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## COMBINATORICS, GROUP THEORY AND GEOMETRIC MODELS

by Jean PEDERSEN

### 0. INTRODUCTION.

The work of Professor Ehresmann was rooted in geometry and was concerned with its interrelations between various parts of mathematics.

I have also been concerned, though in a more modest way, with the same aspects of mathematics, principally at the undergraduate, or even pre-college level. As a professional geometer my point of view is that the principal role geometry plays in the mathematical education of the student is that of providing an immensely rich source of natural motivation, rather than as an ideal system for displaying the power of abstract reasoning. For the latter purpose I think it is more appropriate to look to algebra or analysis.

In this paper I will give some examples of the way I have attempted to present concrete geometry to students and to link it to group theory and elementary real analysis.

In Section 1 a hexaflexagon is used to motivate the introduction of some very practical folding techniques on straight strips of paper such as adding machine tape or the gummed tape ordinarily used to seal packages. Knowing these techniques would enable you to approximate (to any desired degree of accuracy) a regular polygon having, say,  $2^n \pm 1$  sides, by simply folding straight strips of paper in an appropriate iterative sequence<sup>1</sup>.

1. Of course ruler and compass constructions are only «perfect in the mind». This important aspect is frequently overlooked when applying geometric concepts to the real world. Although the folding sequences mentioned here never guarantee perfection they are *convergent* sequences, and as long as you fold approximately correctly each fold will produce a better approximation to its limit than its predecessor. On the other hand, as we all know, in real life the student's ruler and compass construction frequently *diverges*, the accuracy of the final result often being a function of how recently the student has sharpened his or her pencil.

The mathematical by-products of the material covered in this section include group theory and limits. One relevant theorem on limits, of the kind we recommend for student discussion, is contained in Appendix A at the end of the paper.

Section 2 is an account of how the folded tape described in Section 1 can be used to construct some special polyhedra. The construction of such polyhedra then leads very naturally to group theory and combinatorics.

Section 3 closes this paper with a brief mention of some research questions that have grown out of this concrete approach to geometry.

I wish to acknowledge helpful conversations with Peter Hilton during the preparation of this paper.

## 1. ITERATIVE FOLDING SEQUENCES.

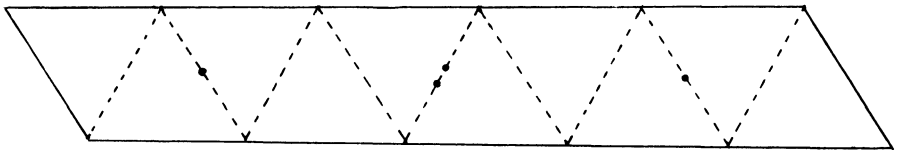


FIGURE 1

A straight strip of ten equilateral triangles scored so that they flex along the nine edges connecting them (as shown in Figure 1) can be folded into an hexagon by «valley folding» each of the transversals marked with a single dot and by «mountain folding» the transversal marked with a double dot. The resulting configuration, with the two overlapping end triangles glued together, is well known as a tri-hexaflexagon. It was invented in 1939 by Arthur H. Stone and is the ancestor of many hundreds of polygon-like figures that can be made to lie in a plane; yet, when flexed appropriately in 3-dimensions, will reveal new faces in an almost hypnotic way (see [1] through [7] for details about constructing this and other flexagons).

To gain an appreciation for the esthetic qualities of this particular

model you should color one «face» (i.e., the collection of six equilateral triangles that lie within a hexagonal boundary when the flexagon is flat) in some symmetrically pleasant way; then turn the model over and color the other visible «face» with another design. Then to bring out the third «face» valley fold along one of the two sets of three axes radiating from the center of the hexagon at 120 degrees from each other, and bring those three axes together below the hexagon (see Figure 2). If you have chosen the right

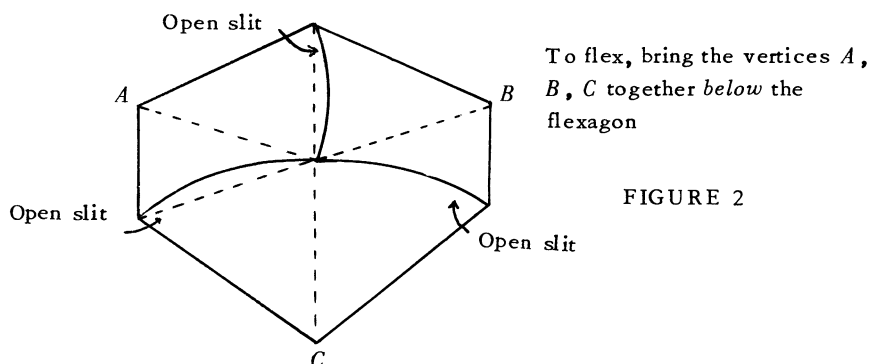


FIGURE 2

set of axes for your valley folds the hexagon will come apart at the top in a rotating fashion and lie flat to reveal the third (uncolored) hexagonal «face». If your flexagon does not come apart at the top when you try this you must choose the other set of three axes to fold as valley folds. This model can easily be shown to be a concrete representation of the symmetric group  $S_3$  with the elements of the group being the transformations in space that permute the three visible «faces» of the model ([8] and [9] provide details of the group theory for this and other flexagons).

