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A CARTESIAN CLOSED EXTENSION OF A CATEGORY OF AFFINE SCHEMES

by Paul CHERENACK

The main result shows that there is a slight extension *ind-aff* of the category *aff* of reduced affine schemes of countable type over a field k which is cartesian closed. Objects which correspond to the jet spaces of Ehresmann [5] but in the context of affine schemes are employed (Section 4) to define the internal hom-functor in *ind-aff*. In addition we show that *ind-aff* is countable complete and cocomplete. Certain commutation properties for the inductive limits which define the objects of *ind-aff* are derived. Using the internal hom-functor in *ind-aff* one can place a topology on the collection of all scheme maps between two affine schemes X and Y. Then under certain restrictions the scheme maps $f: X \to Y$ which are transversal to a closed subscheme W of Y are shown to form a constructible subset of the collection of all scheme maps from X to Y. The methods used here show how one might begin to extend results (see [6]) on transversality for smooth mappings between differentiable manifolds to the setting of affine schemes.

Let $k^{\mathrm{N}} = Spec(k[X_1, \dots, X_n, \dots])$ where $k[X_1, \dots, X_n, \dots]$ is the polynomial ring in a countable number of variables. All objects in *ind-aff* can be identified as we shall see with subsets of k^{N}. They will have the topology inherited from their structure as ind. lim. of objects of *aff* in the category of ringed spaces. The topology is not necessarily that induced from k^{N}. See Example 1.11.

The meaning of cartesian closedness for a category is found in Definition 3.4.

Objects of *ind-aff* are called ind-affine schemes.

To avoid confusion a closed set will always be a closed subset of k^{N} and the closure of a set will be the smallest closed subset of k^{N} containing it unless one specifies that that set is closed in some ind-affine scheme and thus not necessarily in k^{N}.

We say that *ind-aff* is a slight extension of *aff* since the objects of *ind-aff* form (as one can see using Proposition 1.4) the smallest collection of subobjects of k^{N} containing all linear (see Definition 1.3) and closed subsets of k^{N} and intersections of these. Compare this to the somewhat larger extension of Demazure and Gabriel [4], page 63.

Let $X \subset k^{\mathrm{N}}$. $X(k)$ denotes the set of k valued points of X. A morphism $f: X(k) \to Y(k)$ is a tuple $(f_n)_{n \in Z}$ where f_n is a polynomial in $k[X_1, \ldots, X_n, \ldots]$. Let $\underline{V}$ be the category of all $X(k)$ with X in *aff* and morphisms between such objects. Then, if k is algebraically closed and uncountable since the Hilbert Nullstellensatz holds for affine rings $k[X]$, X an object of *aff* (see Lang [10]), $\underline{V}$ will be isomorphic to *aff* and hence $\underline{V}$ also has a slight extension which is cartesian closed. Otherwise one must make some suitable adjustment of $\underline{V}$.

We mention briefly one of the possible applications of the theory developed here. Using the cartesian closedness of *ind-aff* and supposing

$$Y^X : \textit{ind-aff} \times \textit{ind-aff} \longrightarrow \textit{ind-aff}$$

is the internal hom-functor in *ind-aff*, one can form the loop functor

$$E : \textit{ind-aff} \longrightarrow \textit{ind-aff}$$

by setting $E(X) = X^k \cap F$ where F is the closed subscheme of k^{N} provided by the condition that the basepoint $* \in k$ is mapped to the basepoint $*$ of X. As *ind-aff* has coequalizers, one can form the cone functor

$$C : \textit{ind-aff} \longrightarrow \textit{ind-aff}$$

by letting $C(X)$ be the coequalizer of the maps

$$i, * : (\{*\} \times X) \cup (k \times \{*\}) \rightrightarrows k \times X,$$

where i is the inclusion and $*$ maps everything to the basepoint $(*, *)$ of $k \times X$. Then one can show (adding basepoints) that C is left adjoint to E

and thus associate to C or E homotopy groups $\Pi_n(X, Y)$. For the details of this construction see Huber [9]. The direction in which one might want to take this theory can be seen in [2].

We outline the paper. In Section 1 we provide in Proposition 1.4 three different descriptions of the objects of *ind-aff* as subsets of k^{N}. We use the more convenient description as required. The objects of *ind-aff* are then given the structure of ringed spaces and the mappings between them are described. A map $f: X \to Y$ in *ind-aff* is required to preserve the filtration that X and Y have as objects in *ind-aff*. We show (Proposition 1.10) that this is not a severe restriction. In section 2 we show that *ind-aff* has countable limits and coproducts in a fairly straightforward way. In Section 3 we show that an external hom-functor on *aff* to *ind-aff* (see the first paragraph of 3 for the definition of this concept) can be extended to an internal hom-functor on *ind-aff* provided that the inductive limits defining objects of *ind-aff* satisfy certain commutativity relations. The existence of an external hom-functor on *aff* to *ind-aff* is demonstrated in Section 4. The necessary commutativity relations are to be found in Section 5. The reason for proving the results in Section 3 first is to emphasize that the methods appearing here might be used to form a cartesian closed category in a more general context. Categorical methods have made this a more concise paper.

Let X, Y, W be non-singular affine irreducible schemes of finite types over k, let W be parallelisable in Y and X be parallelisable (see Section 6 for definitions). $Y^X(k)$ is identified with the scheme maps $f: X \to Y$. Let k be an algebraically closed field. Then in Section 6 we demonstrate that the set T_W of all f such that f is transversal to W is constructible in $Y^X(k)$. $Y^X(k)$ can be viewed as the directed union of certain closed algebraic subvarieties $A_{\underline{r}}$ of $k^{\mathrm{N}}(k)$. Let $E \cap A_{\underline{r}}$ denote the set of maps in $A_{\underline{r}}$ which extend (see Section 6) to scheme maps from the projective model of X to that of Y. Then with some limitation we show that $T_W \cap E \cap A_{\underline{r}}$ contains an open subset of $E \cap A_{\underline{r}}$.

For schemes the reader might refer to [7, 8] ; for category theory to [11] ; for notions of transversality to [6].

For further applications of ind-affine schemes, see [3].

1. DEFINITION OF *ind-aff*.

We present a definition of ind-affine schemes and then derive some properties of ind-affine schemes.

DEFINITION 1.1. By a *closed linear subscheme of*

$$k^{\mathrm{N}} = Spec(k[X_1, \dots, X_n, \dots])$$

we mean a closed subscheme of k^{N} whose defining ideal is generated by linear polynomials in $k[X_1, \dots, X_n, \dots]$.

Let X be a subset of k^{N} and let $Y \subset X$ be a closed subset of k^{N} for which there is a minimal closed linear subscheme H_Y of k^{N} such that $Y = X \cap H_Y$. Note that the existence of a minimal H_Y follows from the existence of one H_Y. Let T_X denote a maximal subset of the set consisting of all such H_Y with $H_X = \cup H\,(H \in T_X)$ a fixed set.

DEFINITION 1.2. We say that T_X *is directed* if $H, K \in T_X$ implies that for some $M \in T_X$ we have $H \cup K \subset M$.

DEFINITION 1.3. A subset H of k^{N} is said to be *full linear* if:

a) Let V be a closed linear subscheme of k^{N}. Then $V(k) \subset H$ implies $V \subset H$.

b) $H(k)$ is a vector subspace of $k^{\mathrm{N}}(k)$.

c) H is the union of closed linear subschemes.

PROPOSITION 1.4. *The following are equivalent:*

i) $\bar{X} \cap H_X = X$; T_X *is directed.*

ii) $\bar{X} \cap H = X \cap H$ *for* $H \in T_X$. $X \subset H_X$. T_X *is directed.*

iii) $X = X_1 \cap H_1$ *where* X_1 *is a closed affine subscheme and* H_1 *is a full linear subset of* k^{N}. T_X *is the set of* H_Y *such that* $H_Y \subset H_1$.

