CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

LEONARDO CIRLINCIONE

On a family of varieties not satisfying Stoka's measurability condition

Cahiers de topologie et géométrie différentielle catégoriques, tome 24, nº 2 (1983), p. 145-154

<http://www.numdam.org/item?id=CTGDC_1983__24_2_145_0>

CAHIERS DE TOPOLOGIE
ET GÉOMÉTRIE DIFFÉRENTIELLE

Vol. XXIV-2 (1983)

ON A FAMILY OF VARIETIES NOT SATISFYING STOKA'S MEASURABILITY CONDITION

by Leonardo CIRLINCIONE

RÉSUMÉ. On donne un premier exemple de famille mesurable de variétés dont le groupe attaché au groupe maximal d'invariance n'est pas mesurable. On prouve, ainsi, que la condition suffisante de mesurabilité, donnée par M. I. Stoka [3], n'est pas nécessaire.

Let $\mathcal{F}_q$ be a family of varieties, depending on the (essential) parameters $a_1, a_2, \ldots, a_q$, of an homogeneous n-dimensional space $\mathcal{H}_n$ and let G_r be an r-dimensional Lie group of transformations of $\mathcal{H}_n$. Assume that $\mathcal{F}_q$ is G_r-invariant and no element of G_r (except the identity map) fixes every variety of $\mathcal{F}_q$. In case G_r is contained in no other group having the same properties, G_r is called *the maximal group of invariance of* $\mathcal{F}_q$ and its subgroups are the *groups of invariance of* $\mathcal{F}_q$.

Let $T \in G_s$, where G_s is an s-dimensional group of invariance of $\mathcal{F}_q$. If (a_i) and (β_i) $(i = 1, \ldots, q)$ are the parameters characterizing the varieties $\mathcal{V}$ and $T(\mathcal{V})$ of $\mathcal{F}_q$, the map $(a_i) \mapsto (\beta_i)$ defines an s-dimensional Lie group H_s (of the parameter space $\mathcal{H}_q$ of $\mathcal{F}_q$), isomorphic to G_s ([4], page 33) (the *associated group of* G_s *with respect to* $\mathcal{F}_q$). H_s can be either a measurable Lie group ([4], page 12) or not. We say that $\mathcal{F}_q$ has the (elementary) *measure* $\phi(a_i)\,da_i$, with respect to G_s, if H_s is measurable, admitting ϕ as measure. According to M. I. Stoka, $\mathcal{F}_q$ is *measurable* if ϕ is also the measure for each measurable Lie group which is associated to the same group of invariance of $\mathcal{F}_q$.

M. I. Stoka proves (see [3], [4] page 40) that if the group associated to the maximal group of invariance of $\mathcal{F}_q$ is a measurable Lie group, then $\mathcal{F}_q$ is measurable. This condition is also necessary if $q = 1, 2, 3$ ([4] page 41). Nevertheless this is not generally true; in fact we give an example of a 5-dimensional measurable family $\mathcal{F}_5$ of varieties of $\mathfrak{A}_3(\mathrm{R})$ such that the group associated to the maximal group of invariance of $\mathcal{F}_5$ is not measurable.

The first section of this article is devoted to the determination of the subgroups of G_7, the similarity transformation group of $\mathfrak{A}_3(\mathrm{R})$, depending on 5 or 6 parameters. This is helpful in order to give the mentioned example.

1. THE SIMILARITY TRANSFORMATION GROUPS OF $\mathfrak{A}_3(\mathrm{R})$ DEPENDING ON 5 OR 6 PARAMETERS.

Let G_n be an n-dimensional Lie group of $\mathfrak{A}_3(\mathrm{R})$ generated by the infinitesimal transformations $X_1, \dots, X_n$. The operators

(1) $Y_i = a_i^j X_j$ [(1)] $(i = 1, \dots, m;\ m < n,\ a_i^j \in \mathrm{R},\ rank(a_i^j) = m)$

define an m-dimensional subgroup G_m of G_n iff

(2) $(Y_s, Y_t) = k_{st}^i Y_i \quad (s, t = 1, \dots, m;\ k_{st}^i \in \mathrm{R})$.