The hexaflexagon is, of course, interesting enough to pursue as a mathematical object in its own right <sup>1</sup>. However, our purpose here is to use

<sup>1</sup> Martin Gardner informed me, in 1971, that he had in his possession a lengthy manuscript, by Anthony S. Conrad and Daniel K. Hartline, exploring numerous possibilities for other flexagons. That manuscript is dated May, 1962, and Dover Press would like to publish it, but they are unable to locate either of the authors. If any readers know these men, please ask them to contact either the author of this paper, Martin Gardner or Dover Press - otherwise this manuscript is not likely to ever appear in print.

it to motivate other constructions with straight strips of paper. To that end let us suppose that we begin with a long strip of paper having paralleled edges and further suppose that we wish to fold it into equilateral triangles. There is, of course, a perfectly good way actually to construct an angle of  $\pi/3$  by folding, but suppose we don't know that and that we begin with some arbitrary angle  $x_0$  (as shown in Figure 3 (a)) and treat it as though it were  $\pi/3$ . The first step would be to bisect, by folding, the (obtuse<sup>1</sup>) angle at the top of the tape (the result would appear as shown in Figure 3 (b)); next, bisect, by folding, the resulting obtuse angle produced to the right of that on the bottom of the tape (the result would appear as shown in Figure 3 (c)): etc. Of course there was nothing special about the angle  $x_0$  chosen in Figure 3 (a) and it is the case that for any arbitrary  $x_0$  the acute angles on tape folded in this manner will always converge to  $\pi/3$ .

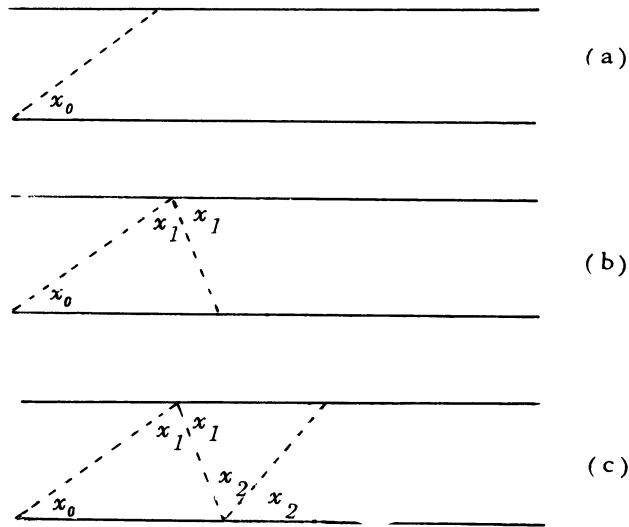


FIGURE 3

<sup>1</sup> All of the examples shown in this paper are illustrated using an initial angle,  $x_0$ , between 0 and  $\pi/2$ , but this is only so that the next obtuse angle will always be to the right of  $x_0$  on the tape. This simplifies the *description* of the folding process. Both the folding and the mathematics are, of course, still valid for  $x_0$  between  $\pi/2$  and  $\pi$ .

This result can be proved in a variety of ways, depending on the level of mathematical sophistication you wish to employ. A fairly direct and elementary approach would be to make use of what we already believe to be true from our observations. Thus, assume that the initial angle  $x_0$  is  $\pi/3 + \epsilon$ , where  $\epsilon$  may be regarded as error. Thus from the angles surrounding the vertex at the top of the tape in Figure 3 (b) we see that

$$2x_1 + x_0 = 2x_1 + \pi/3 + \epsilon = \pi,$$

hence

$$x_1 = \pi/3 + (-1/2)\epsilon.$$

And, likewise, from Figure 3 (c) (using the angles surrounding the vertex at the bottom edge of the tape) it follows that

$$x_2 = \pi/3 + (-1/2)^2\epsilon.$$

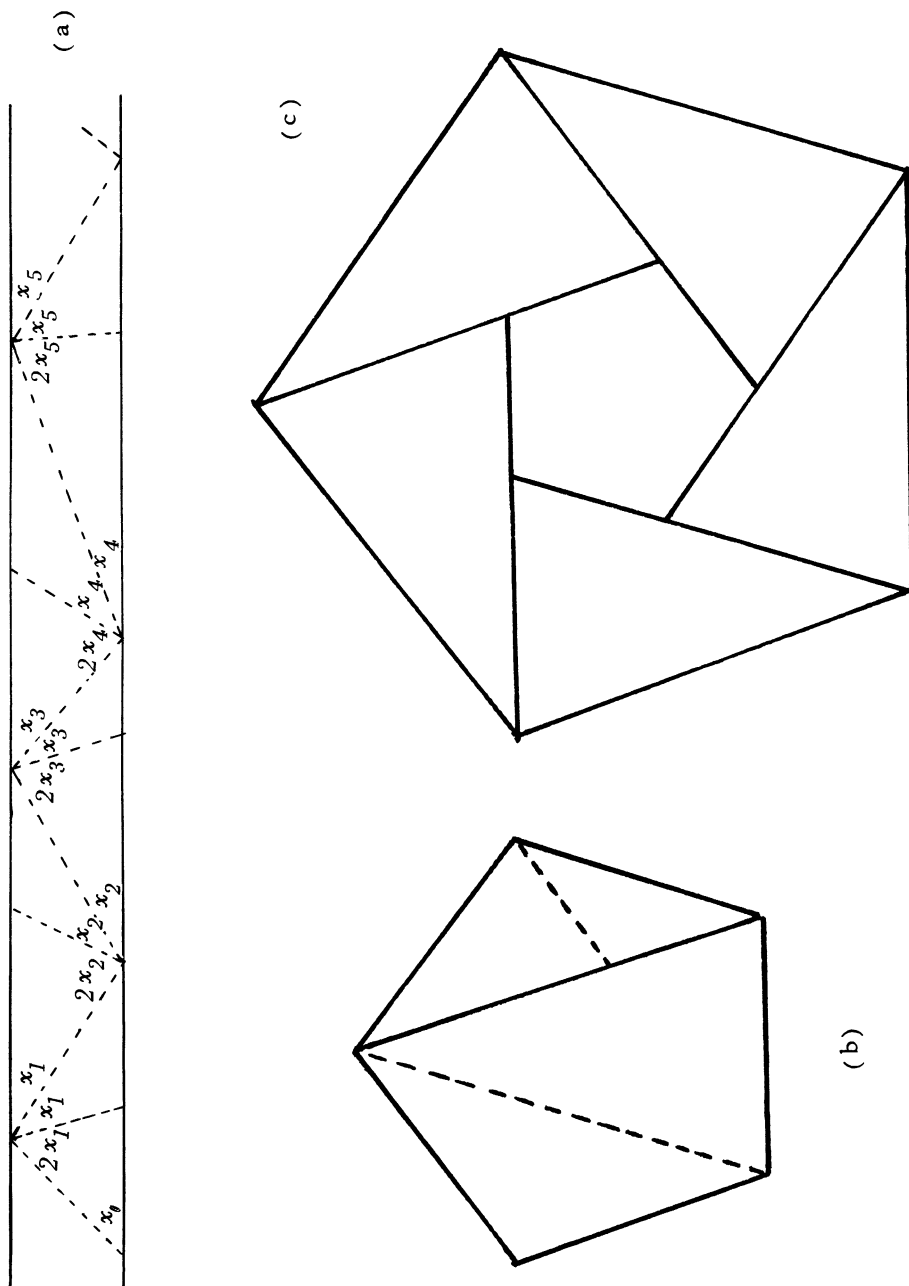
Thus we see that every time a correct fold is made the previous error is cut in half. In general, if  $x_{k-1} + 2x_k = \pi$ , with initial condition  $x_0 = \pi/3 + \epsilon$ , then, by a straightforward mathematical induction,

