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CAUCHY-COMPLETION AND THE ASSOCIATED SHEAF

by R. BETTI and A. CARBONI *)

INTRODUCTION.

We will follow the point of view that categories based on a bicategory B (briefly B -categories) should be thought as general spaces. Such categories arose by considering a variable base for homs and the suggestion for regarding them as spaces comes directly from a paper (Walters [4]) where it is shown that sheaves for a general site are equivalent to symmetric, skeletal, Cauchy-complete categories based on a bicategory constructed out of the site.

For a category, Cauchy-completeness means that any adjoint pair of bimodules can be represented by one functor and, in order to express fundamental constructions of sheaf theory by means of B -category theory, we need to show the existence of the general process of «Cauchy-completion». The experience of metric spaces developed in [3] suggests that this construction should be done by taking adjoint pairs of bimodules. We will prove that in fact we get in this way the general process of Cauchy-completion and that it particularizes to the associated sheaf. This last result will be obtained by showing that such completion is left adjoint to the embedding of a particular kind of symmetric B -categories (called adjoint-inverse, briefly a. i.), and by constructing a «comparison functor». This functor also leads to compare the B -categorical one with an already known one-step construction of the associated sheaf [2].

The following diagram summarizes the whole subject :

$$\begin{array}{ccc}
 \text{Rel}_J(C)\text{-Cat a. i.} & \xrightarrow{\text{comp}} & S^{C^{op}} \\
 \text{Cauchy} \searrow & & \swarrow a \\
 & sh_J(C) &
 \end{array}$$

*) Work partially supported by the Italian C.N.R.

where $Rel_J(C)$ is the base bicategory associated to the site (C, J) .

We will use a different (but equivalent to that of [4]) construction of the base bicategory and because the absence of papers on B-categories we feel the need to give the main definitions, though they are simply translations (from one to many objects) of classical V-category ones.

1. We introduce now a notion which corresponds to that of «polyad» in the terminology of [1].

DEFINITION. When B is a bicategory, a *B-category* X is defined by assigning:

- i) objects $x, y, \dots$;
- ii) to every object x an «underlying» object $e(x)$ of B ;
- iii) to every ordered pair $\langle x, y \rangle$ of objects an «object of morphisms»

$$e(x) \xrightarrow{X(x, y)} e(y)$$

in the category $B(e(x), e(y))$;

- iv) to every ordered triple $\langle x, y, z \rangle$ a «composition» in the category $B(e(x), e(z))$:

$$\begin{array}{ccccc} e(x) & \xrightarrow{X(x, y)} & e(y) & \xrightarrow{X(y, z)} & e(z) \\ & \searrow & \Downarrow & \swarrow & \\ & & X(x, z) & & \end{array}$$

- v) to every object x an «identity» in the category $B(e(x), e(x))$

$$\begin{array}{ccc} & I & \\ e(x) & \xrightarrow{\quad} & e(x) \\ & \Downarrow & \\ & X(x, x) & \end{array}$$

The above data have to be subjected to the associativity and unity laws, which can be expressed by commutative diagrams of 2-cells in B . The base bicategories involved in the following are locally partially-ordered, so that these conditions hold trivially.

DEFINITION. If X and Y are two B-categories, a *B-functor* $f: X \rightarrow Y$ is a function on objects which preserves underlyings: $e(fx) = e(x)$; moreover, for each ordered pair $\langle x, x' \rangle$ of X -objects, a 2-cell must be assigned:

$$\begin{array}{ccc}
 & X(x, x') & \\
 e(x) & \xrightarrow{\quad} & e(x') \\
 & \Downarrow & \\
 & Y(fx, fx') &
 \end{array}$$

which preserves identity and composition (always satisfied in the partially-ordered case).

