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NOTION OF TOPOLOGY FOR BICATEGORIES

by R. BETTI and A. CARBONI *)

INTRODUCTION.

We deal with the approach to categories based on a bicategory introduced by Walters [3, 4] and Betti & Carboni [1]. In view of further developments, where B-category theory becomes relevant to cohomology (Street [2]) and other geometrical applications (as Walters' «glueing data») it seems useful to have an abstract notion of topology for general bicategory.

Here such a notion is given in term of a «closure operator» for a locally partially ordered bicategory, even if it can be easily generalized to any locally left exact bicategory. The theorem is that in the wellknown case when $B = Rel(C)$ (the base bicategory for presheaves) closure operators on B are exactly Grothendieck topologies on C .

Moreover, on the basis of the given definition and of a previous paper [1] an intrinsic notion of sheaf for a closure operator is given. These notions arose in helpful conversations with R.H. Street, while he was visiting Milan.

1. We recall that $Rel(C)$ is the bicategory defined as follows (Walters [4], for C any locally small category):

objects of $Rel(C)$ are those of C , 1-cells $u \mapsto v$ are cibles of spans $u \leftarrow w \rightarrow v$, 2-cells are inclusions.

$Rel(C)$ is a symmetric bicategory, and R^o denotes the opposite 1-cell of R .

DEFINITION. Let B be a bicategory, locally an inf-semilattice. A closure operator on B is a locally left exact lax idempotent monad in B which is the identity on objects.

Any Grothendieck topology J on C determines a closure operator

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on $Rel(C)$ by defining

$$\bar{R} = \{ \begin{array}{c} p \\ \swarrow \searrow \\ u \quad v \end{array} \mid \text{there exists a cover } (p_i \rightarrow p)_i \text{ such that}$$

$$\begin{array}{c} p_i \\ \downarrow \\ p \\ \swarrow \searrow \\ u \quad v \end{array} \in R \} ,$$

as in Walters [4] where all stated properties are proved, but the locally left exactness $\overline{R \wedge S} = \bar{R} \wedge \bar{S}$: just observe that if U and V are coverings of u , then also $U \wedge V$ is a covering.

Let us observe that the axioms of a closure operator cannot be strengthened by requiring strict functoriality. Indeed, let C be a regular category and J the regular epi-topology; if f is a regular epi, then the crible generated by $\langle f, f \rangle$ contains its closure iff f has a section.

Our aim is to show that in fact any closure operator on $Rel(C)$ can be obtained in the previous way from a unique Grothendieck topology J on C .

THEOREM. *Closure operators on $Rel(C)$ correspond bijectively to Grothendieck topologies on C .*

PROOF. If $Rel(C) \rightarrow Rel(C)$ is a closure operator, define the covering cibles as those cibles

$$U = \{ b_i: u_i \rightarrow u \}$$

such that the 1-cell $R_U = \langle b_i, b_i \rangle$ satisfies $1_u \subset \bar{R}_U$. With this definition, we get a topology on C :

i) trivially the maximal crible covers.

ii) Let U be a covering crible of u and $f: v \rightarrow u$ a morphism of C ; observe that

$$R_{f*U} = 1 \wedge f. R_U. f^o$$

(for terminology and properties of $Rel(C)$ see Walters [1]), then

$$1 \subset f f^o \subset f. \bar{R}_U. f^o \subset \overline{f. \bar{R}_U. f^o} = \overline{f. R_U. f^o}$$

(the last equality easily follows from the identity $\overline{\bar{R} \cdot \bar{S}} = \overline{\bar{R} \cdot \bar{S}}$, which is a direct consequence of the axioms). Hence

$$1 \subset 1 \wedge \overline{f \cdot R_U \cdot f^0} \subset \bar{1} \wedge \overline{f \cdot R_U \cdot f^0} = \overline{1 \wedge f \cdot R_U \cdot f^0}$$

(by left exactness). So $1 \subset \bar{R}_f * U$ and $f^* U$ covers.

iii) Let U be a covering crible of u and V a crible such that, for each $f \in U$, $f^* V$ covers, we have

$$\begin{aligned} \overline{f^0 \cdot f} &\subset \overline{f^0 \cdot f \cdot R_V \cdot f^0 \cdot f} \quad (\text{because } f^* V \text{ covers}) \\ &= \overline{f^0 \cdot f \cdot R_V \cdot f^0 \cdot f} \subset \bar{R}_V \quad (\text{because } f^0 \cdot f \subset 1). \end{aligned}$$

So

$$R_U = \bigvee_{f \in U} f^0 \cdot f \subset \bar{R}_V,$$

hence $\bar{R}_U \subset \bar{R}_V$ and thus $1 \subset \bar{R}_V$.