$$x_k = \pi/3 + (-1/2)^k\epsilon, \text{ and thus } \lim_{k \rightarrow \infty} x_k = \pi/3.$$

It is natural to ask what would happen if this folding process were generalized. We will take two steps towards that goal, here, and leave the complete generalization along with some questions to investigate (in Section 3) for the reader to contemplate.

First, one can very quickly verify that if you begin as before with an arbitrary angle  $x_0$  and successively fold  $n$  times at each new obtuse angle created in the right hand direction of the tape, then the smallest angle on the tape converges to  $\pi/(2^n + 1)$ . (Figures 3 and 4 (a) exemplify this in the cases of  $n = 1$  and  $n = 2$ , respectively.) If we take portions of this tape following, say  $x_7$ , you can use it to construct good approximations (of  $n$  different sizes) of regular  $(2^n + 1)$ -gons. In the case of  $n = 2$  for example, folding the tape on successive short transversals yields the pentagon shown in Figure 4 (b) and folding the tape on successive long transversals gives a construction for the «hollow» pentagon shown in Fig-

FIGURE 4



ure 4 (c) (of course when  $n = 1$  the 3-gons are just the triangles of the tape itself).

Verification that these assertions are true follows with an argument analogous to that above; for details see Appendix A.

Notice that when  $n = 3$  the resulting tape can be used to construct approximations to regular 9-gons (which are impossible to construct with a ruler and compass!), and when  $n = 4$  the tape can be used to construct approximations to regular 17-gons (which might have been of interest to Gauss, whose sensational discovery of a ruler and compass construction of a regular 17-gon is legendary).

As a second step towards generalizing the convergent iterative folding sequences on parallel tape you could begin with our usual arbitrary angle of  $x_0$  at the bottom edge of the tape and fold always on the obtuse angles produced towards the right hand end of the tape in the following way:

(i) fold  $n$  times on the obtuse angles lying on the top edge of the tape, thus dividing those obtuse angles into  $n+1$  successive angles of sizes  $2^{n-1}x_i, 2^{n-2}x_i, \dots, 2x_i, x_i, x_i$  (where  $i$  is odd);

(ii) fold  $m$  times on the obtuse angles lying on the bottom edge of the tape, thus dividing those obtuse angles into  $m+1$  successive angles of sizes  $2^{m-1}x_j, 2^{m-2}x_j, \dots, 2x_j, x_j, x_j$  (where  $j$  is even).

For the sake of discussion let us call this an  $(n, m)$ -folding sequence and assume that  $n \leq m$ . Then the following equations hold:

$$x_{2k} + 2^n x_{2k+1} = \pi \quad \text{and} \quad x_{2k+1} + 2^m x_{2k+2} = \pi.$$