DEFINITION. If X and Y are B -categories, a *bimodule* $X \dashrightarrow Y$ assigns to every ordered pair of objects x in X and y in Y a 1-cell

$$e(y) \xrightarrow{\phi(y, x)} e(x)$$

subject to actions

$$\begin{array}{ccccc}
 e(y) & \xrightarrow{\phi(y, x)} & e(x) & \xrightarrow{X(x, x')} & e(x') \\
 & \searrow & \Downarrow & \nearrow & \\
 & & \phi(y, x') & & \\
 e(y') & \xrightarrow{Y(y', y)} & e(y) & \xrightarrow{\phi(y, x)} & e(x) \\
 & \searrow & \Downarrow & \nearrow & \\
 & & \phi(y', x) & &
 \end{array}$$

satisfying unity, associativity and mixed associativity (always true in the partially-ordered case).

If any hom-category of B allows arbitrary sups preserved by compositions, then bimodules $\phi: X \dashrightarrow Y$ and $\psi: Y \dashrightarrow Z$ can be *composed* by

$$(\psi \circ \phi)(z, x) = \bigvee_y \psi(z, y) \phi(y, x).$$

Any B -functor $f: X \rightarrow Y$ becomes a bimodule

$$f_*: X \dashrightarrow Y \quad \text{by} \quad f_*(y, x) = Y(y, fx),$$

and the essential feature of such bimodules is that there exists an adjoint bimodule

$$f^*: Y \dashrightarrow X \quad \text{defined by} \quad f^*(x, y) = Y(fx, y),$$

where *adjointness* $\phi \dashv \psi$ means

$$\begin{aligned}
 X(x, x') &\leq (\psi \circ \phi)(x, x') \quad \text{for each } x, x' \quad \text{and} \\
 (\phi \circ \psi)(y, y') &\leq Y(y, y') \quad \text{for each } y, y'.
 \end{aligned}$$

If u is an object of B , let us denote by $\hat{u}$ the trivial B -category

with just one object over u .

DEFINITION (Lawvere [3]). A B-category Y is said to be *Cauchy-complete* (shortly C.c.) if for each u and each pair of adjoint bimodules

$$\begin{array}{ccc} \hat{u} & \xrightarrow{\phi} & Y \\ & \psi & \\ & \leftarrow & \end{array} \quad (\phi \dashv \psi)$$

is representable by a functor $y: \hat{u} \rightarrow Y$, i. e., $\phi = y_*$ and $\psi = y^*$.

For this reason, in the following, dealing with adjoint pairs of bimodules, we will simply use ϕ to mean the pair, and ϕ_* , ϕ^* to denote the left and right adjoint parts.

We now prove that Cauchy completion still exists in the B-categorical framework.

THEOREM. *The embedding of Cauchy complete B-categories in B-Cat has a left adjoint in the appropriate two-dimensional sense.*

PROOF. When X is a B-category, define its Cauchy-completion $\tilde{X}$, by taking as objects all pairs of adjoint bimodules $\phi: \hat{u} \dashv \vdash X$. The underlying object of ϕ is u and the hom is defined by $\tilde{X}(\phi, \psi) = \phi^* \circ \psi_*$ (observe that $\phi^* \circ \psi_*$ has just one component $u \rightarrow v$ if $e(\psi) = v$).

Easily the adjointness conditions provide the necessary B-category operations in $\tilde{X}$.

There exists a B-functor $c: X \rightarrow \tilde{X}$ sending objects x of X into the adjoint pair it represents; c is fully faithful:

$$\tilde{X}(c(x), c(x')) = \bigvee_x X(x, x'') X(x'', x') \approx X(x, x').$$

Therefore we can identify without any ambiguity each X -object with its image in $\tilde{X}$. A direct calculation shows that:

$$(*) \quad \tilde{X}(\phi, x) \approx \phi^*(x), \quad \tilde{X}(x, \psi) \approx \psi_*(x).$$