Let $G_m^1 = \langle Y_1, \dots, Y_m \rangle$ and $G_m^2 = \langle Z_1, \dots, Z_m \rangle$ be two m-dimensional subgroups of G_n. G_m^1 is conjugate to G_m^2 (in G_n) iff there exist

$$T \in G_n \quad \text{such that} \quad T(Y_i) = b_i^s Z_s \quad (b_i^s \in \mathrm{R}).$$ [(2)]

By supposing in (1) $a_s^t \neq 0$, one can change Y_i into $(a_s^t)^{-1} Y_s$. Thus we may assume $a_s^t = 1$. Then change Y_i $(i \neq s)$ into $Y_i - a_i^t Y_s$; we see that it is not restrictive to put $a_i^t = 0$.

Let G_7 be the similarity transformation group of $\mathfrak{A}_3(\mathrm{R})$. We can write the equations of G_7 in the following way (see [4] page 154)

$$(3)\quad \begin{cases} x = h((1+l^2-m^2-n^2)x' + 2(lm-n)y' + 2(ln+m)z') + t_1, \\ y = h(2(lm+n)x' + (1-l^2+m^2-n^2)y' + 2(mn-l)z') + t_2, \\ z = h(2(ln-m)x' + 2(mn+l)y' + (1-l^2-m^2+n^2)z') + t_3, \end{cases}$$

$h, l, m, n, t_1, t_2, t_3 \in \mathrm{R}$, $h \neq 0$ and $h(1+l^2+m^2+n^2)$ is the homothetic ratio.

(1) We use the Einstein's convention.

(2) More information can be found in [1].

G_7 is generated by the infinitesimal transformations

$$(4)\begin{cases} X_1 f = x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z}, \quad X_2 f = y\dfrac{\partial f}{\partial z} - z\dfrac{\partial f}{\partial y}, \\ X_3 f = -x\dfrac{\partial f}{\partial z} + z\dfrac{\partial f}{\partial x}, \quad X_4 f = x\dfrac{\partial f}{\partial y} - y\dfrac{\partial f}{\partial x}, \\ X_5 f = \dfrac{\partial f}{\partial x}, \quad X_6 f = \dfrac{\partial f}{\partial y}, \quad X_7 f = \dfrac{\partial f}{\partial z}, \end{cases}$$

and its group structure is given by

$$(5)\begin{cases} (X_1, X_2) = (X_1, X_3) = (X_1, X_4) = 0, \quad (X_1, X_5) = -X_5, \\ (X_1, X_6) = -X_6, \quad (X_1, X_7) = -X_7 \,; \; (X_2, X_3) = X_4, \\ (X_2, X_4) = X_3, \quad (X_2, X_5) = 0, \quad (X_2, X_6) = -X_7, \\ (X_2, X_7) = X_6 \,; \; (X_3, X_4) = -X_2, \quad (X_3, X_5) = X_7, \\ (X_3, X_6) = 0, \quad (X_3, X_7) = -X_5 \,; \; (X_4, X_5) = -X_6, \\ (X_4, X_6) = X_5, \quad (X_4, X_7) = 0 \,; \\ (X_5, X_6) = (X_5, X_7) = (X_6, X_7) = 0. \end{cases}$$

REMARK 1. We pointed out that in (4) $X_1 f$ generates the dilatation group; $\langle X_2 f\rangle$, $\langle X_3 f\rangle$, $\langle X_4 f\rangle$ are (resp.) the three rotation groups on the axes x, y, z and $X_5 f$, $X_6 f$, $X_7 f$ generate the translation group.