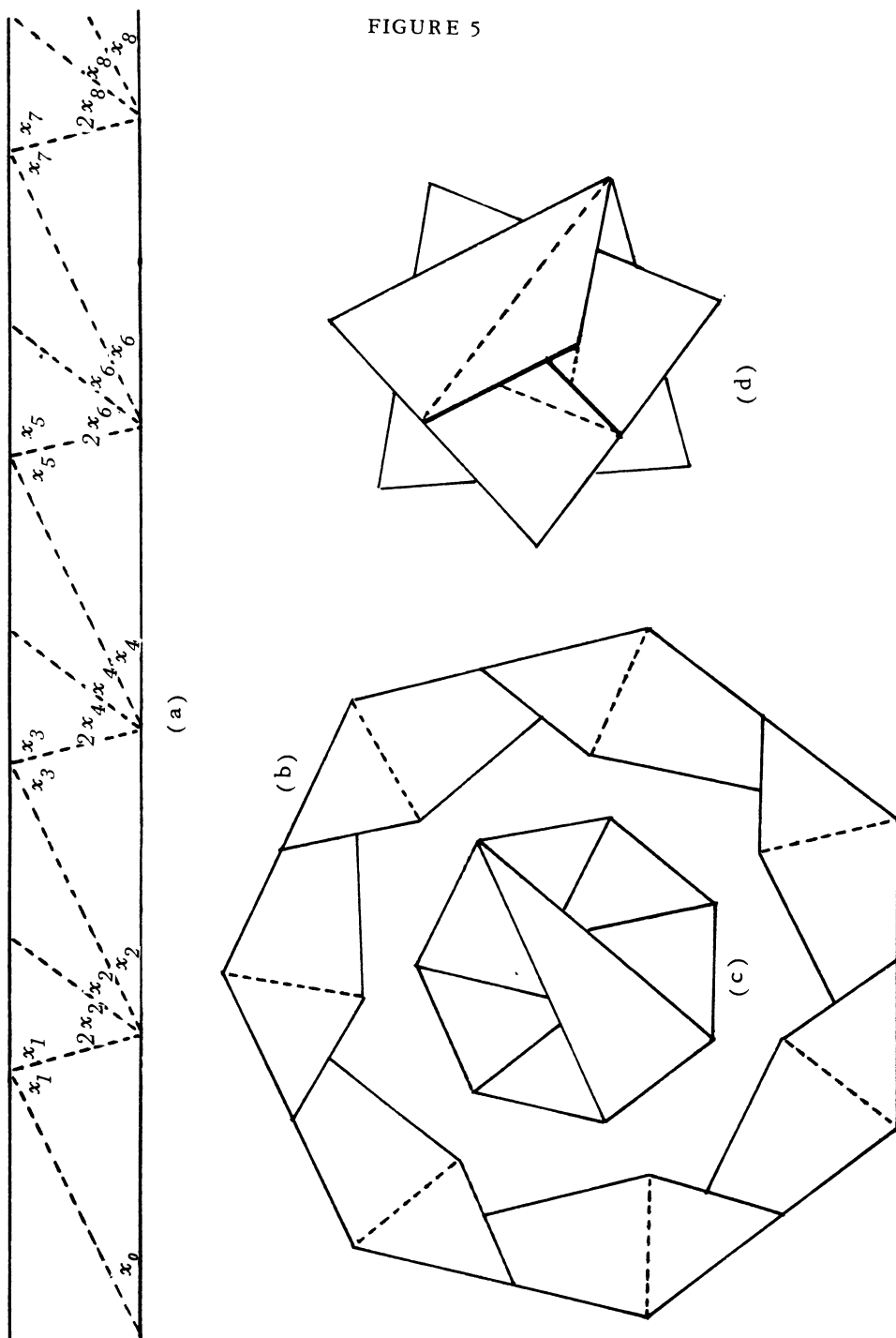
Now, because  $n \leq m$  it follows that the smallest angle on this tape will be determined by  $\lim_{k \rightarrow \infty} x_{2k}$ . If we assume that such a limit exists then it would follow that

$$\lim_{k \rightarrow \infty} x_{2k} = \lim_{k \rightarrow \infty} x_{2k+2},$$

and  $\lim_{k \rightarrow \infty} x_{2k+1}$  also exists, so that by solving the simultaneous linear equations

$$A + 2^n B = \pi, \quad B + 2^m A = \pi,$$

FIGURE 5



for  $A$  we would anticipate that the value for  $\lim_{k \rightarrow \infty} x_{2k}$  is

$$(*) \quad \pi(2^n - 1)/(2^{m+n} - 1).$$

A proof of this, together with the companion statement

$$\lim_{k \rightarrow \infty} x_{2k+1} = \pi(2^m - 1)/(2^{m+n} - 1),$$

will be found in Appendix A.

Figure 5 (a) shows the folded tape when  $n = 1$  and  $m = 2$ . The smallest angle on this tape approaches  $\pi/7$  and if you take portions of this tape past, say  $x_6$ , and fold on all transversals except the shortest ones you get the regular «hollow» 7-gon shown in Figure 5 (b). Likewise, if you fold this tape on all transversals except the medium length ones you obtain the regular 7-gon shown in Figure 5 (c); and if you fold on all transversals except the longest ones it produces the «star» shaped 7-gon of Figure 5 (d).

Observe that if  $n = 1$  in (\*), the smallest angle on this tape will approach  $\pi/(2^{m+1} - 1)$ . This tape will have  $m+1$  kinds of transversals and when  $m$  is even if you fold that tape consistently leaving transversals of any one kind *unfolded*, it will produce a regular  $(m+1)$ -gon (which may be solid, hollow, or a star shaped polygon); but, when  $m$  is odd, then in order to produce the regular  $(m+1)$ -gons you must fold the tape so that you only fold on one particular kind of transversal.

The generalizations of iterative folding sequences can be carried further (see Section 3), but we will now use the tape for other kinds of constructions. Having observed that single pieces of straight strips of paper can be folded to outline the boundary of certain regular polygons in the plane the natural question arises:

*Can several straight strips of paper be oriented in space so that they will either cover the surface, or outline the edges, of a regular polyhedron?*

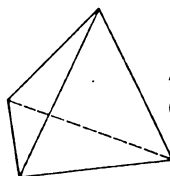
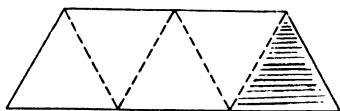
We explore some of the various possibilities in Section 2.

## 2, WOVEN COVERINGS OF THE REGULAR CONVEX POLYHEDRA (PLATONIC SOLIDS<sup>1</sup>).

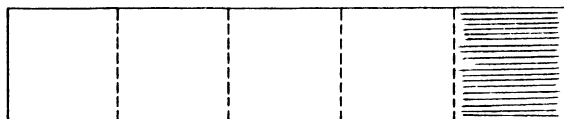
<sup>1</sup> Those convex polyhedra for which all faces are regular, of the same size and shape, forming the same arrangement about each vertex.

