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QUOTIENT SYSTEMS IN GROTHENDIECK TOPOI

by Kimmo I. ROSENTHAL

In this paper, we define a method for constructing Grothendieck topoi from a given Grothendieck topos $\mathcal{E}$. If $\underline{C}$ is a set of generators of $\mathcal{E}$, we define the notion of a quotient system $\mathfrak{D}$ on $\underline{C}$ and construct a topos $\mathcal{E}_{\mathfrak{D}}$ of $\mathfrak{D}$-generated objects of $\mathcal{E}$. There is a geometric morphism $\mathcal{E} \to \mathcal{E}_{\mathfrak{D}}$ which is hyperconnected, and the results of Johnstone [4] are helpful in analyzing the quotient system. On the other hand, quotient systems provide some insight into how hyperconnected geometric morphisms can arise and we investigate the connection between the two concepts. This work is an outgrowth of a suggestion by Lawvere [7] about unifying certain constructions involving G-sets, where G is a group or a monoid, and as examples of topoi of the form $\mathcal{E}_{\mathfrak{D}}$ we obtain the topos of G-sets with finite orbits and the topos of continuous G-sets, where G is a topological group. Finally, we make the observation that any Grothendieck topos $\mathcal{F}$ is equivalent to one of the form $\mathcal{E}_{\mathfrak{D}}$, where $\mathcal{E}$ is an étendue (i.e. if $\mathcal{E}$ has enough points, $\mathcal{E}$ is equivalent to a topos of G-sheaves, where G is an étale topological groupoid [8]).

I would like to thank Bill Lawvere for some helpful comments, Julian Cole for several illuminating conversations and Peter Johnstone for sending me copies of his work.

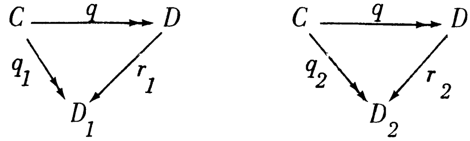
1. QUOTIENT SYSTEMS.

Let $\mathcal{E}$ be a Grothendieck topos and $\underline{C}$ be a set of generators of $\mathcal{E}$. We begin with a series of definitions.

DEFINITION 1.1. A *$\underline{C}$-system of quotients* $\mathfrak{D}$ consists of the following: for every $C \in \underline{C}$, we have a family of quotients D_C such that

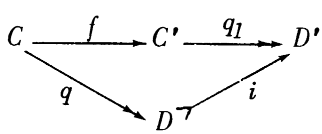
(1) If $q_1: C \twoheadrightarrow D_1$, $q_2: C \twoheadrightarrow D_2$ are in D_C, there is $q: C \twoheadrightarrow D$

in D_C and morphisms $r_1 : D \to D_1$, $r_2 : D \to D_2$ such that



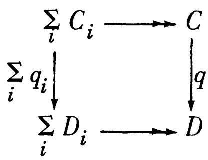
commute.

(2) If $C, C' \in \underline{C}$, $f: C \to C'$ is a morphism in $\underline{C}$ and $q_1 : C' \twoheadrightarrow D'$ is in $D_{C'}$, and if we take the image factorization



then $q : C \twoheadrightarrow D$ is in D_C.

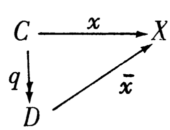
(3) If $\{C_i \to C\}_{i \in I}$ is a covering of C for the canonical topology in $\mathfrak{E}$ and if $q_i : C_i \twoheadrightarrow D_i$ is in D_{C_i} for all i, if we form the pushout



then $q: C \twoheadrightarrow D$ is in D_C.

DEFINITION 1.2. A *quotient system* in $\mathfrak{E}$ is given by a set of generators $\underline{C}$ and a $\underline{C}$-system of quotients $\mathfrak{D}$. (Remark: For most of our examples, we shall only need conditions (1) and (2) of Definition 1.1.)

DEFINITION 1.3. Let $C \in \underline{C}$ and $x: C \to X$ be a morphism in $\mathfrak{E}$. We say *x is a $\mathfrak{D}$-element of X* if there is $q: C \twoheadrightarrow D$ in D_C and $\bar{x}: D \to X$ such that



commutes.