In order to determine the subgroups of G_7 of dimension 5, we consider the transformations

$$Y_i = a_i^j X_j \quad (i = 1, 2, 3, 4, 5, \quad a_i^j \in \mathrm{R}, \quad \mathrm{rank}(a_i^j) = 5),$$

where X_j $(j = 1, 2, 3, 4, 5, 6, 7)$ is given by (4). Assume

$$a_5^1 = a_5^2 = a_5^3 = a_5^4 = a_5^5 = a_5^6 = 0 \;;$$

then $a_5^7 \neq 0$. Thus, we may put $a_5^7 = 1$, $a_i^7 = 0$, $i \neq 5$, obtaining

A: $Y_i = a_i^j X_j \, (i = 1, 2, 3, 4)$, $j \neq 7$, $Y_5 = X_7$.

Under the hypothesis

$$a_5^1 = a_5^2 = a_5^3 = a_5^4 = a_5^5 = 0, \quad a_5^6 \neq 0$$

we have

B: $Y_i = a_i^j X_j$, $j \neq 6$, $Y_5 = X_6 + a_5^7 X_7$.

Likewise one can infer the remaining cases :

$$C: \ Y_i = a^j_i X_j \quad j \neq 5, \quad Y_5 = X_5 + a^6_5 X_6 + a^7_5 X_7,$$

$$D: \ Y_i = a^j_i X_j \quad j \neq 4, \quad Y_5 = X_4 + a^5_5 X_5 + a^6_5 X_6 + a^7_5 X_7,$$

$$E: \ Y_i = a^j_i X_j \quad j \neq 3, \quad Y_5 = X_3 + a^4_5 X_4 + a^5_5 X_5 + a^6_5 X_6 + a^7_5 X_7,$$

$$F: \ Y_i = a^j_i X_j \quad j \neq 2, \quad Y_5 = X_2 + a^3_5 X_3 + a^4_5 X_4 + a^5_5 X_5 + a^6_5 X_6 + a^7_5 X_7$$

$$G: \ Y_i = a^j_i X_j \quad j \neq 1,$$
$$Y_5 = X_1 + a^2_5 X_2 + a^3_5 X_3 + a^4_5 X_4 + a^5_5 X_5 + a^6_5 X_6 + a^7_5 X_7 .$$

In case A (as well as in the remaining cases), we can again put the same assumption on the coefficients of Y_4 . Thus one finds

$$A_a: \ Y_i = a^j_i X_j \quad (i = 1, 2, 3), \ j \neq 6, 7, \quad Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_b: \ Y_i = a^j_i X_j \quad j \neq 5, 7, \quad Y_4 = X_5 + a^6_4 X_6, \quad Y_5 = X_7,$$

$$A_c: \ Y_i = a^j_i X_j \quad j \neq 4, 7, \quad Y_4 = X_4 + a^5_4 X_5 + a^6_4 X_6, \quad Y_5 = X_7,$$

$$A_d: \ Y_i = a^j_i X_j \quad j \neq 3, 7, \quad Y_4 = X_3 + a^4_4 X_4 + a^5_4 X_5 + a^6_4 X_6, \quad Y_5 = X_7,$$

$$A_e: \ Y_i = a^j_i X_j \quad j \neq 2, 7, \quad Y_4 = X_2 + a^3_4 X_3 + a^4_4 X_4 \times a^5_4 X_5 + a^6_4 X_6,$$
$$Y_5 = X_7,$$

$$A_f: \ Y_i = a^j_i X_j, \quad j \neq 1, 7,$$
$$Y_4 = X_1 + a^2_4 X_2 + a^3_4 X_3 + a^4_4 X_4 + a^5_4 X_5 + a^6_4 X_6, \quad Y_5 = X_7.$$

From A_a it follows

$$A_{a_1}: \ Y_i = a^j_i X_j \ (i = 1, 2), \ j \neq 5, 6, 7, \quad Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_{a_2}: \ Y_i = a^j_i X_j \quad j \neq 4, 6, 7, \quad Y_3 = X_4 + a^5_3 X_5, \quad Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_{a_3}: \ Y_i = a^j_i X_j \quad j \neq 3, 6, 7, \quad Y_3 = X_3 + a^4_3 X_4 + a^5_3 X_5,$$
$$Y_4 = X_6, \quad Y_5 = X_7,$$

$$A_{a_4}: \ Y_i = a^j_i X_j \quad j \neq 2, 6, 7, \quad Y_3 = X_2 + a^3_3 X_3 + a^4_3 X_4 + a^5_3 X_5,$$
$$Y_4 = X_6, \quad Y_5 = X_7.$$

$$A_{a_5}: \ Y_i = a^j_i X_j \quad j \neq 1, 6, 7, \quad Y_3 = X_1 + a^2_3 X_2 + a^3_3 X_3 + a^4_3 X_4 + a^5_3 X_5,$$
$$Y_4 = X_6, \quad Y_5 = X_7,$$