FIGURE 6

(a)

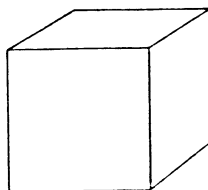


Tetrahedron  
( 2 strips )

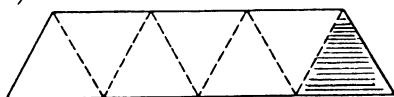


(b)

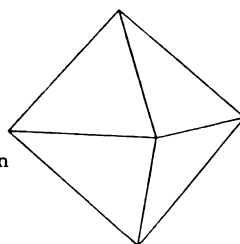
Cube  
( 3 strips )



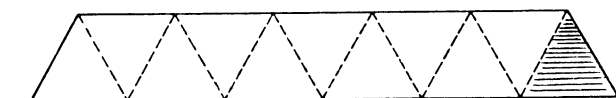
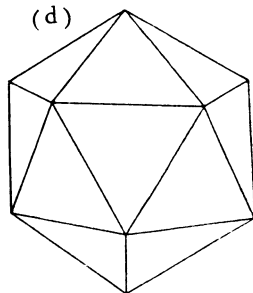
(c)



Octahedron  
( 4 strips )



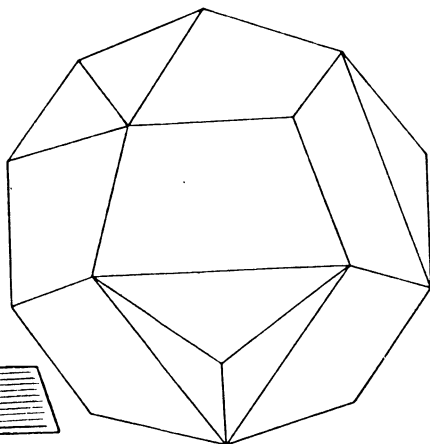
(d)



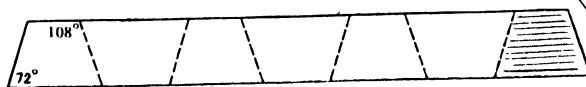
Icosahedron  
( 5 strips )

(e)

Dodecahedron  
( 6 strips )



These strips can be folded as  
shown in Figure 4 (a). Then  
use only the short fold lines.



It is easy to see (though not at all easy to discover, see [10-12]) the following

**THEOREM.** *If you construct each of the Platonic solids with  $n$  congruent straight strips, each of a different color, such that*

*(a) every edge is crossed at least once by a strip (that is, no edge is an open slot),*

*(b) every face is completely covered by at least one thickness of some strip,*

*(c) every color has an equal area exposed on the model's surface,*  
*then it is possible to construct*

$$\text{the } \left[ \begin{array}{c} \text{tetrahedron} \\ \text{cube} \\ \text{octahedron} \\ \text{icosahedron} \\ \text{dodecahedron} \end{array} \right] \text{ with } \left[ \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \right] \text{ strips}$$

Figure 6 shows a typical pattern piece for the required strips and an illustration of the completed model for each of the five solids. In each case it is assumed that the shaded portion of the straight strip will overlap the other end of the same strip so that on the completed model the strips are cylindrical-like rings braided together in space. The construction for the tetrahedron, the cube and the octahedron is relatively simple and I encourage the reader to go ahead and make the pattern pieces out of some fairly substantial paper and assemble them using the requirements imposed by the statement of the theorem. The icosahedron and dodecahedron may prove more difficult to assemble and explicit instructions for their construction (along with instructions for constructing the more simple polyhedra) can be found in [10] through [12].

Having constructed these models they can be used to exemplify Euler's formula connecting the number of vertices ( $V$ ), edges ( $E$ ) and faces ( $F$ ) (that is,

$$V - E + F = 2 \quad \text{for any homeomorph of a sphere)}$$

or Descartes's formula for the sum of the angular deficiencies<sup>1</sup> at each vertex for any given polyhedron (which is always  $4\pi$  for any homeomorph of a sphere)<sup>2</sup>.

Although the relation and generalization of these classic formulae make very interesting mathematics they would take us too far from the theme of this particular paper which involves looking at the relationship of concrete geometric models to group theory and combinatorics. In order to cite just one striking example of this connection let us look at a model that was not mentioned in the theorem, at the beginning of this section. Figure 7 shows a cube woven from four identical strips, each containing seven (including the overlapping tab) right isosceles triangles. This model, which will be referred to as a *diagonal cube*, is an esthetically pleasing woven covering of a cube and it represents, as will be shown, the concrete manifestation of some very beautiful mathematics.

Consider how the diagonal cube relates, for example, to all proper rotations in space of the cube<sup>3</sup>. There are 24 such rotations consisting of

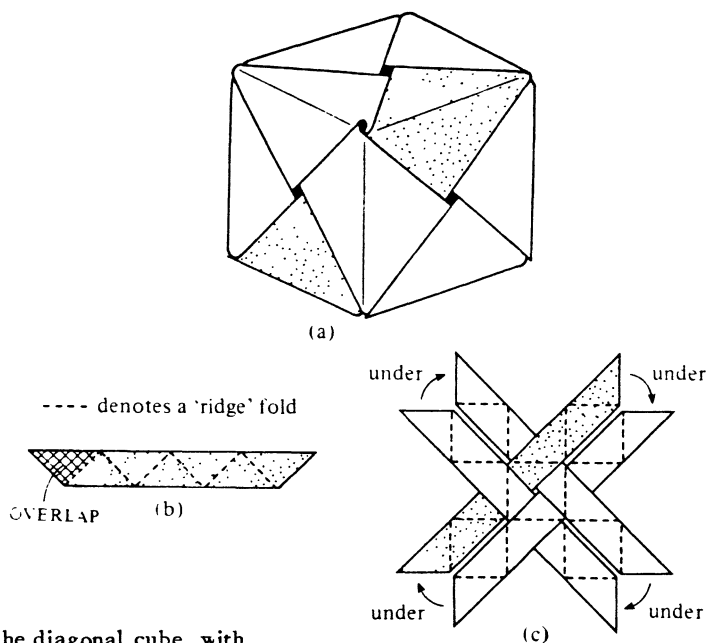
1 identity,

6 rotations about axes joining opposite edges

<sup>1</sup> The angular deficiency at any given vertex is defined to be the difference between  $2\pi$  and the sum of all the face angles surrounding that vertex, thus the angular deficiency at a vertex of the cube is  $2\pi - 3(\pi/2) = \pi/2$ .