PROOF. i => ii: $\bar{X} \cap H = \bar{X} \cap (H_X \cap H) = X \cap H$.

ii => i: $\bar{X} \cap H_X = \bar{X} \cap (\cup H) = \cup(\bar{X} \cap H) = X$.

i => iii: As H_X is the union of closed linear subschemes in T_X and T_X is directed, $H_X(k)$ is a vector subspace of $k^N(k)$. Suppose that $H(k) \subset H_X$ where H is a closed linear subscheme. One can by Zorn's Lemma if T_X contains no maximal element restrict to the case where $\{L_i\}$ $(L_i \in T_X)$ is a countable family totally ordered by inclusion such that

$$H(k) = \cup((L_i \cap H)(k)).$$

Note that the linear polynomials in $k[X_1, \dots, X_n, \dots]$ form a vector space of countable dimension. But an easy argument then shows that $H(k) = (L_i \cap H)(k)$ for some i and thus $H(k) \subset L_i(k)$ for some i. But then $H \subset L_i \subset H_X$.

iii => ii: Every point $P \in X$ belongs to some closed linear subscheme contained in H_1. Let T_X consist of all closed linear subschemes H_Y contained in H_1. Then $X \subset H_X$. It is not difficult to show:

LEMMA 1.5. *If H, K are closed linear subschemes of k^N there is a closed linear subscheme $H+K$ containing H and K and such that*

$$(H+K)(k) = H(k) + K(k).$$

Let $H, K \in T_X$. Clearly $H, K \subset H_1$ and, as H_1 is full linear, $H+K \subset H_1$. There is a minimal closed linear subscheme $L \in T_X$ such that $L \cap X = (H+K) \cap X$. Clearly $L \supset H, K$. As every $H_Y \subset H_X$ is contained in H_1, T_X must be maximal. As every $H \in T_X$ is contained in H_1,

$$X \cap H = X \cap (H_1 \cap H) = (X \cap H_1) \cap H = (X_1 \cap H_1) \cap H =$$
$$= X_1 \cap H = \bar{X} \cap H. \qquad \text{Q.E.D.}$$

DEFINITION 1.6. An *ind-affine subset of k^N* is a subset X of k^N, satisfying any one of the equivalent conditions of Proposition 1.4.

REMARK. The collection of ind-affine subsets is closed under arbitrary intersection but the union of two full linear subschemes need not be an ind-affine subset.

The additional structure which makes an ind-affine subset X into a ringed space will now be introduced.

The set T_X can be viewed as a category where the arrows are inclusions. There is a functor $F_X : T_X \to Rngsp$ from T_X to the category of ringed spaces assigning the affine scheme $H \cap \bar{X}$ to H for each $H \in T_X$, and inclusions to inclusions. The inductive limit of F_X is a ringed space whose underlying set is X.

Note that one obtains the same inductive limit if one replaces T_X by the category whose objects are of the form $H \cap X$ $(H \in T_X)$ and arrows inclusions. Also there may be several T_X for the same X. Whether they define the same element of *Rngsp* is not clear. If X is expressed $X = X_1 \cap H_1$ as in Proposition 1.4, then T_X will be the collection of all H_Y which are closed linear subschemes contained in H_1. $H \cap \bar{X}$ is the affine scheme whose ideal is $A + B$ where A is the ideal of H and B is the ideal of $\bar{X}$. Thus $H \cap \bar{X}$ need not be reduced.

DEFINITION 1.7. An *ind-affine scheme* is a ringed space of the form: $limind F_X$. The category of ind-affine schemes (denoted *ind-aff*) consists of all ind-affine schemes together with morphisms $f: (X, \underline{O}_X) \to (Y, \underline{O}_Y)$ of ringed spaces which are induced from morphisms $k^N \to k^N$ in *aff*, the category of reduced affine schemes of countable type over k, and such that for $H \in T_X$ there is a $K \in T_Y$ such that $f(H \cap X) \subset K \cap Y$. We will usually write X instead of $(X, \underline{O}_X)$.

Let X be an ind-affine scheme. From the definition of $limind\, F_X$ it follows $Y \subset X$ is closed iff $Y \cap H$, $H \in T_X$ is a closed affine subset of k^N.

We will show that under certain weak conditions if $f: X \to Y$ is a map between two objects in *ind-aff* which is the restriction of a map between k^N in *aff*, then f is a map in *ind-aff*.

LEMMA 1.8. *Let Y be an ind-affine scheme, X an irreducible object in aff and $f: X \to Y$ the restriction of a map between k^N in aff. Then there is a $K \in T_Y$ such that $f(X) \subset Y \cap K$.*

PROOF. $X = \cup f^{-1}(Y \cap K)$ $(K \in T_X)$. As X is irreducible, one of the $f^{-1}(Y \cap K)$ contains the generic point of X and hence $X = f^{-1}(Y \cap K)$.

Q. E. D.

DEFINITION 1.9. An ind-affine scheme X is *irreducible* if for each $H \in T_X$ there is a $K \in T_X$ with $K \supset H$ and $X \cap K$ irreducible.

PROPOSITION 1.10. *Let $f: X \to Y$ be a set map between ind-affine schemes, which is the restriction of a map between k^{N} in aff, and X irreducible. Then f induces a map in ind-aff.*

PROOF. Let $H \in T_X$. There is a $K \in T_X$ such that $K \supset H$ with $X \cap K$ irreducible. Then Lemma 1.8 implies that $f(K \cap X) \subset Y \cap L$ for some $L \in T_Y$, and hence that $f(H \cap X) \subset Y \cap L$. Restricted to $H \cap X$, f is a map in *Rngsp* from $H \cap X$ to $Y \cap L$. Taking direct limits one obtains a map $f: X \to Y$ in *ind-aff*. Q.E.D.

EXAMPLE 1.11. The topology on $X \cap H$ need not be that induced from the Zariski topology on k^{N}. Consider $k^k(k)$ which is the set of all (a_i) with

$$a_i \text{ in } k, \quad i \in \mathrm{N} \quad \text{and} \quad a_i = 0 \text{ for all but finitely many } i.$$

Let $k^n(k)$ be the set of all (a_i) in $k^k(k)$ such that

$$a_i = 0 \quad \text{if} \quad i > n,$$

and $\Pi_n : k^k(k) \to k^n(k)$ the projection. Choose a subset $C = \{P_j\}_{j \in \mathrm{N}}$ of $k^k(k)$ such that $C \cap (k^n(k))$ consists of finitely many points and $\Pi_n(C)$ is dense in $k^n(k)$. It is easy to see that this is possible. Also every closed linear subset K of $k^n(k)$ is contained in $k^n(k)$ for some n (just consider the echelon form of the linear equations defining K). Hence by definition C is closed in $k^k(k)$ which has a topology as the inductive limit of the Zariski topologies on the $k^k(k) \cap K$. On the other hand as the closure of C in $k^{\mathrm{N}}(k)$ is $k^{\mathrm{N}}(k)$, C cannot be a closed subset of $k^k(k)$ for the topology induced from $k^{\mathrm{N}}(k)$. If k is the complex numbers, then clearly C will be closed for the inductive limit topology in k^k but not for the topology induced from k^{N}.

2. COUNTABLE LIMITS AND COLIMITS IN *ind-aff*.

We show that *ind-aff* has countable limits by showing that it has countable products and equalizers.

