CAHIERS DE
TOPOLOGIE ET GÉOMÉTRIE DIFFÉRENTIELLE
CATÉGORIQUES

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Cahiers de topologie et géométrie différentielle catégoriques, tome 22, nº 3 (1981), p. 337-342

<http://www.numdam.org/item?id=CTGDC_1981__22_3_337_0>

CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE
Vol. XXII-3 (1981)
3^e COLLOQUE SUR LES CATEGORIES
DEDIE A CHARLES EHRESMANN
Amiens, Juillet 1980

CATEGORICAL ASPECTS OF THE CLASSICAL ASCOLI THEOREMS

by John W. GRAY

0. INTRODUCTION

In a forthcoming paper [4] (summarized below) a version of Ascoli's Theorem for topological categories enriched in bornological sets is proved. In this paper it is shown how the two classical versions as found in [1] or [9] fit into this format.

1. GENERAL THEORY

Let *Born* denote the category of bornological sets and bounded maps. (See [7].) It is well known that *Born* is a cartesian closed topological category. (See [6].) Let $\underline{T}$ denote a topological category (cf. [5] or [11]) which is enriched in *Born* in such a way that the enriched hom functors $\underline{T}(X, -)$ preserve sup's of structures and commute with f^*, with dual assumptions on $\underline{T}(-, Y)$. If $H \subset \underline{T}(X, Y)$ belongs to the bornology we shall call H *bounded.*

1.1 LEMMA. *Let $f: X \to Y$ be a function and let $(Y, \eta_Y) \in \underline{T}$. Given $H \subset Y^X$ then there is a largest $\underline{T}$-structure η_X on X such that $H \subset \underline{T}(X, Y)$ and H is bounded.*

1.2 DEFINITION. i) Let $S \in$ *Sets* and $(Y, \eta_Y) \in \underline{T}$. *The uniform $\underline{T}$-structure $\underline{F}_u(S, Y)$ on Y^S* is the largest structure such that

$$\{pr_s\}_{s \in S} \subset \underline{T}(\underline{F}_u(S, Y), Y)$$

is bounded.

ii) Let $(X, \beta_X) \in$ *Born* and $(Y, \eta_Y) \in \underline{T}$. Then

$$\underline{F}_\beta(X, Y) = \varprojlim_{B \in \beta_X} \underline{F}_u(B, Y).$$

1.3 PROPOSITION. *$\underline{T}$ is tensored and cotensored over Born. The cotensor* *is* $\underline{F}_\beta(X, Y)$ *and the tensor is a suitable structure on* $X \times Y$.

1.4 DEFINITION. i) Let $\underline{BT}$ denote the pullback of $\underline{T}$ and *Born* over *Sets*. $(X, \beta_X, \eta_X) \in \underline{BT}$ is called β_X*-generated* if

$$(X, \eta_X) = \varinjlim_{B \in \beta_X} (B, \eta_B) \quad \text{where} \quad \eta_B = \eta_X | B.$$

ii) Let $(X, \beta_X, \eta_X) \in \underline{BT}$. Then $\underline{T}_\beta(X, Y) \subset \underline{F}_\beta(X, Y)$ denotes the induced $\underline{T}$-structure on the set of $\underline{T}$-maps from (X, η_X) to (Y, η_Y).

iii) Let $G: \underline{T} \to$ *Born* be a functor over *Sets*. (X, η_X) is called a *G-space* if $G(X, \eta_X) = (X, 1_X)$ where 1_X denotes the discrete bornology on X.

iv) Let $i: (A, \eta_X | A) \to (X, \eta_X)$ be an inclusion. A is called *G-closed in* X if $G(A, \eta_X | A) = i^*(G(X, \eta_X))$.

1.5 AXIOMS.

A. $G(\underline{T}_u(X, Y)) \leq \underline{T}(X, Y)$.

B. i) G preserves products.

ii) $\underline{T}_\beta(X, Y)$ is G-closed in $\underline{F}_\beta(X, Y)$.

iii) Let $H \subset \underline{T}(X, Y)$ be bounded. Then $H \subset H' \subset \underline{T}(X,Y)$ where H' is bounded and G-closed with respect to the product structure $\underline{F}_p(X, Y)$. Furthermore, if $pr_X(H) \in G(Y)$, then $pr_X(H') \in G(Y)$.