DEFINITION 1.4. An object X of $\mathfrak{E}$ is called *$\mathfrak{D}$-generated* if for every morphism $x: C \to X$ with $C \in \underline{C}$, x is a $\mathfrak{D}$-element of X.

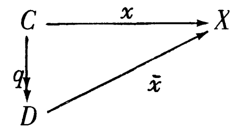
Let $\mathfrak{E}_{\mathfrak{D}} = \{ X \in \mathfrak{E} \mid X$ is $\mathfrak{D}$-generated$\}$. We make this into a category by taking our morphisms to be morphisms in $\mathcal{E}$. We shall show that $\mathcal{E}_{\mathfrak{D}}$ is a topos and we have a surjection of topoi $\mathcal{E} \to \mathcal{E}_{\mathfrak{D}}$ (which is in fact a hyperconnected geometric morphism).

By Giraud's Theorem [2, Chapter 0] we have an inclusion of topoi $i : \mathcal{E} \to \mathcal{S}^{\underline{C}^{op}}$ with $\mathcal{E}$ equivalent to the topos $sh(\underline{C}; J)$ where J is the canonical Grothendieck topology on $\underline{C}$. Using this representation, we shall define a left exact cotriple $G : \mathcal{E} \to \mathcal{E}$ and conclude that $\mathcal{E}_{\mathfrak{D}} \approx \mathcal{E}_G$, the category of G-coalgebras, and hence is a topos.

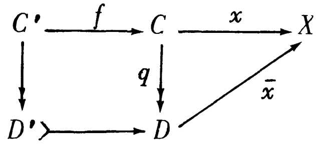
Define a functor $\phi : \mathcal{E} \to \mathcal{S}^{\underline{C}^{op}}$ as follows. If $X \in \mathcal{E}$, $C \in \underline{C}$, let

$$\phi(X)(C) = \{ x : C \to X \mid x \text{ is a } \mathfrak{D}\text{-element of } X \}.$$

$\phi(X)$ is a contravariant functor for if $f : C' \to C$ is a morphism in $\underline{C}$ and $x \in \phi(X)(C)$, suppose $q : C \twoheadrightarrow D$ in $\mathfrak{D}$ and $\bar{x} : D \to X$ are such that



By (2) in Definition 1.1, $im(q \cdot f) = C' \twoheadrightarrow D'$ is in $D_{C'}$. So we have

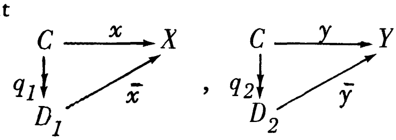


Thus, $x . f \in \phi(X)(C')$ and this is functorial.

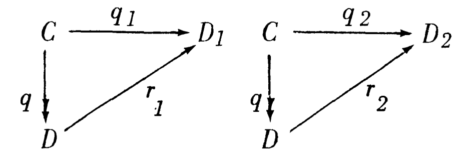
LEMMA 1.5. $\mathcal{E} \to \mathcal{S}^{\underline{C}^{op}}$ *is left exact.*

PROOF. We must show that ϕ preserves products and equalizers. Let X and Y be in $\mathcal{E}$. It is clear that $\phi(X \times Y) \rightarrowtail \phi(X) \times \phi(Y)$, since a $\mathfrak{D}$-element $C \to X \times Y$ gives rise to $\mathfrak{D}$-elements $C \to X$, $C \to Y$ by composition with the projections. We must show the opposite inclusion.

Suppose $x : C \to X$, $y : C \to Y$ are $\mathfrak{D}$-elements of X and Y respectively. Then, there exist $q_1 : C \twoheadrightarrow D_1$, $q_2 : C \twoheadrightarrow D_2$ in D_C and $\bar{x} : D_1 \to X$, $\bar{y} : D_2 \to Y$ such that