PROPOSITION 2.1. *ind-aff has countable products.*

PROOF. Let $\{X_i\}_{i\in\mathrm{N}}$ be objects in *ind-aff*. $X_i = \bar{X}_i \cap H_i$ where H_i is full linear and $\bar{X}_i$ is the closure of X_i in k^{N}. One considers (see Remark 2.6) $\times \bar{X}_i$ as a closed affine and $\times H_i$ as a full linear subset of $k^{\mathrm{N}\times\mathrm{N}} = Spec(k[X_i^j])$ where the X_i^j are indeterminates. But $k^{\mathrm{N}\times\mathrm{N}} \approx k^{\mathrm{N}}$ (by diagonal counting). Hence

$$\times X_i = (\times \bar{X}_i) \cap (\times H_i)$$

is an object in *ind-aff*. Let $p_i : \times X_i \to X_i$ be the projection map. Let $f_i : Z \to X_i$ be ind-affine maps. It is easy to see that the p_i and the unique map $f: Z \to \times X_i$ such that $p_i \circ f = f_i$ belong to *ind-aff*. Q. E. D.

PROPOSITION 2.2. *ind-aff has equalizers.*

PROOF. Let $f, g : X \rightrightarrows Y$ belong to *ind-aff*. Let

$$E = \{ P \in X \mid f(P) = g(P) \}.$$

Then taking unions over $H \in T_X$,

$$\bar{E} \cap H_X = \cup(\bar{E} \cap H) = \cup(\bar{E} \cap \bar{X} \cap H) = \bar{E} \cap (\cup(\bar{X} \cap H)) = \bar{E} \cap X = E$$

where the last equality follows from the fact that $f(Q) = g(Q)$ for $Q \in \bar{E}$ as f, g are induced by maps between k^{N} in *aff*. Thus E can be given the structure of an ind-affine scheme. Clearly the inclusion $i : E \to X$ is a map of ind-affine schemes.

Let $h: Z \to X$ be a map in *ind-aff* such that $f \circ h = g \circ h$. For $K \in T_Z$ there is a $H \in T_X$ such that $h(Z \cap K) \subset X \cap H$. There is an L in T_Y such that

$$g(X \cap H) \subset Y \cap L \quad \text{and} \quad f(X \cap H) \subset Y \cap L.$$

But $E \cap H$ is the equalizer in *aff* of the restricted maps

$$f, g : X \cap H \rightrightarrows Y \cap L$$

and hence there is a unique map $c_K : Z \cap K \to E \cap H$ such that $i \circ c_K = h$ on $Z \cap K$. Taking direct limits one obtains a unique map $c : Z \to E$ such that $i \circ c = h$. The uniqueness of c follows from the fact that a map such as c must induce the c_K again. Q. E. D.

aff has countable products and equalizers but not countable coproducts. On the other hand:

PROPOSITION 2.3. *ind-aff has countable coproducts.*

We outline a proof. Let X_i $(i \in \mathbb{N})$ be ind-affine schemes. Shift the X_i so that they do not contain the origine $0 \in k^{\mathbb{N}}$. Construct an embedding $s_i : X_i \to k^{\mathbb{N} \times \mathbb{N}}$ where on k valued points $s_i(x_j) = (y^j_m)$ and

$$y^j_m = 0 \text{ if } m \neq i, \quad y^j_m = x_j \text{ if } m = i.$$

Let $C = \cup s_i(X_i)$. The objects of T_C are of the form $\times H_i$ where $H_i \in T_{X_i}$. It is easy to see that $C = \bar{C} \cap H_C$ and that the s_i are ind-affine maps defining a coproduct structure on C.

Finally we show that *ind-aff* has countable colimits by showing:

PROPOSITION 2.4. *ind-aff has coequalizers.*

PROOF. Let $f, g : X \rightrightarrows Y$ be two maps in *ind-aff*. f, g induce maps $\bar{f}, \bar{g} : \bar{X} \rightrightarrows \bar{Y}$ in *aff* and $\bar{f}, \bar{g}$ have a coequalizer $\bar{q} : \bar{Y} \to Q'$ in *aff*. Suppose that $H \in T_Y$. Let

$$r^*_H : k[Q'] \to k[\bar{Y}] \to k[Y \cap H]$$

be the composition of the inclusion and natural quotient maps of affine rings. Choose a basis $\{q_i\}_{i \in \mathbb{N}}$ for $k[Q']$. Then $k[Q'] = k[q_i]$ and Q' can be imbedded as a closed affine subscheme of $k^{\mathbb{N}}$ in terms of the generators $\{q_i\}_{i \in \mathbb{N}}$ of $k[Q']$. Let L_H be the closed linear subscheme of $k^{\mathbb{N}}$ defined by the condition that $\Sigma_i a_i q_i$ is sent to 0 under r^*_H. Then the L_H are directed by inclusion (for $H, K \in T_Y$ there is an $M \in T_Y$ such that $L_H \cup L_K \subset L_M$) and hence $L = \cup L_H$ $(H \in T_Y)$ is a full linear subset of $k^{\mathbb{N}}$. Let $Q = Q' \cap L$. Q is an ind-affine scheme. As

$$q(Y \cap H) \subset Q' \cap L_H \subset Q,$$

$q(Y) \subset Q$. Clearly $q : Y \to Q$ is a map of ind-affine schemes. Note that $\overline{q(\bar{Y})} = Q'$. Hence $Q' = \bar{Q}$.

Let $c : Y \to Z$ be a map in *ind-aff* and $\bar{c} : \bar{Y} \to \bar{Z}$ be the corresponding map on the Zariski closures. Suppose that $c \circ f = c \circ g$ and hence that

$\bar{c} \circ \bar{f} = \bar{c} \circ \bar{g}$. Then there is a unique map $\bar{h}: \bar{Q} \to \bar{Z}$ such that $\bar{h} \circ \bar{q} = \bar{c}$. Let $H \in T_Y$. There is a $K \in T_Z$ such that $c(Y \cap H) \subset Z \cap K$. Consider the diagram

$$\begin{array}{ccccc} k[\bar{Z}] & & \xrightarrow{\quad u_1 \quad} & & k[Z \cap K] \\ {\scriptstyle h^*}\downarrow & \searrow^{c^*} & & & \downarrow{\scriptstyle c_H^*} \\ k[Q'] & \xrightarrow[q^*]{} & k[\bar{Y}] & \xrightarrow[u_2]{} & k[Y \cap H] \end{array}$$

where h^*, c^* and c_H^* are the k-algebra maps corresponding to $\bar{h}$, $\bar{c}$, and the restriction $c: Y \cap H \to Z \cap K$ respectively, and where u_1, u_2 are the natural quotient maps. The inner diagrams commute and thus so does the outer. As u_1 is surjective,

$$c_H^*(k[Z \cap H]) \subset u_2 \circ q^*(k[\bar{Q}]) = r_H^*(k[\bar{Q}]).$$

But $r_H^*(k[\bar{Q}]) = k[Q \cap L_H]$. Hence c_H^* maps

$$c_H^*: k[Z \cap K] \to k[Q \cap L_H]$$

which implies that $\bar{h}$ restricts to a map sending $Q \cap L_H$ into $Z \cap K$ and thus to a map h in *ind-aff* sending Q to Z. Clearly $h \circ q = c$. h is unique since it must be the restriction of $\bar{h}$. Q.E.D.

From the above follows:

THEOREM 2.5. *ind-aff is countably complete and cocomplete.*

REMARK 2.6. To any vector space V of $k^{\mathrm{N}}(k)$ one can associate a full linear subset V^* of k^{N} by enlarging it to include points of closed linear subschemes H of k^{N} such that $H(k)$ is a subspace of V. By $\times H_i$ $(i \in \mathrm{N})$ in Proposition 2.1 we mean not the set theoretic product which may not be full linear but $(\times H_i(k))^*$ ($\times$ now in sets). We use this convention as required.