<sup>2</sup> Since Descartes died in 1650 and Euler lived from 1707 to 1783 it might be supposed that Euler was familiar with Descartes' formula. In fact, Descartes' formula was not printed till a hundred years after Euler's death, and there is no evidence that Euler rediscovered Descartes's formula, though he came very close (see [13, 14, 23]). Euler's discovery of his own formula does not appear to have been accompanied by a satisfactory proof. Pólya [13, 14] has given an easily accessible proof of the equivalence of the two formulae, in the sense that the angular deficiency is equal to  $2\pi(V - E + F)$ . This equivalence may be regarded as the forerunner of the celebrated Gauss-Bonnet Theorem (see [21], particularly Section 4). Today, it is clearly understood that the Euler formula is a topological invariant for any (2-dimensional) polyhedron (see, for example, [22]). This whole story offers us a superb example of the fascination to be gained from a study of the history of mathematics!

<sup>3</sup> See [15] for a determination, on pages 149-154, of all finite groups of proper rotations in 3-space.



(a) The diagonal cube, with  
 (b) one of the four strips, from which it is made,  
 and (c) the way in which they are laid out at the beginning of the braiding.

FIGURE 7

8 rotations about axes joining opposite vertices,

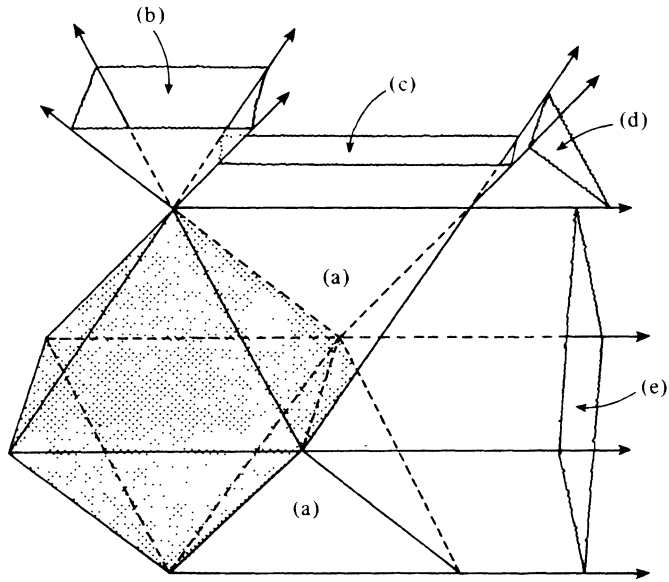
9 rotations about axes joining opposite faces.

This collection of rotations is easily seen to be  $S_4$  by simply observing that all possible permutations of the four strips of the diagonal cube are accounted for with the rotations listed.

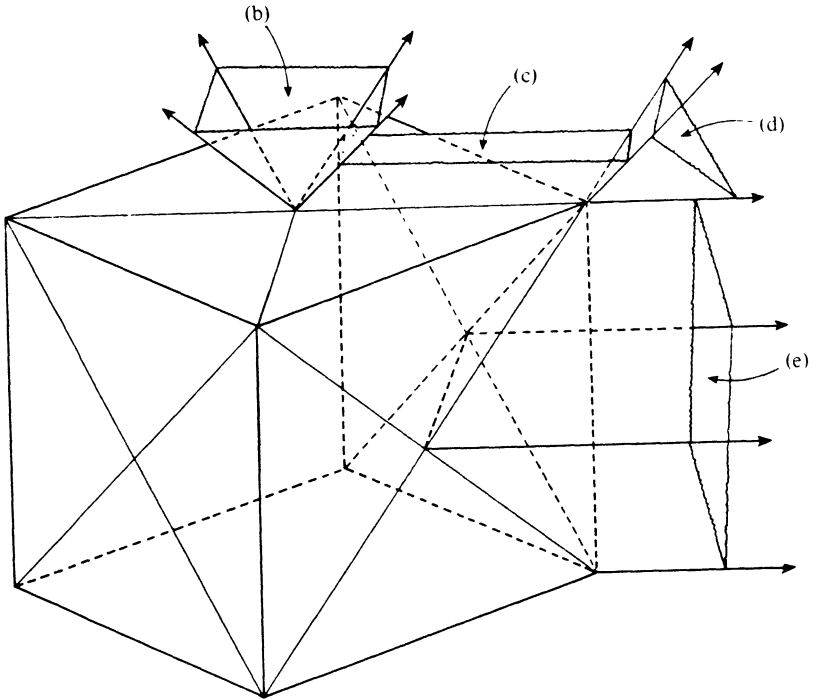
It is a very elementary exercise to compute that the number of ways 4 colors can be arranged, 4 at a time, in a circle is 6 ; and, likewise, that the number of ways 4 colors can be arranged, 3 at a time, in a circle is 8. *All of these combinations appear on the diagonal cube!* When a diagonal cube is constructed from strips of four different colors the centers of the 6 faces are all surrounded by different arrangements of the 4 colors and the 8 vertices are all surrounded by different arrangements of 3 colors each.

Having given us this much information the diagonal cube is still not ready for retirement. If you study Figures 8 (a) and 8 (b) you may be able

FIGURE 8



- (a) Extending the face planes of an octahedron
- (b) The use of the diagonal cube to visualize the unbounded regions formed by the extended face planes of the octahedron.

