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SYMMETRIZATIONS OF CATEGORIES AND CATEGORIES OF RELATIONS

by Giuseppe CONTE

INTRODUCTION.

This paper is concerned with relations in general categories. The classical example is given by the relations in the category *Set*. A relation a between two sets A and B is a subobject of the product $A \times B$. Relations are composable by pullbacks and they form an involution category (i.e. a category provided with a contravariant endofunctor J such that $J.J$ is the identity) in which *Set* is embeddable.

In categories with finite products, the concept of relation between two objects may be generalized as in [11], defining it as a subobject (of the product) with respect to a given bicategory structure.

For a category C without any assumption on products, relations between objects A and B may be defined as pairs of maps $A \leftarrow X \rightarrow B$ under a suitable equivalence relation. When composition of equivalence classes of pairs by pullback is associative, relations so defined form an involutive category in which C is embeddable. This situation was considered by many authors (M.S.Calenko [5], Y. Kawahara [10], A. Klein [12], F. Parodi [14], Coppey & Davar Panah [15]) and the category of relations so obtained is algebraically a sort of category of quotients of C.

If C is abelian any relation $A \leftarrow X \rightarrow B$ can be factorized in an essentially unique way as

$$A \leftarrowtail A' \twoheadrightarrow C \twoheadleftarrow B' \rightarrowtail B$$

where $\rightarrowtail$ and $\twoheadrightarrow$ denote respectively a monic and an epic in C. In an exact category, relations may be defined as diagrams of the above kind

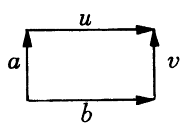
(H.B. Brinkmann & D. Puppe [3], M.S. Calenko [4], P.J. Hilton [9]).

In our approach a category of relations for C is viewed as the quotient, by means of a suitable equivalence relation, of a bigger involution category containing C. In the terminology of [7] this involution category, denoted by C^M, is called the maximum symmetrization of C and any equivalence relation in C^M compatible with composition and involution is said a congruence. Any involution category containing C in an essential way and whose involution changes the isomorphisms of C into their inverse is a quotient of C^M by a suitable congruence. Conversely any congruence of C^M defines such a category.

The generality of this point of view has some advantage as we will try to show in this paper. First we prove that the categories of relations constructed in the above mentioned papers are all defined by congruences of the same kind depending on the choice of a subcategory of C. These congruences are spanned in C^M by two simple conditions involving pullbacks and the commutativity of a particular diagram. In some cases this allows us to give an easy description of relations.

Then we compare, using the corresponding congruences, different constructions of categories of relations. Our results show that in some interesting cases the «natural» order relation between involution categories containing C may be expressed by means of properties of exact squares of C.

NOTATIONS. A square of maps



in C will be denoted by $[a, b, u, v]$. The symbol * means «dual».

1.1. We recall some notions introduced in [7]. A symmetrization of a category C (roughly speaking an embedding of C in an involution category H having the same objects as C) is defined as a pair (s, J) such that the following conditions are satisfied:

S1) $s: C \to H$ is a functor injective on the objects;

S2) J is an involution on H, i.e. a contravariant endofunctor identical on the objects and such that $JJ = 1_H$;

S3) any involutive subcategory of H containing $s(C)$ is equal to H;

S4) $J s(u) = s(u^{-1})$ for any isomorphism u of C.

By abuse of notations we will often denote a symmetrization by $s: C \to H$ or shortly by s. Usually $J(a)$ will be denoted by $\tilde{a}$.

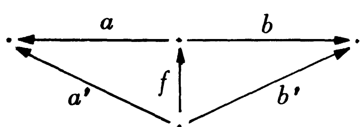
Any category C has a symmetrization $s_{CM}: C \to C^M$ which is maximum in the following sense: for any symmetrization $s: C \to H$ there exists a unique functor h_s compatible with involution such that $s = h_s s_{CM}$ (C^M is a quotient of the category $CWORD(C)$ of [2]).