Now A_{a_1} splits again in four cases. Consider the first

$$A_{a_1}(I): \ Y_1 = a^1_1 X_1 + a^2_1 X_2 + a^3_1 X_3, \quad Y_2 = X_4, \quad Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7$$

In view of (5), (2) yields

$$(Y_1, Y_2) = -a_1^3 X_2 + a_1^2 X_3 = k_{12}^i Y_i ,$$

whence

$$a_1^1 k_{12}^1 = 0, \quad a_1^2 k_{12}^1 = -a_1^3, \quad a_1^3 k_{12}^1 = a_1^2, \quad k_{12}^2 = k_{12}^3 = k_{12}^4 = k_{12}^5 = 0.$$

These equations are compatible if

$$\operatorname{rank} \begin{pmatrix} a_1^1 & 0 \\ a_1^2 & -a_1^3 \\ a_1^3 & a_1^2 \end{pmatrix} < 2.$$

Therefore $(a_1^2)^2 + (a_1^3)^2 = 0$, i.e. $a_1^1 \neq 0$. As it is not restrictive to set $a_1^1 = 1$, we have the first 5-dimensional subgroup of G_7

(6) $$G_5^1 = \langle X_1, X_4, X_5, X_6, X_7 \rangle.$$

In the second case, A_{a_1} (II), we have

$$Y_1 = a_1^1 X_1 + a_1^2 X_2 + a_1^4 X_4, \quad Y_2 = X_3 + a_2^4 X_4, \quad Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7.$$

From $(Y_1, Y_2) = k_{12}^i Y_i$ it follows that

$$a_1^1 k_{12}^1 = 0, \quad a_1^2 k_{12}^1 = a_1^4, \quad a_1^4 k_{12}^1 + a_2^4 k_{12}^2 = -a_1^2,$$
$$k_{12}^2 = k_{12}^3 = k_{12}^4 = k_{12}^5 = 0.$$

A simple computation leads to

$$(a_1^2)^2 + (a_1^4)^2 = 0, \quad \text{i.e. } a_1^2 = a_1^4 = 0.$$

Thus one finds the family of 5-dimensional groups

(7) $$\{ G_5^2(a) = \langle X_1, X_3 + a X_4, X_5, X_6, X_7 \rangle \}_{a \in \mathbf{R}} .$$

In the case A_{a_1} (III) we may assume $a_1^1 \neq 0$. For supposing $a_1^1 = 0$ and $a_1^3 = 0$ (then it must be necessarily $a_1^4 \neq 0$) we obtain a special case of A_{a_1} (I) (since it is not restrictive to put also $a_2^4 = 0$), while if $a_1^3 \neq 0$ a particular case of A_{a_1} (II) occurs. Therefore

$$Y_1 = X_1 + a_1^3 X_3 + a_1^4 X_4, \quad Y_2 = X_2 + a_2^3 X_3 + a_2^4 X_4,$$
$$Y_3 = X_5, \quad Y_4 = X_6, \quad Y_5 = X_7.$$

Compute as usual (Y_1, Y_2); then (2) and (5) imply

$$k^1_{12}=0, \quad k^2_{12} = a^4_1 a^3_2 - a^3_1 a^4_2, \quad a^3_1 k^1_{12} + a^3_2 k^2_{12} = -a^4_1,$$
$$a^4_1 k^1_{12} + a^4_2 k^2_{12} = a^3_1, \quad k^3_{12} = k^4_{12} = k^5_{12} = 0.$$