With these identities we can prove that c_* and c^* are inverse bimodules:

$$\begin{aligned} (c_* \circ c^*)(\phi, \psi) &= \bigvee_x c_*(\phi, x) c^*(x, \psi) = \bigvee_x \tilde{X}(\phi, x) \tilde{X}(x, \psi) \approx \\ &\approx \bigvee_x \phi^*(x) \psi_*(x) = \tilde{X}(\phi, \psi), \end{aligned}$$

$$\begin{aligned}
 (c^* \circ c_*)(x, x') &= \bigvee_{\phi} c^*(x, \phi) c_*(\phi, x') = \bigvee_{\phi} \tilde{X}(x, \phi) \tilde{X}(\phi, x') \approx \\
 &\approx \bigvee_{\phi} \phi_*(x) \phi^*(x') \approx X(x, x'),
 \end{aligned}$$

because

$$\bigvee_{\phi} \phi_*(x) \phi^*(x') \leq X(x, x')$$

follows by adjointness, and in the other direction it suffices to take

$$\phi_* = X(-, x) \quad \text{and} \quad \phi^* = X(x, -).$$

$\tilde{X}$ is Cauchy complete: If $\phi: \hat{u} \dashrightarrow \tilde{X}$ is an adjoint pair of bimodules, the composites $c^* \circ \phi_*$ and $\phi^* \circ c_*$ are still adjoint because c_* and c^* are inverses each other, so give rise to a point ψ of $\tilde{X}$ which lies over u . Consider the B-functor which takes the only object of $\hat{u}$ to ψ . The adjoint pair ϕ is represented by ψ :

$$\begin{aligned}
 \tilde{X}(\psi, \theta) &= \phi^* \circ c_* \circ \theta_* = \bigvee_{\psi'} \phi^*(\psi') (c_* \circ \theta_*)(\psi') = \\
 &= \bigvee_{\psi'} \phi^*(\psi') \bigvee_x c_*(\psi', x) \theta_*(x) = \bigvee_{\psi'} \phi^*(\psi') \bigvee_x \tilde{X}(\psi', x) \theta_*(x) \approx \\
 &\approx \bigvee_{\psi'} \phi^*(\psi') \bigvee_x \psi'^*(x) \theta_*(x) = \bigvee_{\psi'} \phi^*(\psi') \tilde{X}(\psi', \theta) \approx \phi^*(\theta).
 \end{aligned}$$

In the same way it can be shown that $\tilde{X}(\theta, \psi) \approx \phi_*(\theta)$.

To show the universal property of the Cauchy-completion, extend any B-functor $g: X \rightarrow Y$ (Y Cauchy complete) along c to a B-functor

$$\tilde{g}: \tilde{X} \rightarrow Y \quad \text{by} \quad \tilde{g}(\phi) = \text{the object of } Y \text{ which represents } g \circ \phi.$$

The functor $\tilde{g}$ is determined up to invertible 2-cells in $\mathbf{B-Cat}$; in the partially ordered case $\tilde{g}$ equivalent to $\tilde{g}'$ just means that for each ϕ the objects $\tilde{g}(\phi)$ and $\tilde{g}'(\phi)$ are B-isomorphic, i. e.,

$$1 \leq Y(\tilde{g}(\phi), \tilde{g}'(\phi)) \quad \text{and} \quad 1 \leq Y(\tilde{g}'(\phi), \tilde{g}(\phi)).$$

2. Following the line of Lawvere's «Metric spaces» [3], where the pursued aim is that «*fundamental* structures are themselves categories ... by taking account of a certain natural generalization of category theory within itself» (namely V-category theory), the further generalization from V to B leads to consider sheaves also as categories.

If (C, J) is a site, we construct a bicategory $Rel_J(C)$ as follows: objects of $Rel_J(C)$ are those of C , 1-cells $R: u \rightarrow v$ are families of spans

$$u \xleftarrow{h} w \xrightarrow{k} v$$

which are saturated by composition, i. e. if $\langle h, k \rangle \in R$, then $\langle fh, fk \rangle \in R$ for all $f: w' \rightarrow w$. We write $\{\langle h, k \rangle\}$ for the 1-cell in $Rel_J(C)$ generated by $\langle h, k \rangle$. Composition RS is defined as the family of spans $\langle h, k \rangle$ for which there exists a g with

$$\langle h, g \rangle \in R \quad \text{and} \quad \langle g, k \rangle \in S.$$