3. THE EXTENSION OF EXTERNAL HOM-FUNCTORS IN *aff*, TO INTERNAL HOM-FUNCTORS IN *ind-aff*.

All hom-sets are those of *ind-aff* unless specified otherwise.

We suppose that there is a bifunctor (as will be shown in 4)

$$B(X, Y) = Y^X : aff \times aff \to ind\text{-}aff$$

such that a natural equivalence

$$Hom(X \times Y, Z) \approx Hom(X, Z^Y)$$

(where X, Y, Z are restricted to objects in *aff*) exists. In this situation $B(X, Y)$ is called an *external hom-functor on aff to ind-aff.*

Let Y be an object in *ind-aff* and $Y = lim\,ind(Y \cap H)$ where the inductive limit is taken over $H \in T_Y$. Let X be affine. Extend the bifunctor B on objects by letting $Y^X = lim\,ind(Y \cap H)^X$ where the inductive limit is taken over $H \in T_Y$. We'll see that Y^X is an object of *ind-aff* later in Proposition 5.2. Let $f: Y \to W$ belong to *ind-aff*. For each $H \in T_Y$ there is a $K \in T_W$ such that $f(Y \cap H) \subset K \cap W$ and thus a map

$$f^X : (Y \cap H)^X \to (W \cap K)^X$$

in *ind-aff*. Taking inductive limits one obtains a map $f^X: Y^X \to W^X$. Let $g: Z \to X$ be a map in *aff*. Taking inductive limits of the maps

$$(Y \cap H)^g : (Y \cap H)^X \to (Y \cap H)^Z$$

one obtains a map $Y^g: Y^X \to Y^Z$.

One readily verifies that defining f^X and Y^g as above one has extended the bifunctor B to a bifunctor

$$B(X, Y) = Y^X : aff \times ind\text{-}aff \to ind\text{-}aff.$$

See Remark 5.6.

Let now $X = lim\,ind(X \cap K)\ (K \in T_X)$. Define

$$\underline{\underline{Y}}^X = lim\,proj\, Y^{(X \cap K)}$$

where the projective limit is taken over $K \in T_X$. $\underline{\underline{Y}}^X$ is not necessarily an ind-affine scheme (the proofs would be shorter if it was). Both $\underline{\underline{Y}}^X$ and $\bar{Y}^{\bar{X}}$ (as X is reduced; see Proposition 5.5) can be viewed as subsets of $\times \bar{Y}^{(X \cap K)}\ (K \in T_X)$.

DEFINITION 3.1. $Y^X = \underline{\underline{Y}}^X \cap \bar{Y}^{\bar{X}}$.

In Section 5 we will show that Y^X is an ind-affine scheme.

In a manner analogous to that above (but dual) one obtains an extension of the bifunctor B to a bifunctor

$$\underline{\underline{B}}(X, Y) = Y^X : \textit{ind-aff} \times \textit{ind-aff} \to \underline{R}$$

where $\underline{R}$ is the category of ringed spaces. Let $c : \textit{ind-aff} \to \textit{aff}$ be the functor which associates to $X \in \textit{ind-aff}$ the closure $\bar{X}$ of X in k^N and to an arrow $f: X \to Y$ the induced map $\bar{f}: \bar{X} \to \bar{Y}$ in $\textit{aff}$. Then

$$\underline{B}(X, Y) = B(c(X), c(Y)) : \textit{ind-aff} \times \textit{ind-aff} \to \textit{ind-aff}$$

is clearly a bifunctor. Letting $B(X, Y) = \underline{\underline{B}}(X, Y) \cap \underline{B}(X, Y)$ one sees that we have extended the bifunctor B to a bifunctor (see Remark 5.6):

$$B(X, Y) = Y^X : \textit{ind-aff} \times \textit{ind-aff} \to \textit{ind-aff}.$$

If *ind-aff* had the inductive and projective limits that we needed above, we would have used only the following two lemmas in the proof that *ind-aff* was cartesian-closed. Their proofs are to be found in Section 5.

LEMMA 3.2. *Let* $Z = \lim ind(Z \cap L)$ *where the inductive limit is taken over* $L \in T_Z$ *and* X, Y *be affine. Then*

$$Hom(X, Z^Y) = \bigcup_L Hom(X, (Z \cap L)^Y).$$

LEMMA 3.3. *Let* X *be ind-affine and* $Y = \lim ind(Y \cap H)$ $(H \in T_Y)$. *Then*

$$\lim ind(X \times (Y \cap H)) = X \times Y \quad (H \in T_Y).$$

DEFINITION 3.4. A category $\underline{C}$ is cartesian closed if there is a bifunctor $B : \underline{C} \times \underline{C} \to \underline{C}$, $\underline{C}$ has finite products and there is a natural equivalence

$$Hom_{\underline{C}}(X \times Y, Z) \approx Hom_{\underline{C}}(X, B(Y, Z))$$

with X, Y, Z objects in $\underline{C}$.

Then we show:

THEOREM 3.5. *There is a natural equivalence*

$$(\dagger) \qquad Hom_{\underline{I}}(X \times Y, Z) \approx Hom_{\underline{I}}(X, Z^Y)$$

induced from the natural equivalence

$$Hom_{\underline{I}}(\bar{X} \times \bar{Y}, \bar{Z}) \approx Hom_{\underline{I}}(\bar{X}, \bar{Z}^{\bar{Y}})$$

where X, Y, Z *are objects in* $\underline{I}$ = *ind-aff and* $\bar{X}, \bar{Y}, \bar{Z}$ *denote the closure of* X, Y, Z *in* k^{N}. *Thus ind-aff is cartesian closed.*

We omit the subscript $\underline{I}$ below.

PROOF. Let X, Y be affine and Z as in Lemma 3.2. Then

$$Hom(X, Z^Y) = Hom(X, \lim ind(Z\cap L)^Y) = \cup Hom(X, (Z\cap L)^Y) =$$
$$= \cup Hom(X\times Y, Z\cap L) = Hom(X\times Y, \lim ind\, Z\cap L) = Hom(X\times Y, Z)$$

using Lemma 3.2, the assumption that $(\dagger)$ holds for affines and the definitions of mappings between ind-affine schemes.

Note that as the isomorphism between $Hom(X, (Z\cap L)^Y)$ and $Hom(X\times Y, Z\cap L)$ is induced from an isomorphism between $Hom(X, \bar{Z}^Y)$ and $Hom(X\times Y, \bar{Z})$, the isomorphism between

$$Hom(X, Z^Y) \quad \text{and} \quad Hom(X\times Y, Z)$$

is also induced from this isomorphism. See (**) of Remark 2.6.

Next let Y be affine and X, Z be ind-affine schemes. Let $X = \lim ind(X\cap K)$ with $K \in T_X$. There are commutative diagrams

$$\begin{array}{ccc} Hom(\bar{X}, Z^Y) & \xrightarrow{i_1} & \times Hom(X\cap K, Z^Y) \\ \beta\big\downarrow & & \big\downarrow \\ Hom(\bar{X}\times Y, Z) & \xrightarrow{i_2} & \times Hom((X\cap K)\times Y, Z) \end{array}$$

and

$$\begin{array}{ccc} \lim proj\, Hom(X\cap K, Z^Y) & \xrightarrow{j_1} & \times Hom(X\cap K, Z^Y) \\ \big\downarrow & & \big\downarrow \\ \lim proj\, Hom((X\cap K)\times Y, Z) & \xrightarrow{j_2} & \times Hom((X\cap K)\times Y, Z) \end{array}$$

where i_1, i_2, j_1, j_2 are canonical embeddings and the natural isomorphisms which are the vertical mappings we have by the first part of the proof. Again note that i_1, i_2 are embeddings because $\bar{X}$ and $\bar{X}\times Y$ are reduced. As

$$Hom(X, Z^Y) = Hom(\bar{X}, Z^Y) \cap (\lim proj\, Hom(X\cap K, Z^Y))$$

and

$$Hom(X\times Y, Z) = Hom(\bar{X}\times Y, Z)\cap(lim\,proj\,Hom((X\cap K)\times Y,\ Z))$$

(using Lemma 3.3), there is a natural isomorphism between $Hom(X, Z^{Y})$ and $Hom(X\times Y, Z)$ induced from the natural isomorphism β between $Hom(\bar{X}, Z^{Y})$ and $Hom(\bar{X}\times Y, Z)$.

