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CHARACTERIZATION OF INJECTIVE ENVELOPES

by Hans-E. PORST

ABSTRACT. We discuss various sets of conditions which guarantee that a minimal (least) injective extension or a maximal (largest) essential extension is an injective envelope.

INTRODUCTION

Injective envelopes - if they exist - are a useful hull-operation in various mathematical theories. A general categorical description is usually given as follows (cp. [1, 2]): In a category $\underline{K}$ there is fixed some class $\underline{M}$ of morphisms - called extensions - which is assumed to be closed under composition and to contain all sections (= coretractions) of $\underline{K}$, and which moreover fulfills the following conditions: $f \in \underline{M}$ is an isomorphism whenever there is some $g \in \underline{M}$ such that $f \circ g = id$.

With respect to this class $\underline{M}$ one has the well known notion of injectivity: an object $X \in ob\underline{K}$ is called *injective* provided for any

$$m: A \to B \in \underline{M} \quad \text{and} \quad f: A \to X \in Mor\underline{K}$$

there is some

$$g: B \to X \in Mor\underline{K} \quad \text{such that} \quad f = g \circ m.$$

Moreover there is defined the subclass $\underline{M}^* \subset \underline{M}$ of essential extensions by calling an extension $m: X \to X'$ *essential* iff a $\underline{K}$-morphism $g: X' \to A$ belongs to $\underline{M}$ provided $g \circ m$ is in $\underline{M}$.

Any essential extension $m: X \to X'$ with injective codomain X' is then called an *injective envelope (of X).*

It is a well known fact that the following assertions on a morphism $m: X \to X'$ are equivalent provided any object of $\underline{K}$ has an injective envelope:

(i) $m: X \to X'$ is an injective envelope,

(ii) $m: X \to X'$ is a *minimal injective extension*, i.e., $m \in \underline{M}$, X' is *injective* and given any factorization

$$m = X \xrightarrow{m''} X'' \xrightarrow{m'} X'$$

with $m', m'' \in \underline{M}$ and X'' injective, then m' is an isomorphism.

(iii) $m: X \to X'$ is a *least injective extension*, i.e., $m \in \underline{M}$, X' is injective and for any $i: X \to X''$ with X'' injective there exists some

$$j: X' \to X'' \in \underline{M} \text{ such that } i = j \circ m.$$

(iv) $m: X \to X'$ is a *maximal essential extension*, i.e., $m \in \underline{M}^*$ and any morphism $g: X' \to X''$ $(\in \underline{M})$ is an isomorphism if $g \circ m \in \underline{M}^*$.

(v) $m: X \to X'$ is a *largest essential extension*, i.e., $m \in \underline{M}^*$ and for any $i: X \to X'' \in \underline{M}^*$ there exists some

$$j: X'' \to X' \ (\in \underline{M}) \text{ such that } m = j \circ i.$$

Because of this fact injective envelopes are usually constructed either as minimal injective extensions (e.g., in the case of metric spaces [5]) or as maximal essential extensions (e.g., in the case of module theory). Now there are classical and also recent results which show that the notions of injective envelopes, maximal essential extensions, and minimal injective extensions do not coincide in general - even if the latters exist - and that moreover these notions may be equivalent without a general existence of injective envelopes (examples are given in the last section). Hence there is the question for categorical conditions - weaker that the existence of injective envelopes - which ensure the equivalence of the various notions of extensions introduced above.

RESULTS

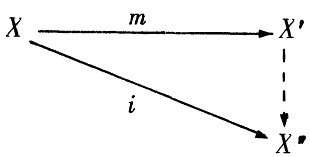
The following lemmata are to be proved easily and already contained on [1].

LEMMA 1. *$\underline{M}^*$ is closed under composition, and if a composite $f \circ g$ is essential then f is essential provided either f and g are in $\underline{M}$ or g is es-*

sential.

LEMMA 2. *If X is injective then any essential extension of X is an isomorphism.*

Moreover there is the obvious observation that any essential extension $m: X \to X'$ is contained (with respect to $\underline{M}$) in any injective extension $i: X \to X''$, since in the following diagram the dotted arrow exists by injectivity of X'' and it belongs to $\underline{M}$ since m is essential:



Hence one has most of the following proposition.

PROPOSITION 1. *Let $m: X \to X'$ be an injective envelope; then m is as well:*

(i) a minimal injective extension,

(ii) a least injective extension,

(iii) a maximal essential extension,

(iv) a largest essential extension.

Moreover m is unique up to isomorphisms over X [1].