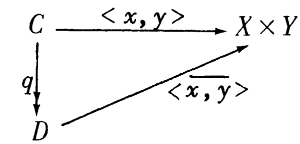
commute. By (1) in Definition 1.1, choose $q: C \twoheadrightarrow D$ in D_C such that there exist $r_1: D \to D_1$, $r_2: D \to D_2$ with



commuting. Define $\overline{\langle x, y\rangle}: D \to X \times Y$ by

$$D \xrightarrow{r_1} D_1 \xrightarrow{\bar{x}} X \quad \text{and} \quad D \xrightarrow{r_2} D_2 \xrightarrow{\bar{y}} Y.$$

Then

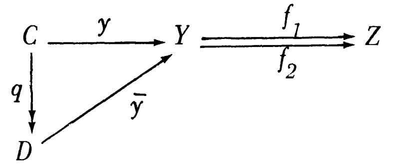


commutes and thus ϕ preserves products.

To show that ϕ preserves equalizers, suppose that

$$X \rightarrowtail Y \underset{f_2}{\overset{f_1}{\rightrightarrows}} Z$$

is an equalizer diagram in $\mathcal{E}$. Let $y: C \to Y$ be a $\mathfrak{D}$-element of Y with $f_1 y = f_2 y$. Since y is a $\mathfrak{D}$-element of Y there is $q: C \twoheadrightarrow D$ in D_C and $\bar{y}: D \to Y$ such that



commutes.

Since $f_1 \bar{y} q = f_2 \bar{y} q$ and q is epi, it follows that $f_1 \bar{y} = f_2 \bar{y}$. Thus, $\bar{y}: D \to Y$ factors through $X \rightarrowtail Y$ and so does $y: C \to Y$, hence it is a $\mathfrak{D}$-element of X. This shows that ϕ preserves equalizers, and completes the proof for left exactness.

LEMMA 1.6. *If $X \in \mathcal{E}$, then $\phi(X)$ is a sheaf for the canonical topology on C.*

PROOF. This will follow directly from condition (3) in Definition 1.1. Suppose $x_i: C_i \to X$ are $\mathfrak{D}$-elements of X where $\{C_i \to C\}_{i \in I}$ is a cover for the canonical topology. Then, there are $q_i : C_i \twoheadrightarrow D_i$ in D_{C_i}, and $\bar{x}_i : D_i \to X$ for all i with

$$\begin{array}{ccc} C_i & \xrightarrow{x_i} & X \\ {\scriptstyle q_i}\downarrow & \nearrow{\scriptstyle \bar{x}_i} & \\ D_i & & \end{array}$$

commuting. Since X is a sheaf for the canonical topology there is a unique $x: C \to X$ such that

$$\begin{array}{ccc} C_i & \xrightarrow{x_i} & X \\ \downarrow & \nearrow{\scriptstyle x} & \\ C & & \end{array}$$

commutes for all i. By (3) in Definition 1.1, if we form the pushout

$$\begin{array}{ccc} \Sigma C_i & \twoheadrightarrow & C \\ {\scriptstyle \Sigma q_i}\downarrow & & \downarrow{\scriptstyle q} \\ \Sigma D_i & \twoheadrightarrow & D \end{array}$$

then $q: C \twoheadrightarrow D$ is in D_C. It follows that there is $\bar{x}: D \to X$ with

$$\begin{array}{ccc} C & \xrightarrow{x} & X \\ {\scriptstyle q}\downarrow & \nearrow{\scriptstyle \bar{x}} & \\ D & & \end{array}$$

commuting. Thus, $\phi(X)$ is a sheaf for the canonical topology on $\underline{C}$.

We are now ready to define our cotriple $G: \mathfrak{E} \to \mathfrak{E}$, whose coalgebras will preceisely be the $\mathfrak{D}$-generated objects of $\mathfrak{E}$. Let $G: \mathfrak{E} \to \mathfrak{E}$ be the functor defined by $GX = i^*(\phi(X))$ for $X \in \mathfrak{E}$, where we recall that $i: \mathfrak{E} \to \mathfrak{S}^{\underline{C}^{op}}$ is the canonical inclusion given by Giraud's Theorem.

PROPOSITION 1.7. *If* $X \in \mathfrak{E}$, $GX \rightarrowtail X$ *is the largest* $\mathfrak{D}$*-generated subobject of* X.