It is straightforward to verify that if J is a topology on C and $\bar{}$ is the associated closure operator then J is a J -cover iff $1 \subset \bar{R}_U$. Conversely, given a closure operator on $Rel(C)$, we need to show

$$\bar{T} = \left\{ \begin{array}{c} \begin{array}{ccc} & w & \\ b \swarrow & & \searrow k \\ u & & v \end{array} \end{array} \right\} \left| \begin{array}{l} \text{there exists a crible } U \text{ on } w \text{ such that} \\ 1 \subset \bar{R}_U \text{ and } b^0 \cdot R_U \cdot k \subset T \end{array} \right.$$

for each 1-cell T of $Rel(C)$. In one direction we have

$$\langle b, k \rangle = b^0 \cdot k \subset \overline{b^0 \cdot k} \subset \overline{b^0 \cdot \bar{R}_U \cdot k} = \overline{b^0 \cdot R_U \cdot k} \subset \bar{T}.$$

In the other one, define R_U as $1 \wedge b \cdot T \cdot k^0$. Then $1 \subset \bar{R}_U$ for

$$1 \subset b \cdot b^0 \cdot k \cdot k^0 \subset b \cdot \bar{T} \cdot k^0 \subset \overline{b \cdot \bar{T} \cdot k^0} = \overline{b \cdot T \cdot k^0}.$$

Moreover

$$b^0 \cdot (1 \wedge b \cdot T \cdot k^0) \cdot k \subset b^0 \cdot b \cdot T \cdot k^0 \cdot k \subset T.$$

In the same way we can translate notions relative to Grothendieck topologies in this more «algebraic» context. For instance it is easy to prove the following

PROPOSITION. *The following conditions are equivalent:*

- i) *Representables are J -sheaves.*
- ii) *If R is a «partial map» (i. e. $R^0 R \subset 1$), then*

$$\bar{R} \cdot \bar{S} = \overline{\bar{R} \cdot \bar{S}} \quad \text{and} \quad \bar{1} = 1.$$

PROOF. A compatible family $u_\alpha \rightarrow v$ on the covering $u_\alpha \rightarrow u$ gives rise to a partial map $R: u \dashrightarrow v$. The hypothesis implies that R is a map.

Given any closure operator $(\bar{}): \mathbf{B} \rightarrow \mathbf{B}$, a new bicategory $\bar{\mathbf{B}}$ is defined by taking the same objects as $\mathbf{B}$ and, as 1-cells, the closed ones, i.e. locally the algebras for the idempotent monad induced by the closure operator. In $\bar{\mathbf{B}}$ the composition is defined by $\overline{R \cdot S}$; identities for such composition are closures of the old ones.

A pair of morphisms of bicategories is obtained: $\mathbf{B} \xrightleftharpoons[(\bar{})]{i} \bar{\mathbf{B}}$, such that $(\bar{})$ is locally left exact left adjoint to i . Observe that i is really a lax morphism (see the above counterexample), while $(\bar{})$ is a strict one (homomorphism). The induced change of base $\mathbf{B}\text{-Cat} \rightarrow \bar{\mathbf{B}}\text{-Cat}$ is also denoted by $(\bar{})$.

The result of [1] motivates the following definitions:

DEFINITION 1. Let $\mathbf{B} \rightarrow \bar{\mathbf{B}}$ be a closure operator and X a $\mathbf{B}$ -category. A bimodule $R: Y \dashrightarrow X$ covers X if $Y \subset \overline{R \cdot R^0}$ and $R^0 \cdot R \subset \bar{X}$.

The above definition amounts to require that $\bar{R}: \bar{Y} \dashrightarrow \bar{X}$ has a right adjoint in $\bar{\mathbf{B}}\text{-Cat}$.

DEFINITION 2. Let X be a $\mathbf{B}$ -category. X is a *sheaf* if each covering $R: Y \dashrightarrow X$ is representable by a functor.

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