Identities are given by $\{\langle 1, 1 \rangle\}$. It is straightforward to verify that in this way we get a category which defines the 1-dimensional part of $Rel_J(C)$. The 2-cells of $Rel_J(C)$ are essentially depending upon the topology:

$$\begin{aligned} R <_J S \quad \text{iff for all } \langle h, k \rangle \in R \text{ there exists a covering family} \\ \mathcal{U} = \{w_i \xrightarrow{g_i} w\}_{i \in I} \in J(w) \text{ such that } \langle g_i h, g_i k \rangle \in S \\ \text{for all } i \in I. \end{aligned}$$

The proof that $Rel_J(C)$ is a bicategory (in fact, a 2-category) involves directly the axioms of the topology J . Moreover $Rel_J(C)$ is locally a lattice and each $Rel_J(C)(u, v)$ is sup-complete: the sup is simply set-theoretical union of families, and it is easy to verify its strict preservation by composition. Observe that $Rel_J(C)$ is a *symmetric* bicategory, in the sense that there exists a natural isomorphism of categories

$$(-)^0: Rel_J(C)(u, v) \rightarrow Rel_J(C)(v, u)$$

such that $(R^0)^0 = R$ and $(RS)^0 = S^0 R^0$.

We have a faithful functor

$$C \rightarrow Rel_J(C) \quad \text{given by } h \mapsto \{\langle 1, h \rangle\}.$$

This functor allows to identify arrows in C with corresponding ones in $Rel_J(C)$. By this identification, arrows h of C satisfy

$$h h^0 \geq_J 1 \quad \text{and} \quad h^0 h \leq_J 1.$$

The symmetry of the base allows to define *symmetric $Rel_J(C)$ -categories* as those for which $X(x, x')^0 = X(x', x)$.

DEFINITION. $L: S^{Op} \rightarrow Rel(C)\text{-Cat}$ ($Rel(C)$ denotes the bicategory associated to the minimal topology) is a functor defined as follows: LF has objects the sections $x \in Fu$ whose underlying object is u . If $x \in Fu$ and $y \in FV$, then $LF(x, y)$ is the family of spans

$$\langle h, k \rangle \text{ such that } x/h = y/k.$$

Let us observe that the functor L takes its images in the full subcategory of symmetric and skeletal $Rel(C)$ -categories, where *skeletal*, for a B-category X , means:

$$I \leq X(x, y) \text{ and } I \leq X(y, x) \text{ implies } x = y.$$

It is easy to check that the property to be skeletal is equivalent to the uniqueness of representability of bimodules $\hat{u} \dashv \vdash X$, and observe that skeletality destroys the 2-dimensional part of $B\text{-Cat}$.

Finally, observe that by construction the partial order of topologies is preserved, i.e., if $J < J'$, then there is a canonical embedding

$$Rel_J(C)\text{-Cat} \rightarrow Rel_{J'}(C)\text{-Cat}$$

which does not preserve skeletality.

We now need some remarks about symmetry and Cauchy-completion. First observe it is not always true that the Cauchy-completion of a symmetric B-category still is symmetric: consider the monoid $M = Set(A, A)$ as a symmetric Set -category with just one object. It is known that Cauchy-completion for ordinary categories is the universal process of splitting idempotents [3, page 164]; this means that $\tilde{M}$ is not symmetric but in trivial cases. However, in particular cases (e.g. metric spaces) the Cauchy-completion of a symmetric B-category is symmetric. So far we don't know whether the same property holds for all $Rel_J(C)$ -categories. The following lemma provides a characterization for the general case.

LEMMA 1. *Let X be a B-category. The Cauchy-completion $\tilde{X}$ is symmetric iff each adjoint pair $\phi: \hat{u} \dashv \vdash X$ is an inverse pair (i.e., $\phi_*(x)^o = \phi^*(x)$).*

PROOF. In one direction the proof comes directly by the definition of $\tilde{X}$. In the other one, just consider the formulas (*) in the proof of the theorem

on Cauchy-completion and take into account the symmetry of X .