Next let X, Y, Z be ind-affine and $Y = lim\,ind\,(Y\cap H)\ (H\in T_Y)$. Then there are commutative diagrams

$$\begin{array}{ccc} Hom(X, Z^{\bar{Y}}) & \xrightarrow{i_1} & \times Hom(X, Z^{Y\cap H}) \\ \downarrow{\scriptstyle a} & & \downarrow \\ Hom(X\times \bar{Y}, Z) & \xrightarrow{i_2} & \times Hom(X\times(Y\cap H), Z) \end{array}$$

$$\begin{array}{ccc} lim\,proj\,Hom(X, Z^{Y\cap H}) & \xrightarrow{j_1} & \times Hom(X, Z^{Y\cap H}) \\ \downarrow & & \downarrow \\ lim\,proj\,Hom(X\times(Y\cap H), Z) & \xrightarrow{j_2} & \times Hom(X\times(Y\cap H), Z) \end{array}$$

where i_1, i_2, j_1, j_2 are canonical embeddings and the natural isomorphisms which are the vertical maps we have by the last part of the proof. As

$$Hom(X, Z^{Y}) = Hom(X, Z^{\bar{Y}})\cap(lim\,proj\,Hom(X, Z^{Y\cap H}))$$

$(Z^{Y}\subset Z^{\bar{Y}}\subset \bar{Z}^{\bar{Y}}$; see $(**)$ of Remark 5.6) and

$$Hom(X\times Y, Z) = Hom(X\times \bar{Y}, Z)\cap(lim\,proj\,Hom(X\times(Y\cap H), Z)$$

(use Lemma 3.3) there is a natural isomorphism between $Hom(X\times Y, Z)$ and $Hom(X, Z^{Y})$ induced by the natural isomorphism a above. Q.E.D.

4. CONSTRUCTING THE EXTERNAL HOM-FUNCTOR OF *aff* INTO *ind-aff*.

All hom-sets are those in *ind-aff*.

We show that the bifunctor

$$B(X, Y) = Y^{X}: aff\times aff \to ind\text{-}aff$$

described at the outset of Section 3 exists.

Let X, Y be affine. A morphism $f: X\to Y$ is given by a countable number of coordinates $f_i\in k[X]$. Suppose $\{e_j\}_{j\in \mathrm{N}}$ is a basis for $k[X]$

and $I(Y)$ is the ideal defining Y. If

$$f(x) = (f_i(x)) = (\sum_i a_i^j e_j),$$

then

$$0 = F(f(x)) = \sum_p F_p(a_i^j) e_p \quad \text{for} \quad F \in I(Y)$$

implies $F_p(a_i^j) = 0$. We let U be the affine closed subscheme of $k^{N \times N}$ defined by the ideal J_Y which is generated by

$$\{ F_p(X_i^j) \mid p \in N,\ F \in I(Y) \}.$$

$k^{N \times N}$ can be identified with k^N (by diagonal counting). Let

$$T = \{ (t_i) \mid t_i \in N,\ i \in N \}.$$

If $t = (t_i)$ consider the ideal A_t generated by the X_i^j for $j \geq t_i$. Then $H = \cup H_t\ (t \in T)$ where $H_t = Spec(k[X_i^j]/A_t)$ is a full linear subset of k^N.

REMARK. $(U \cap H_t)(k)$ can be naturally identified with the set of maps $f \in Hom(X, Y)$ such that if $f = (\sum_i a_i^j e_i)$ then $a_i^j = 0$ if $j > t_i$. Hence $U \cap H_t$ might be described as a t-jet scheme, and this point of view plays an important role in Section 6.

DEFINITION 4.1. $Y^X = U \cap H = lim\, ind(U \cap H_t)$.

It can be seen that a change of basis $\{e_j\}_{j \in N}$ corresponds to a linear map (each coordinate a linear polynomial) mapping Y^X onto an isomorphic copy.

Let $g: Y \to W$ be in *aff*. As

$$g(f(x)) = (g_m(f(x))) = (g_m(\sum_j a_i^j e_j)) = (\sum_p g_m^p(a_i^j) e_p),$$

g induces a map $g^X: Y^X \to W^X$ in *ind-aff* which on k valued points is defined by $g^X(a_i^j) = (g_m^p(a_i^j))$. Clearly with this defintiion of g^X, Y^X is a functor in Y from *aff* to *ind-aff*.

Let $g: W \to X$ be in *aff*. g induces a map $g^*: k[X] \to k[W]$. Let $\{d_m\}_{m \in N}$ be a basis for $k[W]$.

$$f(g(x)) = (f_i(g(x))) = (g^*(f_i(x))) = (\sum_m L_i^m(a_i^j) d_m)$$

where $L_i^m(a_i^j)$ is a linear function in (a_i^j). Define a map $Y^g: Y^X \to Y^W$ in *ind-aff* by setting $Y^g(a_i^j) = (L_i^m(a_i^j))$ on k valued points. Note that if one changes the basis $\{d_m\}_{m\in N}$ then the map in *ind-aff* obtained differs from the first map by the isomorphism between the two copies of Y^W induced by this base change. Thus with this definition of Y^g a functor Y^X: *aff* → *ind-aff* in X is obtained.

Let $f: Z \to X$, $g: X \to Y$ and $h: Y \to W$ be maps in *aff*. As

$$h \circ (g \circ f) = (h \circ g) \circ f$$

one sees that $W^f \circ g^X = g^Z \circ Y^f$ and hence that

$$B(X, Y) = Y^X: aff \times aff \to ind\text{-}aff$$

is a bifunctor.

THEOREM 4.2. *There is a natural equivalence*

$$Hom(X \times Y, Z) \approx Hom(X, Z^Y).$$

Thus B *is an external hom-functor on aff to ind-aff.*

PROOF. Let $\{e_h^x\}_{h\in N}$, resp. $\{e_j^y\}_{j\in N}$, be a basis of $k[X]$, resp. $k[Y]$. Then $\{e_h^x e_j^y\}$ form a basis for $k[X \times Y] = k[X] \otimes k[Y]$.
If $z_h = \sum_{h,j} a_{hj}^i e_h^x e_j^y$ is the h-th tuple of an element of $Hom(X \times Y, Z)$ write

$$F(z_h) = \sum_{p,q} F_{pq}(a_{hj}^i)\, e_p^x e_q^y.$$

Then the mappings in $Hom(X \times Y, Z)$ correspond to the set of all (a_{hj}^i) ($k^{N\times N\times N}$ is identified with k^N) satisfying:

A) $F_{pq}(a_{hj}^i) = 0$ for all $F \in I(Z)$.

B) For fixed i, $a_{hj}^i = 0$ except for a finite number of h, j.