PROOF. To see that GX is $\mathfrak{D}$-generated, let $x: C \to GX$ be a morphism in $\mathfrak{E}$ with $C \in \underline{C}$. By Lemma 1.6, $i_*(GX) \approx \phi(X)$, since $\phi(X)$ is a sheaf and it follows that

$$C \xrightarrow{x} GX \rightarrowtail X$$

is in $\phi(X)(C)$. Thus, there is $q: C \to D$ in D_C and $\bar{x}: D \to X$ with

$$\begin{array}{ccccc} C & \xrightarrow{x} & GX & \rightarrowtail & X \\ {\scriptstyle q}\downarrow & & & \nearrow_{\bar{x}} & \\ D & & & & \end{array}$$

commuting. But, $\bar{x}$ must factor through $GX \rightarrowtail X$. So, GX is $\mathfrak{D}$-generated. If $Y \rightarrowtail X$ is another $\mathfrak{D}$-generated subobject, then $\phi(Y) \approx i_*(Y)$. Since ϕ is left exact, $\phi(Y) \rightarrowtail \phi(X)$. So,

$$Y \approx i^*(i_*(Y)) \rightarrowtail i^*(\phi(X)) \approx GX.$$

THEOREM 1.8. *$G: \mathcal{E} \to \mathcal{E}$ is a left exact cotriple and $\mathcal{E}_G$, the category of G-coalgebras, is precisely $\mathcal{E}_{\mathfrak{D}}$, the category of $\mathfrak{D}$-generated objects. Hence $\mathcal{E}_{\mathfrak{D}}$ is a topos and there is a surjection of topoi $\mathcal{E} \to \mathcal{E}_{\mathfrak{D}}$.*

PROOF. Since $G \approx i^* \circ \phi$, G is left exact, as both i^* and ϕ are. That G is a cotriple follows readily from Proposition 1.7. There is a monomorphism $G \rightarrowtail I$ where I is the identity functor, and G is idempotent, since $G(GX)$ must be the largest $\mathfrak{D}$-generated subobject of GX. It readily follows that X is a G-coalgebra iff $GX = X$ and thus the G-coalgebras and $\mathfrak{D}$-generated objects coincide. Thus, [2, Chapter 2], $\mathcal{E}_{\mathfrak{D}}$ is a topos and there is a surjection $\mathcal{E} \to \mathcal{E}_{\mathfrak{D}}$ of topoi.

REMARKS. (1) One should note the dual nature of the notions of quotient system $\mathfrak{D}$ and $\mathfrak{D}$-generated object with those of j-dense monics for a topology j and j-sheaf.

(2) Condition (3) in Definition 1.1 is not always necessary, for example if $\mathcal{E} \approx \mathcal{S}^{\underline{C}^{op}}$. In those cases, it is easy to verify whether the conditions for a quotient system are satisfied.

(3) If J is any subcanonical topology on C, condition (3) of Definition 1.1 can be modified to deal with J-coverings which will make our $\mathfrak{D}$-quotients dual in some sense to J-coverings.

(4) The concept of $\mathfrak{D}$-generated object is a simple and natural one. It would make sense in categories other than topoi and is essentially purely algebraic not depending on the higher order structure of topoi.

EXAMPLES. 1. Let $\underset{\sim}{G}$ be a group. We look at $\mathcal{E} = \mathcal{S}^{\underset{\sim}{G}^{op}}$, the topos of $\underset{\sim}{G}$-sets. We can take $\{\underset{\sim}{G}\}$ to be our generating set and if we take all finite $\underset{\sim}{G}$-quotients of $\underset{\sim}{G}$, we have a quotient system $\mathfrak{D}$ and $\mathcal{E}_{\mathfrak{D}}$ is the topos of $\underset{\sim}{G}$-sets with finite orbits.