Observe that the a. i. property implies the symmetry of X ; it suffices to particularize the a. i. property to representable bimodules. As we have already remarked, we don't know if the a. i. property is equivalent to the symmetry of X in the $Rel_J(C)$ case.

The previous lemma implies that the Cauchy-completion restricts:

$$\begin{array}{ccc}
 B-Cat & \xrightleftharpoons[e]{\sim} & B-Cat\ C. c. \\
 \uparrow & & \uparrow \\
 B-Cat\ a. i. & \xrightleftharpoons[e]{\sim} & B-Cat\ sym. C. c.
 \end{array}$$

(C.c. = Cauchy-complete) and that $B-Cat\ a. i.$ is the biggest full subcategory through which the adjunction restricts.

In view of Walters result [4], in the case $B = Rel_J(C)$ let us define a functor Γ_J which will provide a useful description of the $\sim$ -process.

$$Rel_J(C)-Cat \xrightarrow{\Gamma_J} sh_J(C)$$

is defined in the following way:

$\Gamma_J X(u)$ = isomorphism classes of adjoint pairs of bimodules

$$\phi: \hat{u} \dashv \vdash X.$$

When $h: v \rightarrow u$ is an arrow in C , the restriction is defined by the adjoint pair over $\hat{v}$: $\phi_*/h(x) = \phi_*(x)h^o$ and $\phi^*/h(x) = h\phi^*(x)$.

Functoriality of $\Gamma_J X$ is an easy matter. For sheaf conditions, let

$$\mathcal{U} = \{u_i \xrightarrow{h_i} u\}$$

be a J -covering family, and $\phi_i: \hat{u}_i \dashv \vdash X$ be a compatible family. Define $\phi: \hat{u} \dashv \vdash X$ by:

$$\phi_*(x) = \bigvee_i \phi_{i*}(x)h_i, \quad \phi^*(x) = \bigvee_i h_i^o \phi_i^*(x).$$

They are adjoint: to check

$$I \leq \bigvee_j x \left[\bigvee_i h_i^o \phi_i^*(x) \bigvee_j \phi_{j*}(x)h_j \right]$$

it is sufficient to take $i = j$, and so to check

$$I \leq \bigvee_j \bigvee_i [h_i^q(\bigvee_x \phi_i^*(x) \phi_{i*}(x) h_i)].$$

But $\phi_{i*} \dashv \phi_i^*$, so it is enough to check $I \leq \bigvee_j h_i^q h_i$, which is true because $\{h_i\}$ is a covering family. To verify the other adjointness condition, first observe that compatibility means that, for each commutative square $k_i h_i = k_j h_j$, it holds

$$\phi_{i*} k_i^q \approx \phi_{j*} k_j^q \quad \text{and} \quad k_i \phi_i^* \approx k_j \phi_j^* \quad \text{for each } i, j.$$

Hence:

$$\phi_{i*} k_i^q k_j \approx \phi_{j*} k_j^q k_j \leq \phi_{j*}.$$

So, for each $\langle k_i, k_j \rangle$ in $h_i h_j^q$, it holds

$$\phi_{i*}(x) k_i^q k_j \phi_j^*(x') \leq \phi_{j*}(x) \phi_j^*(x') \leq X(x, x'),$$

hence $\phi_{i*}(x) \circ \phi_j^*(x') \leq X(x, x')$.

LEMMA 2. *There exists a functor L_J*

$$\begin{array}{ccc} \text{Rel}_J(\mathcal{C})\text{-Cat a. i.} & \xrightleftharpoons[e]{\bar{e}} & \text{Rel}_J(\mathcal{C})\text{-Cat sym. c. c.} \\ \Gamma_J \searrow & & \nearrow L_J \\ & sh_J(\mathcal{C}) & \end{array}$$

such that $\Gamma_J \dashv L_J e$, in the appropriate 2-dimensional sense.