Let $w_{ij} = \sum_i b_{hj}^i e_h^x$ be the ij-th coordinate of an element in $Hom(X, Z^Y)$. As a consequence of the definition of Z^Y ($j \geq t_i$ implies $w_{ij} = 0$) the b_{hj}^i satisfy condition B. Let $u_i = \sum_j w_{ij} e_j^y$ be the i-th coordinate of an element of $Z^Y(k)$ and $F \in I(Z)$. Then

$$0 = F(u_i) = F(\sum_j(\sum_h b_{hj}^i e_h^x) e_j^y) = \sum_{p,q} F_{p,q}(b_{hj}^i) e_p^x e_q^y$$

and thus the b_{hj}^i satisfy condition B. Conversely it is clear that to every b_{hj}^i satisfying A and B there is a unique element of $Hom(X, Z^Y)$. Thus there is a «natural identification» of both $Hom(X\times Y, Z)$ and $Hom(X, Z^Y)$ with the set of all (a^i_{hj}) satisfying A and B. We leave it to the reader to see that from this «natural identification» there comes a natural isomorphism between $Hom(X\times Y, Z)$ and $Hom(X, Z^Y)$. Q.E.D.

5. COMMUTATION PROPERTIES OF THE INDUCTIVE LIMIT.

We prove the lemmas and show that the hom-functor exists as required in Section 3. The lemmas describe commutation properties of the inductive limit with certain operations. Before we begin the proof of Lemma 3.2 we will describe the inductive limit $Z^Y = limind(Z\cap L)^Y$ where Y is affine and $L \epsilon T_Z$.

First $(Z\cap L)^Y = U_{Z\cap L}\cap H$ where H is a fixed full linear subset of k^{N}, $L \epsilon T_Z$ and $U_{Z\cap L}$ is defined by the ideal $J_{Z\cap L}$ where H and $U_{Z\cap L}$ are chosen as they were in Definition 4.1. Clearly $J_{Z\cap L} = J_{\bar{Z}} + J_L$, and thus $U_{Z\cap L} = U_{\bar{Z}} \cap U_L$. Let

$$M = \cup(U_L \cap H) \quad (L \epsilon T_Z).$$

As U_L is a closed linear subscheme and the $U_L\cap H$ are directed by inclusion the following lemma implies that M is a full linear subset of k^{N}.

LEMMA 5.1. *Let $A(k)\subset M$ be a Zariski closed subset of $M(k)$. Then $A\subset U_L$ for some $L \epsilon T_Z$ if A is a closed linear subscheme of k^{N}.*

PROOF. Recall that the relation

$$F(f(x)) = \sum_p F_p(a^j_i)\, e_p \qquad \text{where } f(x) = (\sum_j a^j_i e_j)$$

provides the generators and linear polynomials $F_p(a^j_i)$ for J_L when the F are restricted to a set of linear generators of $I(L)$ and p varies. Let E_L be the ideal generated by linear F in $I(L)$ or where F_p vanishes on A for all p and $E = \cap E_L$. The closed linear subschemes which are the zero sets of E and E_L are related by $V(E) = \overline{\cup V(E_L)}$. Clearly $V(E_L)\subset L$. Hence:

1° $V(E) \subset \cup L \ (L \epsilon T_Z)$.

From the definition of T_Z and 1° follows

2° $V(E) = R$ for some $R \in T_Z$.

It is not difficult to see that

3° E is the ideal generated by linear polynomials F where F_p vanishes on A for all p or F vanishes on $\bar{H}_Z$.

Next we show

4° For R as in 2°, $A \subset U_R$.

As $A \subset \cup(U_L)$ if $A \cap U_R \neq A$ there is a $K \in T_Z$ $(K \supset R)$ and a point $Q \in U_K \cap A$ such that $Q \notin U_R$. Hence one can find a linear polynomial $G \in I(R)$ such that the G_p vanishes on U_R but $G_p(Q) \neq 0$ for some p. Then by 3° $G \notin E$. This contradicts 2°. Q.E.D.

Let $B = U_{\bar{Z}} \cap M$.

PROPOSITION 5.2. $B = Z^Y$.

PROOF. Let Q_L be the set of all closed linear subschemes of $U_L \cap H$. Then

$$\begin{aligned} Z^Y &= \lim ind(U_{\bar{Z}} \cap U_L \cap H) \quad (L \in T_Z) \\ &= \lim ind(\lim ind(U_{\bar{Z}} \cap A)) \quad (A \in Q_L,\ L \in T_Z) \\ &= \lim ind(U_{\bar{Z}} \cap A) \quad (A \in T_B) \qquad \text{by Lemma 5.1} \\ &= B. \end{aligned}$$

Q.E.D.

Here as before all hom-sets consist of morphisms in *ind-aff* between two objects in *ind-aff*.

PROPOSITION 5.3 *(Lemma 3.2). Let* $Z = \lim ind(Z \cap L)$ *where the limit is taken over* $L \in T_Z$ *and* X, Y *be affine. Then*

$$Hom(X, \lim ind((Z \cap L)^Y)) = \lim ind\, Hom(X, (Z \cap L)^Y).$$

PROOF. Note that the inductive limit in sets here is just union. Let $f: X \to Z^Y$ be a map in ind-affine schemes. Then for $D = Z^Y$ and some $K \in T_D$ $f(X) \subset Z^Y \cap K$. Lemma 5.1 implies that $K \subset U_L \cap H$ for some $L \in T_Z$. Hence

$$f(X) \subset Z^Y \cap K \subset U_L \cap H \cap B = U_L \cap U_{\bar{Z}} \cap H = (Z \cap L)^Y.$$

Q.E.D.

PROPOSITION 5.4 *(Lemma 3.3). Let* X *be an ind-affine scheme and* $Y = \lim ind(Y \cap H)$ $(H \in T_Y)$. *Then*

$$X \times Y = \lim ind(X \times (Y \cap H)) \quad (H \in T_Y).$$

PROOF. Let $X = \bar{X} \cap H_X$ as in Proposition 1.1.

$$X \times Y = (\bar{X} \cap H_X) \times (\bar{Y} \cap H_Y) = (\bar{X} \times \bar{Y}) \cap (H_X \times H_Y).$$

Let A be linear closed in $H_X \times H_Y$. Then there is a $H \in T_Y$, $K \in T_X$ such that $A \subset K \times H$. Let Q_H be the set of all closed linear subschemes contained in $H_X \times H$. Then for fixed H

$$(1) \qquad \lim ind((\bar{X} \times \bar{Y}) \cap L) = (\bar{X} \times \bar{Y}) \cap (H_X \times H) = X \times (\bar{Y} \cap H)$$

where the inductive limit is over $L \in Q_H$. Applying *limind* taken over $H \in T_Y$ the left side of (1) becomes

$$\lim ind((\bar{X} \times \bar{Y}) \cap M) \quad (M \in T_{X \times Y})$$

which equals $X \times Y$. Q.E.D.

We show now the last requirement for the fundamental result of this paper.

PROPOSITION 5.5. Y^X *is an affine scheme for ind-affine schemes* X, Y.