The relation defined in C^M by $\alpha \,\mathfrak{R}_s\, \beta$ iff $h_s(\alpha) = h_s(\beta)$ gives an equivalence relation on any $Hom(A, B)$ compatible with composition and involution. Such a relation is called a congruence and any congruence $\mathfrak{R}$ of C^M defines a symmetrization $s(\mathfrak{R})$ of C, given by the quotient $C^M/\mathfrak{R}$. We consider the congruences of C^M ordered as «parts» of $C^M \times C^M$. This induces a preorder relation, denoted by $>$, between symmetrizations of C. The associated equivalence relation between symmetrizations will be denoted by $\approx$.

1.2. Given a category C let $E \subset C$ be any subcategory. E defines the congruence $\mathfrak{R}_E$ of C^M spanned by the following two conditions:

CPB) if $[a', b', b, a]$ is a pullback in C, then $b'\tilde{a}' \,\mathfrak{R}_E\, \tilde{a} b$;

CE) if the following diagram in C

1.2.1 

commutes and $f \in E$, then $b\tilde{a} \,\mathfrak{R}_E\, b'\tilde{a}'$.

Then for any category C we can consider the symmetrization $s(\mathfrak{R}_E): C \to C^M/\mathfrak{R}_E$ which depends on the choice of a subcategory E. Obviously $E' \subset E$ implies $s(\mathfrak{R}_{E'}) > s(\mathfrak{R}_E)$.

Specializing the subcategory E we obtain some classical constructions. The first example is given by Proposition 1.3 below. First we recall that a span of C is a pair (a, b) of maps with the same domain. We denote by $(a, b)f$ the span (af, bf).

Throughout the rest of the paper we assume that C has pullbacks.

1.3. PROPOSITION. *Let $Iso(C) \subset C$ be the subcategory of isomorphisms. Then the symmetrization $s(\mathfrak{R}_{Iso(C)})$ is given by the embedding of C in the involution category $SPN(C)$ of its spans.*

Any map in $SPN(C)$ is a class $(\overline{a, b})$ of spans modulo isomorphism. Composition is given by pullback.

1.4. Let us introduce the following conditions concerning the subcategory E :

P1) $Iso(C) \subset E$;

P2) if $[a', b', b, a]$ is a pullback in C and $a \in E$, then $a' \in E$.

1.5. PROPOSITION. *Assume that $E \subset C$ verify P1, P2. Then the equivalence relation $\approx$ defined in any $Hom(A, B)$ in $SPN(C)$ by:*

$$\alpha = (\overline{a, b}) \approx \beta = (\overline{a', b'}) \quad \textit{iff there exist } f, g \in E \textit{ such that } (a, b)f = (a', b')g$$

is compatible with composition and involution and is spanned by CE.

As a consequence the symmetrization $s(\mathfrak{R}_E)$ is given by the canonical functor $C \to SPN(C)/\approx$.

PROOF. $\approx$ is an equivalence relation compatible with composition and involution by P2 as it is proved in [11], 2.5. Denote again by $\mathfrak{R}_E$ the relation spanned in $SPN(C)$ by CE. It is implied by $\approx$ and so coincides with it. The second part is obvious by P1.

1.6. The description of $s(\mathfrak{R}_E)$ given by 1.5 shows that the categories of relations constructed by various authors in [5, 6, 11, 12, 14, 15] for some classes of categories can all be obtained as quotients of C^M by $\mathfrak{R}_E$ for a suitable choice of E.

To illustrate our point of view we consider the following example.

Let N be the semigroup of positive integers (or non-zero integers). As shown in [11], the group of positive (non-zero) rational numbers is a category of relations for N. It is of the form $N^M/\Re_N$ as can easily be proved expressing the condition CN in the form:

$$m = m'p \text{ and } n = n'p \text{ implies } m/n \,\Re_N\, m'/n'.$$

This shows immediately that any map of $N^M/\Re_N$ can be represented in an essentially unique way as a span (m, n) (rational fraction m/n) such that the g.c.d. of m and n is 1.

This situation is generalized by the following proposition:

1.7. PROPOSITION. *Let $E \subset C$ be a subcategory and let F be a family of spans such that any span (a, b) of C has a unique up to isomorphism factorization*

$$(a, b) = (a', b')f \quad \textit{where} \quad (a', b') \in F \textit{ and } f \in E.$$

Then any map of $C^M/\Re_E$ has a uniquely determined representation given by a span of F.

PROOF. By 1.5 and CE.