Thus

$$(1 + (a^3_2)^2 + (a^4_2)^2) k^2_{12} = 0, \quad \text{whence} \quad a^3_1 = a^4_1 = 0.$$

Hence we have the family of subgroups of G

$$(8) \qquad \{ G^3_5(b, c) = < X_1, X_2 + bX_3 + cX_4, X_5, X_6, X_7 > \}_{b, c \in R}.$$

$A_{a_1}(IV)$ splits further in three cases: yet they fall under those above examined. The study of the remaining cases is a routine computation. One finds that the early groups fill up the class of subgroups of G_7 depending on five parameters.

THEOREM 1. *Let G_5 be a 5-dimensional subgroup of the similarity transformation group G_7 of $\mathfrak{A}_3(R)$. Then G_5 is conjugate in G_7 to $G^3_5(b, c)$, for suitable $b, c \in R$.*

PROOF. The change of coordinates $T(x, y, z) = (z, x, y)$ induces the operator permutation

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 \\ X_1 & X_4 & X_2 & X_3 & X_7 & X_5 & X_6 \end{pmatrix}.$$

Thus we note that $(G^1_5)^T = G^2_5(0)$. Likewise one checks that $G^2_5(a)$ is conjugate to $G^3_5(a, 0)$.

We can make use of the same processes of Theorem 1 to classify the 6-dimensional subgroups of G_7. The following theorem is the result we obtain:

THEOREM 2. *There exists exactly one 6-dimensional subgroup of G_7 the orthogonal group $G_6 = < X_2, X_3, X_4, X_5, X_6, X_7 >$.*

2. A 5-DIMENSIONAL MEASURABLE FAMILY OF VARIETIES OF $\mathfrak{A}_3(R)$

Let $\mathfrak{V}$ be the variety of $\mathfrak{A}_3(R)$

$$(9) \qquad (a_1 x + a_2 y + a_3 z + 1)((a_2 a_4 + a_3 a_5) x - a_1 a_4 y - a_1 a_5 z - a_1) = 0$$

where $a_i \in R$ and $a_1(a^2_4 + a^2_5) \neq 0$. Clearly (9) is a pair of orthogonal planes of $\mathfrak{A}_3(R)$.

Let $\mathfrak{F}_5 = \mathfrak{V}^{G_7}$ be the family of varieties obtained from $\mathfrak{V}$ through the transformations of G_7. The maximal group of invariance of $\mathfrak{F}_5$ is, of course, G_7. We prove

THEOREM 3. *The group associated to* G_7, *with respect to* $\mathfrak{F}_5$, *is a non-measurable group.*

PROOF. Let T be the transformation (3) of G_7. The parameters β_i of the variety $T(\mathfrak{V})$ define the associated group H_7:

$$(10)\quad \begin{cases} \beta_1 = b\dfrac{(1+l^2-m^2-n^2)a_1 + 2(lm+n)a_2 + 2(ln-m)a_3}{t_1a_1 + t_2a_2 + t_3a_3 + 1} \\ \beta_2 = b\dfrac{2(lm-n)a_1 + (1-l^2+m^2-n^2)a_2 + 2(mn+l)a_3}{t_1a_1 + t_2a_2 + t_3a_3 + 1} \\ \beta_3 = b\dfrac{2(ln+m)a_1 + 2(mn-l)a_2 + (1-l^2-m^2+n^2)a_3}{t_1a_1 + t_2a_2 + t_3a_3 + 1} \\ \beta_4 = b\dfrac{2(lm-n)(a_2a_4+a_3a_5) - (1-l^2+m^2-n^2)a_1a_4 - 2(mn+1)a_1a_5}{t_1(a_2a_4+a_3a_5) - t_2a_1a_4 - t_3a_1a_5 - a_1} \\ \beta_5 = b\dfrac{2(ln+m)(a_2a_4+a_3a_5) - 2(mn-l)a_1a_4 - (1-l^2-m^2+n^2)a_1a_5}{t_1(a_2a_4+a_3a_5) - t_2a_1a_4 - t_3a_1a_5 - a_1} \end{cases}$$