PROOF. Let $X = \lim ind(X \cap K)$ $(K \in T_X)$. As we have seen in Section 4 the maps $b_K : \bar{Y}^{\bar{X}} \to \bar{Y}^{X \cap K}$ induced by the inclusions $X \cap K \to \bar{X}$ are linear. Hence $W = \cap b_K^{-1}(Y^{X \cap K})$ is easily seen to be an ind-affine scheme ($Y^{X \cap K} \subset \bar{Y}^{X \cap K}$. See (*) of Remark 5.6). We show that $W = Y^X$. Let

$$S = k^{\mathrm{N}}, \quad R = S^{\bar{X}} \quad \text{and} \quad R_K = S^{X \cap K}.$$

There is a commutative diagram

$$(\dagger\dagger) \qquad \begin{array}{ccccc} W & \longrightarrow & \bar{Y}^{\bar{X}} & \longrightarrow & \bar{R} \\ {\scriptstyle a_K}\downarrow & & {\scriptstyle b_K}\downarrow & & \downarrow{\scriptstyle c_K} \\ Y^{X \cap K} & \longrightarrow & \bar{Y}^{X \cap K} & \longrightarrow & \bar{R}_K \end{array}$$

for each K. It is easily seen that c_K is surjective and hence the induced maps $c_K^* : k[\bar{R}_K] \to k[\bar{R}]$ are injective. As the $k[\bar{R}_K]$ are directed by inclusion

$$lim\,ind\, k[\bar{R}_K] = \cup\, k[\bar{R}_K] \quad (K \in T_X).$$

Suppose $f \in k[\bar{R}]$ but $f \notin k[\bar{R}_K]$. Recalling the definition of $S^{\bar{X}}$, we see that f is a polynomial in a finite number of variables a_i^j where $(c_i^j) \in R(k)$ arises from $(\Sigma\, c_i^j e_j) \in Hom(\bar{X}, k^{\mathrm{N}})$ and the e_j form a basis for $k[\bar{X}]$. As X is dense in $\bar{X}$ there is a K such that

$$\{\, e_j \mid a_i^j \text{ is a variable in } f \text{ for some } i \,\}$$

is linearly independent in $k[X \cap K]$. One has thus a basis for $k[\bar{R}_K]$ such that c_K is projection and $c_K^*(a_i^j) = a_i^j$ if a_i^j is a variable in f. But then $f \in k[\bar{R}_K]$. Thus $limind\, k[\bar{R}_K] = k[\bar{R}]$ and hence $\bar{R} = limproj\, \bar{R}_K$.

Taking projective limits, diagram ($\dagger\dagger$) becomes

$$\begin{array}{ccccc}
W & \longrightarrow & \bar{Y}^{\bar{X}} & \longrightarrow & \bar{R} \\
\downarrow a & & \downarrow b & & \| \, c \\
limproj\, Y^{X \cap K} & \longrightarrow\; limproj\, \bar{Y}^{X \cap K} & & \longrightarrow & \bar{R}
\end{array}$$

As the horizontal arrows are injective, so are a and b. Thus

$$W \subset \bar{Y}^{\bar{X}} \cap limproj(Y^{X \cap K})$$

(note that strictly speaking one should speak of pullback rather than intersection). If

$$P \in \bar{Y}^{\bar{X}} \cap limproj\ (Y^{X \cap K})$$

then $b(P) = (b_K(P)) \in \times\, Y^{X \cap K}$ and hence $P \in W$. Q.E.D.

REMARK 5.6. Let Y be affine and Z be ind-affine (as in the discussion preceding Lemma 5.1). Then

$$(Z \cap L)^Y = U_{Z \cap L} \cap H \subset U_{\bar{Z}} \cap H \subset \bar{Z}^Y$$

and hence

$$(*) \qquad Z^Y \subset \bar{Z}^Y.$$

Let $f: Z \to W$ be a map in *ind-aff* and $\bar{f}: \bar{Z} \to \bar{W}$ the corresponding map in *aff*. Then $f^Y: Z^Y \to W^Y$ is induced from $\bar{f}^Y$ and hence f^Y is the restriction of a map in *aff* between k^{N}. Because $f(Z \cap L) \subset W \cap K$ for some $K \in T_W$ and Lemma 5.1, f^Y respects filtration in Z^Y and hence belongs to *ind-aff*. $g: X \to Y$ in *ind-aff* induces

$$(Z \cap L)^g : U_{Z \cap L} \cap H \to U'_{Z \cap L} \cap H'$$

which is the restriction of $\bar{Z}^g : U_{\bar{Z}} \cap H \to U'_{\bar{Z}} \cap H'$ and hence Z^g is the restriction of $\bar{Z}^g$. Again Lemma 5.1 implies that Z^g preserves filtration.

Next let X, Y be ind-affine. Suppose that $f: Y \to W$, $g: Z \to X$ are in *ind-aff* and $\bar{f}: \bar{Y} \to \bar{W}$, $\bar{g}: \bar{Z} \to \bar{X}$ are the corresponding maps in *aff*. Then from the definition of Y^X via the b_K it follows that f^X (resp. Y^g) is the restriction of $\bar{f}^{\bar{X}}$ (resp. $\bar{Y}^{\bar{g}}$). Because the b_K are linear and hence map closed linear subschemes to closed linear subschemes, the filtration preserving maps $f^{X \cap H}$ (resp. $Y^{g(L)}$ where $g(L)$ is the restriction of g to $Z \cap L$ for some $L \in T_Z$ and $g(Z \cap L) \subset X \cap H$ for some $H \in T_X$) lift to filtration preserving map f^X (resp. Y^g). The fact that

$$(**) \qquad Y^X \subset Y^{\bar{X}} \subset \bar{Y}^{\bar{X}}$$

follows from the commutativity of

$$\begin{array}{ccc} Y^{\bar{X}} & \longrightarrow & \bar{Y}^{\bar{X}} \\ {\scriptstyle d_K}\downarrow & & \downarrow{\scriptstyle b_K} \\ Y^{X \cap K} & \longrightarrow & \bar{Y}^{X \cap K} \end{array}$$

(which commutes because we have shown that $d_K = Y^I$ is the restriction of $b_K = \bar{Y}^I$ where $I: X \cap K \to \bar{X}$ is the inclusion).

6. TRANSVERSALITY IN Y^X.

Let X, Y, W be affine non-singular irreducible schemes of finite type over an algebraically closed field k, W a closed proper subscheme of Y and $f: X \to Y$ a map of schemes. For an affine non-singular scheme Z of finite type over k by $T_z Z$ we denote the tangent space to Z at z. For the notions that we use see [1, 6]. We assume that k is algebraically closed.

DEFINITION 6.1. *f is transversal to W* if for each $x \in X(k)$, $f(x) \notin W$ or

$$T_{f(x)} Y = T_{f(x)} W + (df)_x (T_x).$$

Let $r = dim\, W$ and $Y \subset k^m$.

DEFINITION 6.2. *W is parallelisable in Y* if there is a map of schemes $a: Y \to (k^m)^r$ such that for each $y \in W(k)$, $a(y)$ is an r-tuple of vectors generating $T_y W$. The map a will be called a *parallelising map*. We say *X is parallelisable* if X is paralellisable in X.

If W is not parallelisable in Y it is not difficult to find a finite open covering $\{U_i\}$ of Y such that $U_i \cap W$ is parallelisable in U_i.

Recall that $Y^X(k)$ is just the collection of all scheme maps $f: X \to Y$. Let $\{b_p\}$ be a basis for $k[X]$. The elements of $Y^X(k)$ have the form $f = (\sum_p a_q^p b_p)$ where $a_q^p \in k$, for fixed q one has

$$a_q^p = 0 \text{ if } p >> 0 \text{ and } q = 1, \dots, m.$$

Let $\underline{r} = (r_1, \dots, r_m) \in \mathrm{N}^m$ and

$$A_{\underline{r}} = \{ f \in Y^X(k) \mid a_q^p = 0 \text{ if } p > r_q \}.$$

DEFINITION 6.3. A subset C of $Y^X(k)$ *is constructible* if, for all $\underline{r}$, $C \cap A_{\underline{r}}$ is constructible, i.e., if for some $n \in \mathrm{N}$,

$$A_{\underline{r}} \cap C = \bigcup_{m=1}^{n} (K_m \cap U_m)$$

where K_m is closed and U_m is open in $k^{\mathrm{N}}(k)$ $(m = 1, \dots, n)$.

PROPOSITION 6.4. *Let X be parallelisable and W be parallelisable in Y. Then*

$$T_W = \{ f \in Y^X(k) \mid f \text{ is transversal to } W \}$$

is constructible.