PROOF. Since (ii) and (iv) follow by the preceding remark and (iii) by Lemma 2 we only have to prove (i). Hence consider a factorization

$$m = X \xrightarrow{m''} X'' \xrightarrow{m'} X'$$

with $m', m'' \in \underline{M}$ and X'' injective. Then m' has a left inverse l since X'' is injective; l belongs to $\underline{M}$ since m is essential; hence l is an isomorphism by the assumptions on $\underline{M}$, and so is m'.

Let us introduce now some conditions which will enable us to prove the converse implications of Proposition 1.

DEFINITION. (i) $X \in ob\underline{K}$ has the *transferability property* (cp. [7]) provided for any pair of morphisms

$$(f: Z \to X,\ m: Z \to Y) \quad \text{with} \quad m \in \underline{M}$$

there exists a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\;m\;} & Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ X & \xrightarrow{\;j\;} & T \end{array}$$

with $j \in \underline{M}$.

(ii) $X \in ob\underline{K}$ is called *essentially extendable* if for any extension $m: X \to X'$ there exists a morphism $g: X' \to Y$ such that $g \circ m$ is essential.

(iii) An extension $m: X \to X'$ is called *thick* provided any endomorphism $f: X' \to X'$ with $f \circ m = m$ is an automorphism.

(iv) $X \in ob\underline{K}$ is called *essentially normal* if any largest essential extension of X is thick.

Obviously, if X has an injective envelope $m: X \to X'$, then X has the transferability property, is essentially extendable, and essentially normal, and m is thick. The following result is due to Banaschewski who first has discussed conditions (i) and (ii) in this context.

PROPOSITION 2 [*1*]. *Assume that X has the transferability property and is essentially extendable. Then X is injective if X is a maximal essential extension of some object Y.*

Moreover we have the following results:

PROPOSITION 3. *Let $m: X \to X'$ be an extension.*

(i) m is an injective envelope iff X is injective and m is thick.

(ii) Largest essential extensions of X are maximal essential extensions iff X is essentially normal.

PROOF. To prove the missing implication of (i) assume that $m: X \to X'$ is injective and thick and that $g: X' \to Y$ is a morphism such that $g \circ m \in \underline{M}$. Since X' is injective there is some

$$f: Y \to X' \text{ such that } f \circ g \circ m = m.$$

Hence $f \circ g$ is an isomorphism and $g \in \underline{M}$.

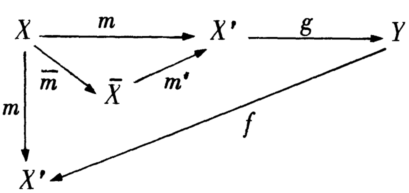
To prove (ii) observe first that maximal essential extensions are thick. If now $m: X \to X'$ is a largest essential extension, hence thick by hypothesis, and $g: X' \to Y$ an extension such that $g \circ m$ is essential, then there is some extension

$$j: Y \to X' \text{ such that } j \circ g \circ m = m.$$

Now $j \circ g$ is an isomorphism and so is j since $j \in \underline{M}$, hence g is an isomorphism.

PROPOSITION 4. *Let X be essentially extendable. Then any injective extension of X contains a largest essential eetension.*

PROOF. Let $m: X \to X'$ be an injective extension. By hypothesis there exists some $g: X' \to Y$ such that $g \circ m \in \underline{M}^*$. Since X' is injective there is a morphism $f: Y \to X'$ such that the outer triangle of the following diagram commutes:



Since $g \circ m$ is essential, f is an extension. To prove that $g \circ m: X \to Y$ is even a largest essential extension, consider an arbitrary essential extension $\bar{m}: X \to \bar{X}$. Injectivity of X' yields $m': \bar{X} \to X'$ such that the inner triangle commutes. Now

$$g \circ m' \circ \bar{m} = g \circ m \in \underline{M}^* \text{ and } \bar{m} \in \underline{M}^*$$

imply $g \circ m' \in \underline{M}$.

Combining these results we get the following theorem:

THEOREM. *Assume that X is essentially extendable and that X' has the the transferability property and is essentially extendable. Then for an extension $m: X \to X'$ the following assertions are equivalent:*

(i) m is an injective envelope,

(ii) m is a minimal injective extension,

(iii) m is a least injective envelope,

(iv) m is a maximal essential extension,

(v) m is a largest essential extension.