1.8. Two examples in which the hypothesis of 1.7 is verified are the following ones:

a) C has finite products, E is the subcategory of epimorphisms in the sense of [1] and F is the family of spans (a, b) such that the product map $\begin{bmatrix} a \\ b \end{bmatrix}$ is monic (see [6]); in this case the maps of $C^M/\Re_E$ are the subobjects of the products.

b) C and E are such that any span (a, b) has a unique factorization $(a, b) = (a', b')f$ with $f \in E$ and (a', b') is a dividing span (i.e.,

$$(a', b')g = (a', b')h \text{ implies } g = h),$$

F is the family of dividing spans (such categories were considered in [5, 6]); in this case the maps of $C^M/\Re_E$ are the dividing spans.

2.1. The conditions CPB and CE of 1.1 can be dualized as CPO and C*E

in the obvious way. CPO and C^*E span the congruence $\mathfrak{R}^*_E$ of C^M and all the properties of $s(\mathfrak{R}_E)$ hold, in the dual formulation, for $s(\mathfrak{R}^*_E)$.

Given two subcategories E and G of C we want now to compare $s(\mathfrak{R}^*_E)$ and $s(\mathfrak{R}_G)$.

We will show that the order relation between $s(\mathfrak{R}^*_E)$ and $s(\mathfrak{R}_G)$ is related to properties of exact squares in C. First we introduce the following definitions (compare with Guitart [16]).

2.2. DEFINITION. Let $E \subset C$ be a subcategory and let $[a, b, u, v]$ be a square of maps. The above square is said *E-exact* iff:

a) there exists the pullback $[v', u', u, v]$ and

b) the map f such that $a = v'f$ and $b = u'f$ belongs to E.

The above square is said *E-coexact* iff a* and b* hold.

2.3. PROPOSITION. *Let C be a category with pushouts and let E, G be subcategories such that G consists of epics only. If pullbacks in C are E-coexact, then $s(\mathfrak{R}_G) > s(\mathfrak{R}^*_E)$.*

PROOF. Suppose that $[v', u', u, v]$ is a pullback; then $u'\tilde{v}'\mathfrak{R}_G \tilde{v}u$. Let $[v', u', u'', v'']$ be a pushout; by CPO, coexactness and C^*E we have

$$u'\tilde{v}'\mathfrak{R}^*_E \check{v}''u''\mathfrak{R}^*_E \tilde{v}u .$$

Suppose that $(a', b') = (a, b)f$ with $f \in G$; then $b\tilde{a}\mathfrak{R}_G b'\tilde{a}'$. Since f is epic, by CPO, $f\tilde{f}\mathfrak{R}^* 1$. Therefore $b'\tilde{a}' = bf\tilde{f}\tilde{a}\mathfrak{R}^*_E b\tilde{a}$, so $\mathfrak{R}_G \subset \mathfrak{R}^*_E$.

2.4. COROLLARY. *Let C be a category with pushouts. If any pullback in C is a pushout and conversely any pushout is a pullback, then there exists an isomorphism between the involution categories $SPN(C)$ and $COSPNS(C)$ which commutes with the embeddings of C.*

Taking E and G equal respectively to the subcategories of monics and epics, the hypotheses of 2.3 and 2.3* are verified in any abelian category C (see [6]). As a consequence, if C is abelian,

$$s(\mathfrak{R}^*_{Mon(C)}) = s(\mathfrak{R}_{Epi(C)}).$$

2.5. Now suppose that (C, E, G) is a bicategory not necessarily provided with pullbacks and denote by $\mathfrak{R}_W$ the congruence spanned in C^M by the following conditions:

2.5.1. If $[a', b', b, a]$ is a pullback in E, then $b'\tilde{a}'\,\mathfrak{R}_W\,\tilde{a}b$;

2.5.1*. If $[a, b, b', a']$ is a pushout in G, then $b\tilde{a}\,\mathfrak{R}_W\,\tilde{a}'b'$;

2.5.2. If $[a', b', b, a]$ is a pullback in C and $a, a' \in E$ and $b, b' \in G$, then $b'\tilde{a}'\,\mathfrak{R}_W\,\tilde{a}b$.