The coefficients of the infinitesimal transformations generating H_7 are

$$\xi_1^1 = a_1,\quad \xi_1^2 = a_2,\quad \xi_1^3 = a_3,\quad \xi_1^4 = a_4,\quad \xi_1^5 = a_5,$$

$$\xi_2^1 = 0,\quad \xi_2^2 = 2a_3,\quad \xi_2^3 = -2a_2,\quad \xi_2^4 = 2a_5,\quad \xi_2^5 = -2a_4,$$

$$\xi_3^1 = -2a_3,\quad \xi_3^2 = 0,\quad \xi_3^3 = 2a_1,\quad \xi_3^4 = 0,\quad \xi_3^5 = -2\frac{a_2a_4+a_3a_5}{a_1},$$

$$\xi_4^1 = 2a_2,\quad \xi_4^2 = -2a_1,\quad \xi_4^3 = 0,\quad \xi_4^4 = 2\frac{a_2a_4+a_3a_5}{a_1},\quad \xi_4^5 = 0,$$

$$\xi_5^1 = -a_1^2,\quad \xi_5^2 = -a_1a_2,\quad \xi_5^3 = -a_1a_3,\quad \xi_5^4 = a_4\frac{a_2a_4+a_3a_5}{a_1},$$
$$\xi_5^5 = a_5\frac{a_2a_4+a_3a_5}{a_1},$$

$$\xi_6^1 = -a_1a_2,\quad \xi_6^2 = -a_2^2,\quad \xi_6^3 = -a_2a_3,\quad \xi_6^4 = -a_4^2,\quad \xi_6^5 = -a_4a_5,$$

$$\xi_7^1 = -a_1a_3,\quad \xi_7^2 = -a_2a_3,\quad \xi_7^3 = -a_3^2,\quad \xi_7^4 = -a_4a_5,\quad \xi_7^5 = -a_5^2.$$

H_7 is a measurable group if the following system of equations

$$
(11)\quad \left\{\begin{array}{l}
a_1\dfrac{\partial\phi}{\partial a_1} + a_2\dfrac{\partial\phi}{\partial a_2} + a_3\dfrac{\partial\phi}{\partial a_3} + a_4\dfrac{\partial\phi}{\partial a_4} + a_5\dfrac{\partial\phi}{\partial a_5} = -5\phi \\
a_3\dfrac{\partial\phi}{\partial a_2} - a_2\dfrac{\partial\phi}{\partial a_3} + a_5\dfrac{\partial\phi}{\partial a_4} - a_4\dfrac{\partial\phi}{\partial a_5} = 0 \\
a_1\left(a_3\dfrac{\partial\phi}{\partial a_1} - a_1\dfrac{\partial\phi}{\partial a_3}\right) + (a_2a_4 + a_3a_5)\dfrac{\partial\phi}{\partial a_5} = -a_3\phi \\
a_1\left(a_2\dfrac{\partial\phi}{\partial a_1} - a_1\dfrac{\partial\phi}{\partial a_2}\right) + (a_2a_4 + a_3a_5)\dfrac{\partial\phi}{\partial a_4} = -a_2\phi \\
a_1^2\left(a_1\dfrac{\partial\phi}{\partial a_1} + a_2\dfrac{\partial\phi}{\partial a_2} + a_3\dfrac{\partial\phi}{\partial a_3}\right) - (a_2a_4 + a_3a_5)\left(a_4\dfrac{\partial\phi}{\partial a_4} + a_5\dfrac{\partial\phi}{\partial a_5}\right) = \\
\qquad = \left(3(a_2a_4 + a_3a_5) - 4a_1^2\right)\phi \\
a_2\left(a_1\dfrac{\partial\phi}{\partial a_1} + a_2\dfrac{\partial\phi}{\partial a_2} + a_3\dfrac{\partial\phi}{\partial a_3}\right) + a_4\left(a_4\dfrac{\partial\phi}{\partial a_4} + a_5\dfrac{\partial\phi}{\partial a_5}\right) = -(4a_2 + 3a_4)\phi \\
a_3\left(a_1\dfrac{\partial\phi}{\partial a_1} + a_2\dfrac{\partial\phi}{\partial a_2} + a_3\dfrac{\partial\phi}{\partial a_3}\right) + a_5\left(a_4\dfrac{\partial\phi}{\partial a_4} + a_5\dfrac{\partial\phi}{\partial a_5}\right) = -(4a_3 + 3a_5)\phi
\end{array}\right.
$$