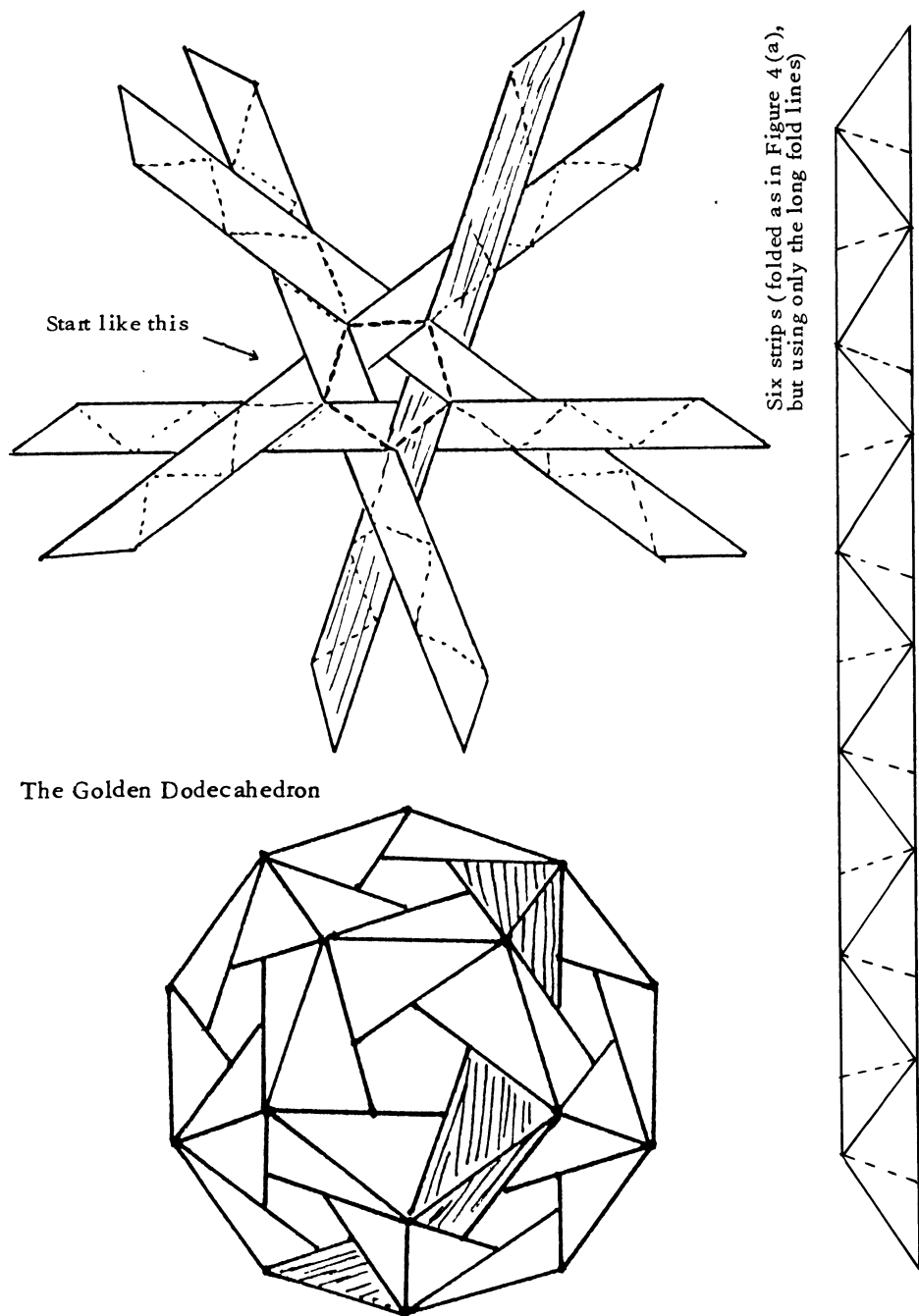


to see how the diagonal cube may be used to count the various kinds of *unbounded* regions created when the face planes of the regular octahedron are extended in space (see [16] for details). The complete count (including the bounded regions) is:

$$\begin{aligned}
 & 1 \text{ octahedron} \\
 & + (a) \ 8 \text{ regular tetrahedra on its faces} \\
 & + (b) \ 6 \text{ unbounded tetrahedral regions from its vertices} \\
 & + (c) \ 24 \text{ unbounded wedges from the edges of the eight tetrahedra} \\
 & \quad \text{which do not coincide with edges of the original octahedron} \\
 & + (d) \ 8 \text{ unbounded trihedral regions from the outside vertices of the} \\
 & \quad \text{eight tetrahedra} \\
 & + (e) \ 12 \text{ unbounded regions having two finite faces on adjacent te-} \\
 & \quad \text{trahedra and four infinite faces} \\
 & = \quad 9 \text{ bounded regions} + 50 \text{ unbounded regions} \\
 & = \quad 59 \text{ regions.}
 \end{aligned}$$

What this model exemplifies is that models braided together with cylindrical-like strips, having a maximum number of two thicknesses (holes are now permitted, like those that appeared in the «hollow» polygons) can be used to count the number of unbounded regions created by extending the planes determined by the edges of those straight strips. Thus, as another example, the «golden dodecahedron» in Figure 9 can be used to account for the 122 unbounded regions created in space when the 12 planes defined by the edges of the six strips of that model are extended in space (see [16,17,18]). Of course one might wish to know what would happen if you looked closer to the «core» of this configuration, rather than at the infinite regions. You could then argue that since you have the intersection of 12 planes in space you know they might contain the faces of a 12 faced polyhedron. This would not be sufficient to guarantee that they intersect to form the faces of the regular pentagonal dodecahedron (since many other 12 faced polyhedra are known to exist). It is the fact that the original configuration belongs to the proper rotation group  $A_5$  that provides the necessary information from which we can conclude that the 12 planes must contain the faces of the

FIGURE 9



regular pentagonal dodecahedron. In a similar way we could have deduced that the 8 planes determined by the four bands of the diagonal cube must contain the faces of the regular octahedron.

Many other woven models exist and many other connections between the models and combinatorics can be made. We will be content to let these examples serve our present purpose and close by proposing some open questions.

### 3. RESEARCH QUESTIONS.

(A) It is clear from Section 1 that it is possible to approximate, by folding, regular polygons having  $2^n \pm 1$  sides.

(?) Which rational multiples of  $\pi$  *can* be constructed by folding tape, iteratively as described in Section 1?

(?) Which regular polygons *cannot* be constructed from tape folded iteratively?