The symmetrization of C given by $s(\mathfrak{R}_W): C \to C^M/\mathfrak{R}_W$ was introduced in [8, 13]. It generalizes the construction of a category of relations developed in [3] for the abelian and exact cases. Any map of $C^M/\mathfrak{R}_W$ can be represented by a diagram

$$A \longleftarrow\!\!< A' \longrightarrow\!\!\!\!\rightarrow C \twoheadleftarrow B' \rightarrowtail B,$$

where $\rightarrowtail$ and $\twoheadrightarrow$ denote respectively a map of E and of G. This representation is unique up to isomorphism under some hypotheses on C (see [8, 13]) which are verified for instance by the category of groups.

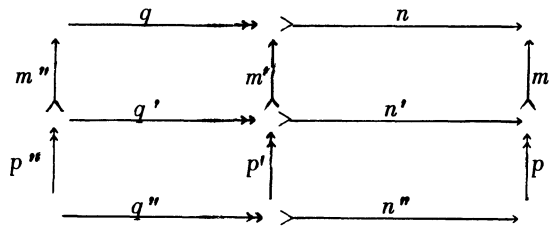
Comparing $s(\mathfrak{R}_G)$ with $s(\mathfrak{R}_W)$ we obtain the following results which can also be stated in the dual formulation.

2.6. PROPOSITION. *Given (C, E, G) suppose that any pushout in G is G-exact. Then $s(\mathfrak{R}_W) > s(\mathfrak{R}_G)$.*

PROOF. The binary relation defined by 2.5.1 and 2.5.2 is contained in the one defined by CPB. Suppose that $[p, q, q', p']$ is a pushout in G; then $q\tilde{p}\,\mathfrak{R}_W\,\tilde{p}'q'$. Let $[p'', q'', q', p']$ be a pullback. By exactness, CPO and CG, we have $q\tilde{p}\,\mathfrak{R}_G\,q''\tilde{p}''\,\mathfrak{R}_G\,\tilde{p}'q'$. So $\mathfrak{R}_W \subset \mathfrak{R}_G$.

2.7. PROPOSITION. *Given (C, E, G) suppose that any pullback in G is a pushout and that given a pullback $[p', q', q, p]$, $p \in G$ implies $p' \in G$. Then $s(\mathfrak{R}_G) > s(\mathfrak{R}_W)$.*

PROOF. Suppose that $[a', b', b, a]$ is a pullback; then $b'\tilde{a}'\,\mathfrak{R}_G\,\tilde{a}b$. Let $a = mp$ and $b = nq$ be E-G factorizations and consider the following diagram (page 8) where each square of maps is constructed by pullback.
We have $a' = m''p''h$ and $b' = n''q''h$ for a unique isomorphism h. Therefore:



$$\tilde{a}b = \tilde{p}mnq\ \mathfrak{R}_W\ \tilde{p}n'\tilde{m}'q\ \mathfrak{R}_W\ n''\tilde{p}'q'\tilde{m}''\ \mathfrak{R}_W\ n''q''\tilde{p}''\tilde{m}''\ \mathfrak{R}_W\ b'\tilde{a}',$$

and so $b'\tilde{a}'\mathfrak{R}_W\ \tilde{a}b$.

Suppose that diagram 1.2.1 commutes; then $b\tilde{a}\,\mathfrak{R}_G\ b'\tilde{a}'$. Since $f \in G$ is epic, $[f,f,1,1]$ is a pushout in G and then $f\tilde{f}\,\mathfrak{R}_W\ 1$. Therefore

$$b'\tilde{a}' = bf\tilde{f}\tilde{a}\ \mathfrak{R}_W\ b\tilde{a}.$$

So $\mathfrak{R}_G \subset \mathfrak{R}_W$.

2.8. The hypotheses of 2.6 and 2.7 are verified in any abelian category C as well as in the category *Grp* of groups. Hence in particular there is an isomorphism between the symmetrizations

$$s(\mathfrak{R}_W) \quad \text{and} \quad s(\mathfrak{R}_{Epi(Grp)})$$

of *Grp* which commutes with the embeddings of *Grp*. Dualizing 2.6 and 2.7 one can prove that there exists an isomorphism between the symmetrizations

$$s(\mathfrak{R}^*_W) \quad \text{and} \quad s(\mathfrak{R}^*_{Mon(Set)})$$

of the category *Set* of sets which commutes with the embeddings.

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