admits exactly one non-trivial solution (up to a constant), see Deltheil [2] page 28.

The last three equations of (11) are equivalent to

$$
(12)\quad \left\{\begin{array}{l}
a_1\dfrac{\partial\phi}{\partial a_1} + a_2\dfrac{\partial\phi}{\partial a_2} + a_3\dfrac{\partial\phi}{\partial a_3} = -4\phi \\
a_4\dfrac{\partial\phi}{\partial a_4} + a_5\dfrac{\partial\phi}{\partial a_5} = -3\phi\,.
\end{array}\right.
$$

Clearly the equations (12) and the first equation of (11) form a system of equations admitting no solution. Therefore H_7 is not measurable.

REMARK 2. The first equation of (11) arises from the dilatation parameter, while the remaining equations correspond, in order of sequence, to the rotation and translation parameters.

Let G_n be an n-dimensional group of G_7, where $n \leq 5$ and let H_n be the associated group with respect to $\mathcal{F}_5$. If $n \leq 4$, then H_n is intransitive, whence it is not measurable ([4] page 15).

In case $n = 5$, from Theorem 1 it follows that G_n is conjugate to $G_5^3(b, c)$, for suitable parameters $b, c \in R$. In view of Remarks 1 and 2, we note that the Deltheil's system of H_5 contains the first and the last three equations of (11). But we have already seen in the proof of Theorem 3 that they admit no solutions. Therefore:

PROPOSITION 1. *$\mathcal{F}_5$ admits no measure with respect to any n-group of invariance, if $n \leq 5$.*

We turn now to the orthogonal group G_6. Let H_6 denote the corresponding associated group. On the ground of Remark 2, we can obtain the Deltheil's system of H_6 by suppressing in (11) the first equation.

It is not difficult to check that this system has only one solution (up to a constant)

$$\phi(a_1, a_2, a_3, a_4, a_5) =$$
$$= a_1^2(((a_2 a_4 + a_3 a_5)^2 + a_1^2(a_4^2 + a_5^2))(a_1^2 + a_2^2 + a_3^2))^{-3/2}.$$

Hence we have

PROPOSITION 2. *With respect to the orthogonal group $\mathcal{F}_5$ admits the elementary measure*

$$a_1^2(((a_2 a_4 + a_3 a_5)^2 + a_1^2(a_4^2 + a_5^2))(a_1^2 + a_2^2 + a_3^2))^{-3/2}$$
$$da_1 \wedge da_2 \wedge da_3 \wedge da_4 \wedge da_5 .$$

As a consequence we have the main result (see also Theorems 2, 3 and Proposition 1):

COROLLARY. *The family of varieties $\mathcal{F}_5$ is measurable in spite of the non-measurability of the group associated to its maximal group of invariance.*

REFERENCES.

1. BIANCHI L., *Lezioni sulla teoria dei gruppi continui finiti di transformazioni*, Zanichelli, Bologna 1928.

2. DELTHEIL R., Sur la théorie des probabilités géométriques, Thèse, *Ann. Fac. Sc. Univ. Toulouse* (3) 11 (1919), 1-65.

3. STOKA M.I., Geometria integrale in uno spazio Euclideo E_n, *Boll. Un. Mat. Ital.* 13-4 (1958), 470-485.

4. STOKA M. I., *Geometria integrale*, Ed. Acad. R. S. Romania, Bucarest 1967.

Istituto di Matematica
dell'Università
Via Archirafi 34
90123 PALERMO. ITALIE