PROOF. L_J is defined as in Walters [4], Proposition 1, by the composition:

$$sh_J(\mathcal{C}) \longrightarrow S^{C^{op}} \xrightarrow{L} \text{Rel}(\mathcal{C})\text{-Cat} \longrightarrow \text{Rel}_J(\mathcal{C})\text{-Cat}$$

where it is shown that it factorizes through $\text{Rel}_J(\mathcal{C})\text{-Cat sym. c. c.}$ and that it is fully-faithful. Observe now that for each X in $\text{Rel}_J(\mathcal{C})\text{-Cat a. i.}$, it holds $\tilde{X} \approx L_J(\Gamma_J X)$. Clearly both categories agree on elements (up to isomorphisms); for homs:

$$\tilde{X}(\phi, \psi) = \phi^* \circ \psi_* = \{ \langle h, k \rangle \mid \phi_* \circ h^o \approx \psi_* \circ k^o \} = L_J(\Gamma_J X(\phi, \psi)).$$

Indeed, let $\langle h, k \rangle$ be such that $\phi_* \circ h^o \approx \psi_* \circ k^o$; then

$$\phi^* \circ \phi_* \circ h^o \circ k \approx \phi^* \circ \psi_* \circ k^o \circ k \leq \phi^* \circ \psi_*;$$

but $I \leq \phi^* \circ \phi_*$ and $\langle h, k \rangle \in h^0 k$, thus $\langle h, k \rangle \in \phi^* \circ \psi_*$. Conversely, let $\langle h, k \rangle \in \phi^* \circ \psi_*$; then $h^0 k \leq \phi^* \circ \psi_*$, therefore

$$\phi_* \circ h^0 \leq \phi_* \circ h^0 \circ k^0 \leq \phi_* \circ \phi^* \circ \psi_* \circ k^0 \leq \psi_* \circ k^0.$$

Now

$$\psi_* \circ k^0 \leq \psi_* \circ k^0 \circ h^0 \circ h^0 \leq \psi_* \circ \psi^* \circ \phi_* \circ h^0 \leq \phi_* \circ h^0,$$

because $h^0 k \leq \phi^* \circ \psi_*$ and the a.i. property implies

$$k^0 h \leq \psi_*^0 \circ \phi^{*0} = \psi_*^* \circ \phi_*.$$

Now the following chain of equivalences proves the adjunction:

$$\begin{array}{ll} \underline{X \longrightarrow e(L_J F)} & \text{by the theorem on } \sim \\ \underline{\bar{X} \longrightarrow L_J F} & \text{by the previous remark} \\ \underline{L_J(\Gamma_J X) \rightarrow L_J F} & L_J \text{ is 2-fully-faithful} \\ \Gamma_J X \longrightarrow F. & \end{array}$$

THEOREM (Walters [4] Proposition 2). *The functor L_J is a 2-equivalence.*

A direct proof may be obtained by considering the adjunction: $\Gamma_J \dashv L_J e$. Because $L_J e$ is 2-fully-faithful, it is enough to prove that $\eta_X: X \rightarrow e(L_J(\Gamma_J X))$ is an equivalence iff X is Cauchy-complete.

By this theorem, we will call Γ_J simply $\sim$.

3. We want now to compare the previous adjunction with the associated sheaf functor.

THEOREM. *There exists a comparison functor L' :*

$$\begin{array}{ccc} \text{Rel}_J(C)\text{-Cat a.i.} & \xrightleftharpoons[L_J e]{\sim} & sh_J(C) \\ \uparrow L' & \nearrow a & \\ SC^{op} & & \end{array}$$

To prove the theorem we need a suitable description of the associated sheaf functor a . For our purpose we found the best one to be that

in [2], where $aF(u)$ is given by « u -locally compatible families of elements of F , with covering support and closed», which means:

DEFINITION 2. If F is a presheaf and u an object of C , a u -locally compatible family with covering support is the assignment for each arrow $i: v \rightarrow u$ of a family $\mathcal{F}_i \subset Fv$ such that:

- 1° if $x \in \mathcal{F}_i$, for each $h: w \rightarrow v$, $x/h \in \mathcal{F}_{hi}$;
- 2° the crible $\{i: v \rightarrow u \mid \mathcal{F}_i \neq \emptyset\}$ is J -covering («covering support»);
- 3° if $x, y \in \mathcal{F}_i$, $\{k: w \rightarrow v \mid x/k = y/k\}$ is J -covering («local compatibility»).