(B) The icosahedron woven from five strips of 11 equilateral triangles each does not itself possess icosahedral symmetry. This is clear since the 10 triangles that «count» in each of the 5 strips provide a total of 50 triangles; and since the number of faces on the icosahedron, namely 20, does not divide 50 there is no way those 50 triangles can be distributed with the same number covering each face. In fact, the 5 triangles about the «North and South poles» are covered by 3 triangles each and the 10 triangles around the «equator» are covered by 2 triangles each.

(?) Is it possible to weave together 5 straight strips (connected to make the required cylindrical-like rings) so that every face of the completed polyhedron is covered by the same number of strips?

(C) [16] employs baseless antiprisms (the cylindrical-like bands in Figures 8 and 9) braided together to form «Platonic arrangements» that delineate unbounded regions in space defined by the parallel planes of the antiprisms' bases.

(?) Are there braided models suitable for delineating the unbounded

regions created by extending the face planes of the Archimedean solids<sup>1</sup>?

(D) The braided models used to determine the parallel divisions of space for the extended face planes of the octahedron and dodecahedron are not the only braided models possible for that purpose. For example, four identical straight strips can be braided to form a model which looks like a truncated octahedron (with holes instead of square faces) and the intersection of the planes determined by the edges of the braided strips is the regular octahedron.

(?) What are *all* such possible braided models for each of the Platonic solids having faces which lie in parallel planes?

(E) Grünbaum and Shephard have discussed, in detail, weavings of straight strips in the plane (see [19] and [20]). They discuss there a finite number of regular types of weavings in the plane (with an infinite number of varieties within each type). In particular they discuss two very common types of weaves called twills and satins. Each of these has a similar appearance on both sides of the weave (differing at most by reflections).

(?) Can designs like satins and twills be woven on the surface of polyhedra<sup>2</sup>? If so, which polyhedra admit this kind of weaving and what determines whether or not twills and satins are admissible weaves? Will the weaves always appear similar on both sides (i.e., inside and outside)? Will it always be necessary to use three (or more) colors to achieve the desired effect?

These represent just a few of the many questions which have come from the concrete models presented in this paper. It seems fairly certain that the answers to these questions however must utilize group theory, combinatorics and perhaps many other mathematical concepts. In this sense

<sup>1</sup> Archimedean solids are those convex polyhedra for which all faces are regular, with edges of the same length - though different faces are not necessarily of the same shape; moreover, the faces form the same arrangement about each vertex.

<sup>2</sup> You might cut each of the three strips in Figure 6(b) into, for example, 5 parallel strips and weave those together to form a checkerboarded cube, or a twill cube, etc.

these questions serve to emphasize the coherence of the entire mathematical discipline and, in particular, they highlight the fact that geometry plays a key role in mathematics as a rich source of ideas and questions, that is, as a source of inspiration.

## APPENDIX A.

In this appendix we prove a general result which implies the various results cited in Section 1 on the construction of certain angular measures by means of iterative folding sequences on parallel tape. This theorem, and its proof, can be appreciated by any student of the elements of real analysis.

**THEOREM A.** *Let a sequence  $\{x_k\}$ ,  $k = 0, 1, 2, \dots$ , be generated by the recurrence relation*

$$x_{k-1} + ax_k = b, \quad k = 1, 2, \dots;$$

*then*

- (i) *if  $|a| > 1$ ,  $x_k \rightarrow b/(1+a)$ ,*
- (ii) *if  $|a| < 1$ ,  $x_k$  diverges to infinity unless  $x_0 = b/(1+a)$  when the sequence is stationary at  $b/(1+a)$ ;*
- (iii) *if  $a = -1$ ,  $x_k = x_0 - kb$  and so diverges to infinity unless  $b = 0$ ;*
- (iv) *if  $a = 1$ ,  $x_k = x_0$ ,  $k$  even;  $x_k = b - x_0$ ,  $k$  odd; so  $x_k$  oscillates finitely.*

**PROOF.** (i) Set  $x_k = b/(1+a) + y_k$ . Then

$$\frac{b}{1+a} + y_{k-1} + \frac{ab}{1+a} + ay_k = b,$$

so that  $y_{k-1} + ay_k = 0$ . It follows that  $y_k = (-1/a)^k y_0$  so that  $y_k \rightarrow 0$  as  $k \rightarrow \infty$ . Hence  $x_k \rightarrow b/(1+a)$ .

We may leave to the reader the demonstration of the remaining parts of this theorem, observing only that the result in case (i) does not depend on the initial value  $x_0$ . It is instructive to observe that this is also almost true of case (ii), there being a single exceptional starting value, namely  $b/(1+a)$ .

It is, of course, only case (i) which arises in our paper-folding sequences. Our theorem tells us that we would, using the specified rules for folding, always converge to the desired angle, but naturally the rate of convergence would depend on our initial angle not being too «wild». Since an angle is always between 0 and  $\pi$ , it never can be too terribly wild; and since, in our folding experiments,  $|a|$  is always a power of 2, the rate of convergence is always rapid.

We reduce the  $(n, m)$ -folding sequence to Theorem A by writing:  
 $u_k = x_{2^k}$ ,  $v_k = x_{2^{k+1}}$ . Then

$$u_k + 2^n v_k = \pi, \quad v_k + 2^m u_{k+1} = \pi,$$

so

$$u_k - 2^{m+n} u_{k+1} = \pi(1 - 2^n)$$

and similarly

$$v_k - 2^{m+n} v_{k+1} = \pi(1 - 2^m).$$

Thus

$$x_{2^k} \rightarrow \frac{\pi(2^n - 1)}{2^{m+n} - 1}, \quad x_{2^{k+1}} \rightarrow \frac{\pi(2^m - 1)}{2^{m+n} - 1}.$$

Further generalizations of the  $(n, m)$ -folding procedure, which will doubtless occur to the reader, may be handled by a device similar to that above.

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