2. Let $\mathcal{E}$ be the topos of sets equipped with an endomorphism, which is equivalent to the topos $\mathcal{S}^{N^{op}}$, where N is the monoid of natural numbers. Let $\{N, \sigma\}$ be our generating set, where $\sigma: N \to N$ is the successor function. Define quotients of (N, σ) as follows. Let (S_n, ρ_n) be the object where $S_n = \{0, 1, 2, \ldots, n-1\}$ and

$$\rho_n(k) = k+1 \text{ for } k \leq n-1 \quad \text{and} \quad \rho_n(n-1) = n-1.$$

Define $q_n : N \to S_n$ by

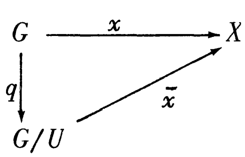
$$q_n(k) = k \text{ for } k \leq n-1, \quad q_n(k) = n-1 \text{ for } k > n-1.$$

Then, it is not hard to see that $\{q_n : (N, \sigma) \to (S_n, \rho_n) \mid n \in N\}$ form a quotient system $\mathfrak{D}$ in $\mathcal{E}$ and $\mathcal{E}_{\mathfrak{D}}$ is the topos of sets equipped with a locally eventually constant endomorphism, i. e. $(X, \tau) \in \mathcal{E}_{\mathfrak{D}}$ iff $\tau : X \to X$ and

$$\forall x \in X, \ \exists n \in N \text{ such that for all } k \geq n, \ \tau^k(x) = \tau^n(x).$$

(This example was first observed by F. W. Lawvere.)

3. Let G be a topological group and let $\mathcal{E} = \mathcal{S}^{G^{op}}$. It is well known that given a G-set X, the action of $\underset{\sim}{G}$ on X is continuous with respect to the topology on G iff for every $x \in X$, the isotropy group of x in G is an open subgroup. Let $x: G \to X$ be an element of X. If U is an open subgroup of G and we have a diagram



commuting in $\mathcal{E}$, then U must be contained in the isotropy group of x. Since the identity is in U, it follows that the isotropy group must be open. Let $\{G \to G/U \mid U \text{ an open subgroup of } G\}$ be our quotient system (conditions (1) and (2) of Definition 1.1 are easily verified). Then, $\mathcal{E}_{\mathfrak{D}}$ from the above reasoning must be the topos of continuous G-sets, $\mathcal{C}(G)$.

4. Let $\mathcal{E}$ be a Grothendieck topos and $\underline{C}$ be a set of generators. Given $C \in \underline{C}$, let D_C be all quotients of C, which are quotients of decidable objects. Since quotients of decidable objects are closed under subobjects, finite products, quotients and coproducts [5], Conditions (1), (2) and (3) of Definition 1.1 are readily verified and we have a quotient system $\mathfrak{D}$. The topos $\mathcal{E}_{\mathfrak{D}}$ is the topos Johnstone has denoted $\mathcal{E}_{qd}$, the topos of quotients of decidable objects. (Again, see [5].)

Examples such as this will become clear in the next section as we establish the connection between quotient systems and hyperconnected geometric morphisms.

2. QUOTIENT SYSTEMS AND HYPERCONNECTED GEOMETRIC MORPHISMS.

The important class of hyperconnected geometric morphisms was introduced in [3] and studied extensively in [4] by P. Johnstone. They are orthogonal to the localic geometric morphisms and together with these provide a factorization system which is stable under pullback along bounded geometric morphisms. The geometric morphism $\mathcal{E} \to \mathcal{E}_{\mathfrak{D}}$ in Section 1 is hyperconnected and in this section we investigate the connection between these morphisms and quotient systems.

DEFINITION 2.1. A geometric morphism $f: \mathcal{F} \to \mathcal{E}$ is *hyperconnected* iff any of the following equivalent conditions holds:

(1) f^* is full and faithful, and its image is closed under subobjects in $\mathcal{F}$.

(2) f^* is full and faithful, and its image is closed under quotients in $\mathcal{F}$.

(3) The unit and co-unit of the adjunction $(f^* \dashv f_*)$ are both mono.

(4) f_* preserves Ω, i.e. the comparison map for $f_*(\Omega_{\mathcal{F}}) \to \Omega_{\mathcal{E}}$ is an isomorphism.

For a proof of the equivalence of these conditions, see [4, Proposition 1.5]. From (3), since the unit of the adjunction is mono, we have a surjection of topoi. Since every surjection is equivalent to one given by a left exact cotriple [2; 4], we see from (3) that a hyperconnected geometric morphism corresponds to a left exact cotriple with monic co-unit. We shall phrase our discussion in terms of cotriples, since there is a ready connection between coalgebras and $\mathfrak{D}$-generated objects.