Such a family is *closed* if moreover:

- 4° if $x \in Fv$ and $\{k: w \rightarrow v \mid x/k \in \mathcal{F}_{ki}\}$ is a J -covering family, then $x \in \mathcal{F}_i$.

Define $L'F$ as LF but thought in $Rel_J(C)$ -Cat sym.

LEMMA 1. If $\phi: \hat{u} \dashrightarrow L'F$ is a bimodule, and $\{<h, k>\} \leq_J \phi(x)$ then $k \leq_J \phi(x/h)$.

PROOF. Directly we have $k \leq_J h \{<h, k>\}$.

By the bimodule property and the assumption:

$$h \{<h, k>\} \leq_J L'F(x/h, x) \phi(x) \leq_J \phi(x/h).$$

LEMMA 2. Isomorphism classes of pairs of adjoint bimodules $\phi: \hat{u} \dashrightarrow L'F$ are in 1-1 correspondance with u -locally compatible, with covering support, closed families of parts of F .

PROOF. Let us consider such a u -family $\mathcal{F}$. Define a pair of adjoint bimodules $\phi = \phi(\mathcal{F})$ by

$$\phi_*(x) = \bigvee_{x' \in \mathcal{F}_i} L'F(x, x') i, \quad \phi^*(x) = \phi_*(x)^0.$$

The proof of the adjointness condition $1 \leq_J \bigvee_x \phi^*(x) \phi_*(x)$ is the same as that in the proof of sheaf conditions for Γ_J , by using condition 2 on $\mathcal{F}$. The other adjointness condition $\phi_*(x) \phi^*(y) \leq_J L'F(x, y)$ holds because for each $x' \in \mathcal{F}_i$ and $x'' \in \mathcal{F}_j$ we have

$$L'F(x, x') i j^0 \leq L'F(x, x'') :$$

$\langle r, s \rangle \in L'F(x, x') i j^0$ means that there exists t such that $x/r = x'/t$, $t i = s j$; by 1,

$$x/r = x'/t \in \mathcal{F}_{ti} \quad \text{and} \quad x''/s \in \mathcal{F}_{sj} = \mathcal{F}_{ti};$$

by 3, x/r and x''/s agree on a covering.

Conversely, given an adjoint pair of bimodules $\phi: \hat{u} \dashrightarrow L'F$, define $\mathcal{F} = \mathcal{F}(\phi)$ by

$$\mathcal{F}_i = \{ y \in Fv \mid i \leq \phi_*(y) \text{ and } i^0 \leq \phi^*(y) \}$$

for each $i: v \rightarrow u$.

Condition 1: if $i \leq \phi_*(y)$, then for each $k: w \rightarrow v$ also $\langle k, ki \rangle \leq \phi_*(y)$ which by Lemma 1 implies $ki \leq \phi_*(y/k)$; $i^0 k^0 \leq \phi^*(y/k)$ follows in a similar way from $i^0 \leq \phi^*(y)$ and a «dual» of Lemma 1.

Condition 2: by the adjointness condition $1 \leq \phi_* \circ \phi^*$, it follows that there exists a covering

$$\mathcal{U} = \{ u_\alpha \xrightarrow{k_\alpha} u \}$$

such that for each α there exists x_α with $\langle k_\alpha, k_\alpha \rangle \in \phi^*(x_\alpha) \phi_*(x_\alpha)$.

This means that there exists m_α such that

$$\langle k_\alpha, m_\alpha \rangle \in \phi^*(x_\alpha) \quad \text{and} \quad \langle m_\alpha, k_\alpha \rangle \in \phi_*(x_\alpha).$$

By Lemma 1 we have $k_\alpha \leq \phi_*(x_\alpha/m)$. So the family

$$\{ i: v \rightarrow u \mid \mathcal{F}_i \neq \emptyset \}$$

contains a covering family, namely $\mathcal{U}$.