PROOF. By Propositions 1, 2 and 3 we already know (i) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). Assume now that m is minimal injective. By Proposition 4 there is a largest essential extension $m': X \to Y$ such that

$$m = X \xrightarrow{m'} Y \xrightarrow{i} X' \quad \text{with} \quad i \in \underline{M},$$

which is maximal essential by Proposition 3, hence Y is injective again by Proposition 2. Minimality of m makes m an injective envelope.
Similarly in a least injective extension there is contained a largest essential, hence injective extension $m': X \to Y$ such that

$$m = X \xrightarrow{m'} Y \xrightarrow{i} X'.$$

Injectivity of Y yields an extension

$$j: X' \to Y \quad \text{such that} \quad j \circ i \circ m' = m'.$$

Since X is essentially normal, $j \circ i$ is an isomorphism and so is i.

There are two additional results which relate the notions of minimal and least injective extensions and of maximal and largest essential extensions respectively.

PROPOSITION 5. *Assume that in any injective extension of X there is contained an injective envelope. Then for an extension $m: X \to X'$ the following assertions are equivalent:*

(i) m is an injective envelope,

(ii) m is a minimal injective extension,

(iii) m is a least injective extension.

The proofs are straightforward and the implication (iii) $\Rightarrow$ (ii) uses only the weaker assumption that in a given injective extension there is contained a minimal one.

PROPOSITION 6. *If X has a maximal and a largest essential extension, then these extensions coincide and hence both types of extensions are unique. If any $X \in ob\underline{K}$ has a unique maximal essential extension, then this is a largest essential extension, too.*

The proofs are also straightforward using Lemma 1 and the definition of maximality.

EXAMPLES

1. Let *Met* denote the category of metric spaces and distance non-increasing maps, $\underline{M}$ the class of isometries. Due to [5] any object in *Met* has a minimal injective extension which is thick. Hence these extensions are injective envelopes.

2. Let Top_0 denote the category of T_0-spaces and continuous maps, $\underline{M}$ the class of embeddings. It is shown in [3] that any object in Top_0 has a unique maximal essential extension, which hence is also a largest essential extension, but which in general fails to be an injective envelope. Any T_0-space has the transferability property (since Top_0 has enough injectives) and is essentially normal, but it fails to be essentially extendable in general.

3. Denote by $Cat_{\underline{X}}$ the category of concrete categories over the category $\underline{X}$ (i.e., faithful functors with codomain $\underline{X}$) with functors over $\underline{X}$ as morphisms. Let $\underline{M}$ be the class of full embeddings. It is shown in [4] that injective envelopes do not exist in general. All kinds of extensions discussed here are equivalent (cp. [6]), since in any injective extension there is contained an injective envelope, any maximal essential extension is injective, and any object is essentially normal (it is in fact a simple observation that any essential extension in $Cat_{\underline{X}}$ is thick).

4. In the category $\underline{F}$ of fields let $\underline{M}$ be the class of algebraic extensions. Then $\underline{M}^* = \underline{M}$ and injective envelopes exist (these are the algebraic closures). Any essential extension is thick.

5. In order to discuss a dual situation denote by Ab^{op} the dual of the category of abelian groups, and let $\underline{M}$ be the class of surjective group homomorphisms (considered as maps in Ab^{op}). For any cyclic group Z_n the canonical map $Z \to Z_n$ is minimal but not least injective, Z_n has countably many essential extensions but neither a maximal nor a largest one, and any essential extension of Z_n is thick. Z_n fails to be essentially extendable, but is essentially normal.

REMARK AND PROBLEM

From the concrete examples there arises the question if any essential extension is thick or if at least any object is essentially normal. If this would be true, we would have the remarkable fact that Banaschewski's transferability and extendability conditions - introduced only to ensure that maximal essential extensions are injective - make all kinds of extensions discussed above equivalent.

REFERENCES

1. BANASCHEWSKI, B., Projective covers in categories of topological spaces and topological algebras, *General Topology and its relations to modern Analysis and Algebra* (Proc. Kanpur Top. Conf. 1968), Academia, Prague, 1971, 63-91.

2. BANASCHEWSKI, B., Injectivity and essential extensions in equational classes of algebras, *Queen's papers in Pure and Applied Mathematics* 25 (Proc. Conf. on Universal Algebra Kingston 1969), Kingston (1970), 131-147.

3. BANASCHEWSKI, B., Essential extensions of T_0-spaces, *Gen. Top. and Appl.* 7 (1977), 233-246.

4. HERRLICH, H., Initial completions, *Math. Z.* 150 (1976), 101-110.

5. ISBELL, J. R., Six theorems about injective metric spaces, *Comment. Math. Helv.* 39 (1964), 65-76.

6. PORST, H.-E., Characterizations of MacNeille completions and topological functors, *Bull. Austral. Math. Soc.* 18 (1978), 201-210.

7. TAYLOR, W., Residually small varieties, *Algebra Univ.* 2 (1972), 33-53.

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