DEFINITION 2.2. Let $\mathfrak{E}$ be a Grothendieck topos. Let $G: \mathfrak{E} \to \mathfrak{E}$ be a left exact cotriple such that for every $X \in \mathfrak{E}$, the co-unit map $GX \rightarrowtail X$ is mono. We shall call G *an interior operator on* $\mathfrak{E}$

REMARK. The terminology follows from the fact that if X is a set and we consider the poset 2^X, then the left exact cotriples on 2^X correspond precisely to interior operators taking «interior» in the topological sense.

Thus, hyperconnected geometric morphisms are equivalent to interior operators.

PROPOSITION 2.3. *Let* $G: \mathfrak{E} \to \mathfrak{E}$ *be an interior operator.*

(1) $X \in \mathfrak{E}$ *is a G-coalgebra iff* $GX = X$.

(2) Every subobject of a G-coalgebra is a G-coalgebra.

(3) Every quotient of a G-coalgebra is a G-coalgebra.

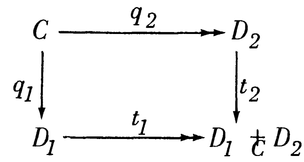
(4) An arbitrary coproduct of G-coalgebras is a G-coalgebra.

PROOF. (1) follows immediately since $GX \rightarrowtail X$ is mono. (2) and (3) are merely restatements of Definition 2.1, (2) and (3). For (4), if $G(X_i) = X_i$ for $i \in I$, the maps $X_i \to \sum_i X_i$ induce a map $\sum_i X_i \to G(\sum_i X_i)$ and since $G(\sum_i X_i) \rightarrowtail \sum_i X_i$ we are done.

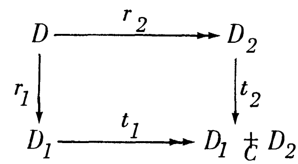
We have seen that a quotient system $\mathfrak{D}$ on $\mathfrak{E}$ gives rise to an interior operator (i.e. hyperconnected geometric morphism). Now, we shall show that given a choice of generators for $\mathfrak{E}$, the converse holds.

THEOREM 2.4. *Let* $\mathfrak{E}$ *be a Grothendieck topos and* $\underline{C}$ *a set of generators for* $\mathfrak{E}$. *If* $G: \mathfrak{E} \to \mathfrak{E}$ *is an interior operator, then the G-coalgebra quotients of objects in* $\underline{C}$ *form a quotient system in* $\mathfrak{E}$.

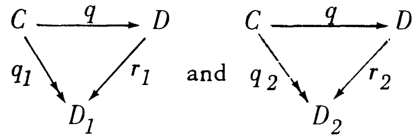
PROOF. We must check that the conditions of Definition 1.1 are satisfied. Suppose $q_1 : C \twoheadrightarrow D_1$, $q_2: C \twoheadrightarrow D_2$ are epimorphisms with $C \in \underline{C}$ and D_1, D_2 are G-coalgebras. Form the pushout in $\mathfrak{E}$



Since pushouts of epis are epis, from Proposition 2.3 (3), it follows that $D_1 +_C D_2$ is a coalgebra. Now, from the pullback



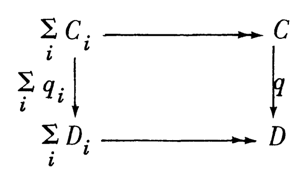
D is a G-coalgebra, and from the commutativity of the pushout diagram and properties of pullbacks, it follows that there is a map $q : C \to D$ such that



If $q : C \to D$ is not epi, take the image of q. It is a subobject of D and hence by Proposition 2.3 a G-coalgebra.

For the second condition, if $f: C \to C'$ and $q': C' \twoheadrightarrow D'$ is epi with D' a G-coalgebra, then the image of q' is a subobject of D' and hence a G-coalgebra quotient of C, again by Proposition 2.3.