Condition 3: if $x, y \in \mathcal{F}_i$, then $ii^0 \leq \phi_*(x) \phi^*(y)$. By the adjointness condition

$$\phi_*(x) \phi^*(y) \leq L'F(x, y)$$

and because $1 \leq ii^0$, we have $1 \leq L'F(x, y)$, which proves condition 3.

Condition 4: we have to show that for each $i: v \rightarrow u$ and each $y \in Fv$, if

$$\mathcal{U} = \{ k: w \rightarrow v \mid y/k \in \mathcal{F}_{ki} \}$$

is a covering family, then $i \leq \phi_*(y)$ and $i^0 \leq \phi^*(y)$. So

$$k i \leq \phi_*(y/k) \quad \text{and} \quad i^0 k^0 \leq \phi^*(y/k).$$

Because $k^0 \in L'F(y, y/k)$, then

$$\{<k, ki>\} = k^0 k i \leq L'F(y, y/k) \phi_*(y/k) \leq \phi_*(y)$$

holds for each k in $\mathcal{U}$. It follows $i \leq \phi_*(y)$. Analogously $i^0 \leq \phi^*(y)$.

Let us check that the two correspondances are inverse each other, when the first one is restricted to closed families. If $\mathcal{F}'$ is the family associated to $\phi(\mathcal{F})$, then $\mathcal{F}_i \subset \mathcal{F}'_i$ for each $i: v \rightarrow u: \mathcal{F}'_i$ being

$$\{y \in Fv \mid i \leq \bigvee_{x'} L'F(y, x')k\}$$

it is enough to take $k = i$ and $x' = y$. Suppose now $\mathcal{F}$ closed; let

$$y \in \mathcal{F}'_i, \quad \text{i.e.} \quad i \leq \bigvee_{x'} L'F(y, x')k,$$

which means there exists a covering $\mathcal{U} = \{h_a: w_a \rightarrow v\}$ such that for each a there exist x_a and k_a with

$$x_a \in \mathcal{F}_{k_a} \quad \text{and} \quad <h_a, h_a i> \in L'F(y, x_a)k_a.$$

It follows there exists t with

$$y/h_a = x_a/t \quad \text{and} \quad h_a i = t k_a.$$

So by condition 1, $y/h_a \in \mathcal{F}_{t k_a} = \mathcal{F}_{h_a i}$. Therefore the family

$$\{h: w \rightarrow v \mid y/h \in \mathcal{F}_{h i}\}$$

contains a covering family. Because $\mathcal{F}$ is closed, $y \in \mathcal{F}_i$.

In the other direction, if ϕ' is the adjoint pair of bimodules corresponding to $\mathcal{F}(\phi)$, it is easy to check $\phi \approx \phi'$: for each $y \in \mathcal{F}_i$ it holds

$$L'F(x, y) i \leq L'F(x, y) \phi_*(y) \leq \phi_*(x),$$

thus

$$\phi'_*(x) = \bigvee_{y \in \mathcal{F}_i} L'F(x, y) i \leq \phi_*(x);$$

conversely, if $\{<h, k>\} \leq \phi_*(x)$, then by Lemma 1 we get $k \leq \phi_*(x/h)$, i.e. $x/h \in \mathcal{F}_k$; so $<h, k> \in \phi'_*(x)$, because it belongs to $L'F(x, x/h)k$.

PROOF OF THE THEOREM. The proof of Lemma 2 shows that $L'F$ is an

a.i. category. The stated bijectivity proves one commutativity, namely $aF = L'F$. The other one is trivial.

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ADDENDUM IN PROOFS.

After this work was submitted, it has been shown that the a.i. hypothesis of Lemma 2 and of the Theorem of Section 3 is not necessary, because from a result by Betti and Walters (The symmetry of the Cauchy-completion of a category, to appear on the Proc. of 1981 Hagen Conference) it follows that in the $Rel_J(C)$ case the Cauchy-completion preserves symmetry (see the remark after Lemma 1, where this problem was posed).

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