PROOF. Let $x \in X(k)$ and $\beta: X \to (k^n)^s$ be the parallelising map of X where $X \subset k^n$ and $s = dim\, X$. Let a as above be the parallelising map of W in Y and $\xi: X \to (k^m)^r$ the composite $a \circ f$. We restrict to $f \in A_{\underline{r}} \cap Y^X(k)$. Form all $r \times r$ determinants from the array $((df)_x(\beta(x)), \xi(x))$ calling these $D_1(a_q^p, x, \underline{r}), \dots, D_a(a_q^p, x, \underline{r})$. Suppose that $F_1, \dots, F_b$ generate the ideal of W and let

$$D_{i+a}(a_q^p, x, \underline{r}) = F_i(f(x)) \quad \text{for} \quad i = 1, \dots, b.$$

Then clearly $f \in A_{\underline{r}}$ will be transversal to W if, with $c = a + b$,

$$D_1(a_q^p, x, \underline{r}) = 0, \quad \ldots, \quad D_c(a_q^p, x, \underline{r}) = 0$$

do not have a common zero on X. The equations

$$D_1(a_q^p, x, \underline{r}) = 0, \quad \ldots, \quad D_c(a_q^p, x, \underline{r}) = 0$$

define a closed subset $S_{\underline{r}}$ of $A_{\underline{r}} \times X$. Let $p_{\underline{r}}: A_{\underline{r}} \times X \to A_{\underline{r}}$ be the projection on the first factor. Then by Chevalley's Theorem [8, page 94] $p_{\underline{r}}(S_{\underline{r}})$ is a constructible subset of $A_{\underline{r}}$ and hence its complement $T_W \cap A_{\underline{r}}$ is also constructible. Q.E.D.

Let Y^* (resp. W^*) be the projective scheme which is the closure of Y (resp. W) in projective m-space P^m defined over k. Similarly let X^* be the projective scheme which is the closure of X in P^n. Suppose that X^*, Y^* and W^* are non-singular. Let F be an element of the projective ring of X^* of a given degree and suppose that

$$X = \{ P \in X^* \mid F(P) \neq 0 \}.$$

There is a smallest integer $m(\underline{r})$ such that, for all $f \in Y^X(k) \cap A_{\underline{r}}$, f can be written in homogeneous coordinates

$$f^* = (F^{m(\underline{r})}, G_1, \ldots, G_m)$$

where $F^{m(\underline{r})}, G_1, \ldots, G_m$ are elements of the same degree in the projective ring of X^*. We call f^* the extension of f (relative to F and $\underline{r}$) if $F^{m(\underline{r})}, G_1, \ldots, G_m$ do not have a common zero in X^*. Let

$$\underline{r} = (r_1, \ldots, r_m), \; \underline{s} = (s_1, \ldots, s_m) \in \mathbb{N}^m.$$

We write $\underline{s} \geq \underline{r}$ if $s_i \geq r_i$ for each i.

We assume that β (resp. α) extends to a map

$$\beta^*: X^* \to (P^n)^s \quad (\text{resp. } \alpha^*: Y^* \to (P^m)^r)$$

of schemes and that restricted to suitable covering affine opens β^* is a parallelising map (resp. α^* is a parallelising map of W^* in Y^*).

PROPOSITION 6.5. *Let $E \cap A_{\underline{r}}$ be the set of elements in $Y^X(k) \cap A_{\underline{r}}$ which extend to scheme maps $X^* \to Y^*$. Suppose that $E \cap A_{\underline{r}}$ contains a map which* *extends to a scheme map* $X^* \to Y^*$ *which is transversal to* W^*. *Then,* $T_W \cap E \cap A_{\underline{r}}$ *contains an open non-empty subset of* $E \cap A_{\underline{r}}$ *(with respect to the subspace topology).*

PROOF. Working in homogeneous coordinates let $\xi^*: X^* \to (\mathrm{P}^m)^r$ be the composite $a^* \circ f^*$ (where $f \in E \cap A_{\underline{r}}$ and f extends to $f^*: X^* \to Y^*$). Form all $r \times r$ determinants of the array $(x \in X^*)\ ((df^*)_x(\beta^*(x)), \xi^*(x))$ restricting to $f \in A_{\underline{r}} \cap E$ and call these

$$D_1(a_q^p, x, \underline{r}),\ \ldots,\ D_a(a_q^p, x, \underline{r}).$$

Suppose that $F_1, \ldots, F_b$ are homogeneous polynomial generating the ideal of W^* and let

$$D_{i+a}(a_q^p, x, \underline{r}) = F_i(f^*(x)) \text{ for } i = 1, \ldots, b.$$

Then clearly if the

$$D_1(a_q^p, x, \underline{r}) = 0,\ \ldots,\ D_c(a_q^p, x, \underline{r}) = 0 \qquad (c = a + b)$$

do not have a common root other than zero then $a_q^p \in A_{\underline{r}} \cap E \cap T_W$.

Let $I(X^*)$ be the homogeneous ideal defining X^* in P^n, and $q: k[T_0, \ldots, T_n] \to k[X^*]$ the quotient by $I(X^*)$. There are homogeneous polynomials $E_j(a_q^p, T, \underline{r})$ such that

$$q(E_j(a_q^p, T, \underline{r})) = D_j(a_q^p, x, \underline{r}) \quad \text{for} \quad j = 1, \ldots, c.$$

Let $H_j(T)$ $(j = 1, \ldots, d)$ generate $I(X^*)$ and

$$E_{j+c}(a_q^p, T, \underline{r}) = H_j(T) \quad \text{for} \quad j = 1, \ldots, d.$$

Set $e = d + c$. Then we apply the following result to be found in van der Waerden [12, page 8].

LEMMA 6.6. e *homogeneous polynomials with indeterminate coefficients possess a resultant system of integral polynomials* $b_1, \ldots, b_\phi$ *in these coefficients such that for special values of the coefficients in an arbitrary field the vanishing of the resultants is necessary and sufficient in order that the homogeneous polynomials have a solution distinct from the zero solution.*

Applying this lemma to the $E_j(a_q^p, T, \underline{r})$ for $j = 1, \ldots, e$ in our case the indeterminate coefficients of the lemma being replaced by polynomials in the a_q^p obtained from the $E_j(a_q^p, T, \underline{r})$ one obtains polynomials $b_1(a_q^p, \underline{r}), \ldots, b_\phi(a_q^p, \underline{r})$ in the a_q^p such that: Let

$$V_{\underline{r}} = \{ a_q^p \epsilon E \cap A_{\underline{r}} \mid b_1(a_q^p, \underline{r}) = 0, \ldots, b_\phi(a_q^p, \underline{r}) = 0 \}.$$

Then $U_{\underline{r}} = (A_{\underline{r}} \cap E) - V_{\underline{r}}$ is an open subset of $E \cap A_{\underline{r}}$ such that if $(c_q^p) \epsilon U_{\underline{r}}$ the equations $E_j(c_q^p, T, \underline{r}) = 0 \quad (j = 1, \ldots, e)$ have no common root. Thus $T_W \cap E \cap A_{\underline{r}} \supset U_{\underline{r}}$, $U_{\underline{r}} \neq \emptyset$ by assumptions. Q.E.D.

EXAMPLE 6.7. One can see that even in the simplest cases, for instance C^C and $W = \{0\}$, that T_W is not an open subset of $Y^X(k)$. For instance let P_2 be the collection of polynomials $f(X) = aX^2 + bX + c$. Then the collection of f not transversal to $\{0\}$ corresponds to the set N of (a, b, c) such that

$$b^2 - 4ac = 0 \quad \text{if } a \neq 0, \qquad a = b = c = 0 \quad \text{if } a = 0.$$

If P_2 is identified with k^3 then clearly $T_2 - N$ is not open with respect to the Zariski topology on k^3 nor the usual topology if $k =$ R or C.

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