin{array}{ccc} & M_{\mathfrak{A}}(X, Y^*) & \\ \theta \swarrow & & \searrow \sim \\ (X \times_{\mathfrak{A}} Y)^* & \xleftarrow{\;\ll 1 \gg\;} & <X \times_{\mathfrak{A}} Y>. \end{array} \qquad \square$$

2. Let $\mathfrak{A}$ be the class whose only element is N_∞, the one-point compactification of the discrete space N of natural numbers.

In this section we shall prove that the $\mathfrak{A}$-open topology and the $\mathfrak{A}$-product do not determine a monoidal closed structure on *Top* ; this gives a negative answer to the question posed by Booth and Tillotson in [1], Section 6, Example (i) page 46.

Clearly, for 1.7 d it suffices to find two spaces X and Y such that

$$(X \times_{\mathfrak{A}} Y)^* \underset{\neq}{<} <X \times_{\mathfrak{A}} Y>.$$

Let $X = Y = N_\infty$, it is easy to see that $N_\infty \times_{\mathfrak{A}} N_\infty = N_\infty \times N_\infty$, where $\times$ is the topological product. From the definition of the $\mathfrak{A}$-open topology, the open sets of $(N_\infty \times N_\infty)^*$ are generated by the sets of the form

$$W(s) = W(r, t) = \{U,\ U \in Op(N_\infty \times N_\infty) \mid U \supseteq s(N_\infty)\},$$

where s is any continuous map $N_\infty \to N_\infty \times N_\infty$, with components r, t: $N_\infty \to N_\infty$. For 1.7 d the subbasic open sets of $<N_\infty \times N_\infty>$ are all the sets of the form

$$\hat{W}(r, t) = \{U,\ U \in Op(N_\infty \times N_\infty) \mid U \supseteq r(N_\infty) \times t(N_\infty)\},$$

where $r, t \in hom(N_\infty, N_\infty)$. The one-point set $\{N_\infty \times N_\infty\}$ is clearly open in $<N_\infty \times N_\infty>$, being $\hat{W}(r, t)$ where $r = t = 1: N_\infty \to N_\infty$. However, it is not open in $(N_\infty \times N_\infty)^*$, for every non-empty open set in $(N_\infty \times N_\infty)^*$ contains elements other than $N_\infty \times N_\infty$. To see this, consider a basic open set

$$I = W(r_1, t_1) \cap \ldots\ldots \cap W(r_h, t_h), \text{ where } h \geq 1.$$

Choose $m \in N$ such that each $r_i(\infty)$ for $1 \leq i \leq h$ is either $< m$ or is ∞.

Then each

$$s_i(N_\infty) = \{(r_i(x), t_i(x)) \mid x \in N_\infty\}$$

has only a finite number of points in common with $\{m\} \times N_\infty$. So there is some $n \in N$ such that (m, n) lies in no $s_i(N_\infty)$. Thus the open set I contains $N_\infty \times N_\infty / \{(m, n)\}$. This completes the proof. □

Since the $\mathfrak{A}$-structure, for $\mathfrak{A} = \{N_\infty\}$, is not monoidal closed it follows that the $\mathfrak{A}$-product is not associative and θ^{-1} is not continuous; further, $M_{\mathfrak{A}}(Y, Z)$ *(cs-open topology = convergent sequence-open topology)* is generally different from $M_{\mathcal{CHC}}(Y, Z)$ ([1], Section 8, Example (VII), page 50 and Section 9 (i) page 52).

The authors would like to thank Max Kelly for the improvements suggested to the first version of the paper.

REFERENCES.

1. P. BOOTH & J. TILLOTSON, Monoidal closed, cartesian closed and convenient categories of topological spaces, *Pacific J. Math.* 88 (1980), 35-53.
2. S. EILENBERG & G.M. KELLY, Closed categories, *Proc. Conf. on Categorical Algebra* (La Jolla, 1965), Springer 1966, 421-562.
3. G. GREVE, How many monoidal closed structures are there in *Top*? *Arch. Math.* 34, Basel (1980), 538-539.
4. J. R. ISBELL, Function spaces and adjoints, *Math. Scand.* 36 (1975), 317-339.
5. P. WILKER, Adjoint product and hom functors in general topology, *Pacific J. Math.* 34 (1970), 269-283.
6. O. WYLER, Convenient categories for topology, *Gen. Top. and Appl.* 3 (1973), 225-242.

Istituto di Matematica
Universita di Trieste
Piazzale Europa 1
34100 TRIESTE. ITALIE