For the last condition of Definition 1.1, suppose $\{C_i \to C\}_{i \in I}$ is a canonical cover of C and $q_i: C_i \twoheadrightarrow D_i$ are G-coalgebra quotients of C_i for $i \in I$. Then, $\sum_i D_i$ is a G-coalgebra by Proposition 2.3 and when we form the pushout

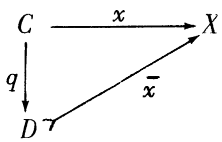


then D is a G-coalgebra being a quotient of $\sum_i D_i$ and hence $q: C \twoheadrightarrow D$ is a G-coalgebra quotient of C.

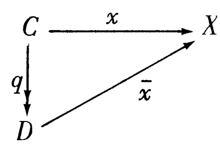
So, every interior operator on a Grothendieck topos $\mathcal{E}$ induces a system of quotients $\mathfrak{D}$ on $\mathfrak{E}$. Now, we shall show that the $\mathfrak{D}$-generated objects coincide with the G-coalgebras.

THEOREM 2.5. *Let $G : \mathfrak{E} \to \mathfrak{E}$ be an interior operator and let $\mathfrak{D}$ be the system of quotients it induces on a set of generators $\underline{C}$ of $\mathfrak{E}$. Then, the topos $\mathfrak{E}_{\mathfrak{D}}$ of $\mathfrak{D}$-generated objects is precisely the topos $\mathfrak{E}_G$ of G-coalgebras.*

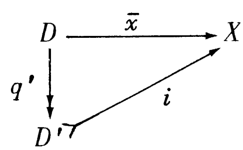
PROOF. Suppose X is a G-coalgebra. Let $C \in \underline{C}$ and $x: C \to X$ be a morphism in $\mathfrak{E}$. If we take the image factorization



then $\bar{x}: D \rightarrowtail X$ is a G-coalgebra since they are closed under subobjects, and hence X is $\mathfrak{D}$-generated. Conversely, suppose X is $\mathfrak{D}$-generated. To show that X is a G-coalgebra, since G is an interior operator it suffices to show that $GX = X$. Suppose $GX \rightarrowtail X$ is not equal to X. Since $\underline{C}$ is a set of generators, choose $x: C \rightarrowtail X$ such that x does not factor through $GX \rightarrowtail X$. Since X is $\mathfrak{D}$-generated, there is a G-coalgebra quotient $q: C \twoheadrightarrow D$ of $\underline{C}$ and $\bar{x}: D \to X$ such that



commutes. If we take the image factorization



then D' is a G-coalgebra, being a quotient of D and hence is a G-coalgra subobject of X. Since G is left exact and idempotent, it follows that $GX \rightarrowtail X$ is the largest G-coalgebra subobject of X, hence $i : D' \rightarrowtail X$ factors through $GX \rightarrowtail X$. Therefore, so does $x: C \to X$. This is contradiction. Hence, $GX = X$ and X is a G-coalgebra.

From these results, we see that the concepts of quotient system $\mathfrak{D}$ and $\mathfrak{D}$-generated object provide another perspective on hyperconnected geometric morphisms. If $f\colon \mathcal{E} \to \mathcal{F}$ is hyperconnected with corresponding interior operator $f^* \circ f_*\colon \mathcal{E} \to \mathcal{E}$, the above method of obtaining a quotient system on $\mathcal{E}$ is by no means always the most efficient.

For instance in [4] it was shown that if a functor $\alpha\colon \underline{C} \to \underline{B}$ between small categories is full and essentially surjective on objects, then the induced geometric morphism $\mathcal{S}^{\underline{C}^{op}} \to \mathcal{S}^{\underline{B}^{op}}$ is hyperconnected. So, $\mathcal{S}^{\underline{B}^{op}} \approx (\mathcal{S}^{\underline{C}^{op}})_{\mathfrak{D}}$ for a quotient system $\mathfrak{D}$. In fact, we can choose $\mathfrak{D}$ so that $D_{\hat{C}}$ consists of a single quotient for each $C \in \underline{C}$ (where $\hat{C}$ denotes the associated representable functor). If $f, g\colon C' \to C$ are maps in $\underline{C}$, say $f \sim g$ iff $\alpha(f) = \alpha(g)$. Let $[f]$ denote the equivalence class of such an f. For $C \in \underline{C}$, define a quotient $\hat{C} \twoheadrightarrow D$ in $\mathcal{S}^{\underline{C}^{op}}$ by

$$D(C') = \{\, [f] \mid f\colon C' \to C \,\} \quad \text{for } C' \in \underline{C}.$$

It is not hard to see that this is a quotient system and that

$$(\mathcal{S}^{\underline{C}^{op}})_{\mathfrak{D}} \approx \mathcal{S}^{\underline{B}^{op}}.$$

3. QUOTIENT SYSTEMS, ETENDUES AND THE DIACONESCU COVER.

In this section, we wish to briefly argue that given a Grothendieck topos $\mathcal{E}$, there is an étendue $\mathcal{E}'$ (for a discussion of étendues, see [8]) and a quotient system $\mathfrak{D}$ on $\mathcal{E}'$ such that $\mathcal{E} \approx \mathcal{E}'_{\mathfrak{D}}$.

In [1], Diaconescu offered a simple proof of Barr's Theorem that every Grothendieck topos $\mathcal{E}$ admits a surjection from a topos of the form $sh(\underline{B})$, where $\underline{B}$ is a complete Boolean algebra. As an initial step, Diaconescu constructed a poset $\underline{D}$ from a set of generators $\underline{C}$ of $\mathcal{E}$ and obtained a pullback diagram

$$\begin{array}{ccc} sh(\underline{H}) & \longrightarrow & \mathcal{S}^{\underline{D}^{op}} \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow{\;i\;} & \mathcal{S}^{\underline{C}^{op}} \end{array}$$

where i is the canonical inclusion, $\underline{H}$ is a Heyting algebra and p is an open surjection (see [6], [9] for discussions of open surjections). In [8] it was observed that p factors through an étendue $\mathcal{E}'$. If $\tilde{\underline{C}}$ denotes the free category generated by the graph $\underline{C}$, then the poset $\underline{D}$ is $\tilde{\underline{C}}/|\underline{C}|$ and we have

$$\begin{array}{ccc}
sh(\underline{H}) & \longrightarrow & \mathcal{S}^{\underline{D}^{op}} \\
\downarrow & & \downarrow \\
\mathcal{E}' & \longrightarrow & \mathcal{S}^{\tilde{\underline{C}}^{op}} \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle f} \\
\mathcal{E} & \xrightarrow{\quad i \quad} & \mathcal{S}^{\underline{C}^{op}}
\end{array}$$

Since maps in $\tilde{\underline{C}}$ are monic, $\mathcal{S}^{\tilde{\underline{C}}^{op}}$ is an étendue and hence so is $\mathcal{E}'$ [8], and $p : \mathcal{E}' \to \mathcal{E}$ is again an open surjection. We would in fact like to show that p' is hyperconnected. This follows easily. From [4; Proposition 2.3], we know that pullback along a bounded morphism preserves hyperconnectedness, so it suffices to see that $f\colon \mathcal{S}^{\tilde{\underline{C}}^{op}} \to \mathcal{S}^{\underline{C}^{op}}$ is hyperconnected. This is clearly true from the discussion following Theorem 2.5 about functor categories. Thus, we have the following equivalent statements:

THEOREM 3.1. *(1) Given a Grothendieck topos $\mathcal{E}$, there is a hyperconnected geometric morphism $f\colon \mathcal{E}' \to \mathcal{E}$, where $\mathcal{E}'$ is an étendue.*

(2) Given a Grothendieck topos $\mathcal{E}$, there is an étendue $\mathcal{E}'$ and a quotient system $\mathfrak{D}$ on $\mathcal{E}'$ such that $\mathcal{E} \approx \mathcal{E}'_{\mathfrak{D}}$.

Every étendue with enough points is equivalent to a topos $sh(X; G)$ of G-sheaves where G is an étale topological groupoid on a topological space X and these topoi have a natural set of generators [8]. These topoi are geometrically interesting and an investigation of the possible quotient systems on these generators should shed light on exactly how an arbitrary Grothendieck topos with enough points is obtainable from G-sheaves.

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