C. If (X, η_X) is a G-space and $H \subset \underline{T}(X, Y)$ is bounded, then

$$\underline{F}_u(X, Y)|H = \underline{F}_p(X, Y)|H.$$

1.6 THEOREM. *Let* (X, β_X, η_X) *be* β_X*-generated and let* $(Y, \eta_Y) \in \underline{T}$.

i) (Weak Ascoli) If G, X *and* Y *satisfy A, then*

$$G(\underline{T}_\beta(X, Y)) \leq \underline{T}(X, Y) \cap \bigcap_{x \in X} pr_x^* G(Y).$$

ii) (Strong Ascoli) If β_X *has a cofinal subset consisting of G-spaces and* G, X *and* Y *satisfy B and C, then the opposite inclusion holds.*

2. UNIFORM SPACES

The most straightforward example of this theory is given by the category *Unif* of uniform spaces and uniformly continuous maps. It is well known to be a topological category and one defines an enrichment in *Born* by calling a set $H \subset Unif(X, Y)$ bounded if it is *uniformly equicontinuous* (cf. [1]), i.e., if $V \in U_Y$ is an entourage for Y, then

$$[\bigcap_{h \in H} (h \times h)^{-1}(V)] \in U_X .$$

To see that this is a hom-functor, observe that if

$$H \subset Unif(X, Y) \text{ and } K \subset Unif(Y, Z)$$

are bounded, then so is $K \circ H \subset Unif(X, Z)$ since

$$\bigcap_{\substack{h \in H \\ k \in K}} (kh \times kh)^{-1}(W) = \bigcap_{h \in H} (h \times h)^{-1}[\bigcap_{k \in K} (k \times k)^{-1}(W)].$$

The other properties are easily checked.

If S is a set and Y is a uniform space, then the uniformity of uniform convergence on Y^S has as a basis the sets

$$\begin{aligned} W(S, V) &= \{ (f, g) \mid (f(s), g(s)) \in V \text{ for all } s \in S \} \\ &= \bigcap_{s \in S} (pr_s \times pr_s)^{-1}(V) ; \end{aligned}$$

i.e., it is the smallest uniformity (= largest structure in the sense of topological categories) such that $\{pr_s\}_{s \in S}$ is uniformly equicontinuous. The definition of $\underline{F}_\beta(X, Y)$ agrees with what is called the unifomity of G-convergence in [1]. It is a standard calculation that $\underline{F}_\beta(X, Y)$ is the cotensor, i.e., given $(X, \beta_X) \in Born$ and (Y, μ_Y), $(Z, \mu_Z) \in Unif$ then there is a bijection

$$\frac{(X, \beta_X) \to Unif((Z, \mu_Z), (Y, \mu_Y)) \text{ in } Born}{(Z, \mu_Z) \to \underline{F}_\beta(X, (Y, \mu_Y)) \text{ in } Unif} .$$

The tensor product has not been discussed before, but it follows immediately from Wyler's Taut Lift Theorem [11] that given $(X, \beta_X) \in Bom$ and (Y, μ_Y) in *Unif*, then $X \otimes Y$ is the largest unifomity (smallest structure) on $X \times Y$ such that $\eta_Y : Y \to \underline{F}_\beta(X, X \times Y)$ is uniformly continuous.

The other important ingredient is the functor G. Here we take $Tb: Unif \to Born$ where $Tb(Y, \mu_Y)$ denotes the set of totally bounded (precompact) subsets of Y. It is immediate that these form a bornology which is functorial in Y. It is proved in [9], I, 5.10 that Tb preserves arbitrary initial structures, thus inf's and f^*. Hence, by [11] again, Tb has a left adjoint $\check{Tb}$; namely, given $(X, \beta_X) \in Born$, then $\check{Tb}(X, \beta_X)$ is the largest uniformity (smallest structure) on X whose totally bounded sets include β_X. Note well however that neither Tb nor $\check{Tb}$ is an enriched functor. A Tb-space is a totally bounded space in the usual sense. Furthermore, an inclusion map $i: (A, \mu_X | A) \to (X, \mu_X)$ is an equalizer and hence preserved by Tb, so every such A is Tb-closed. Therefore the properties in Axiom B are automatically satisfied by Tb. Axiom A says that a totally bounded subset of uniformly continuous functions for the uniformity of uniform convergence is uniformly equicontinuous, which is a standard result. Axiom C says that if X is totally bounded and H is equicontinuous, then the uniformities of uniform convergence and of pointwise convergence coincide on H, which is also a standard result. Theorem 1.6 then is a restatement of [1], X, 2.5, Theorem 2, for the case of uniform structures.

3. MIXED TOPOLOGICAL AND UNIFORM STRUCTURES

The pre-Bourbaki version of Ascoli's Theorem refers to compact subsets of the set of continuous maps from a topological space to a uniform space (or, even more classically, to a metric space). In order to deal with such mixed structures one has to consider more general tensor-hom-cotensor situations as described in [3]. In this case, instead of a single topological category enriched in $Born$, one has a pair of topological categories, $\underline{T}_1$ and $\underline{T}_2$, together with a «hom»-functor $H': \underline{T}_1^{op} \times \underline{T}_2 \to Born$. The tensors and cotensors are functors

$$T: Born \times \underline{T}_1 \to \underline{T}_2 \quad \text{and} \quad C: Born^{op} \times \underline{T}_2 \to \underline{T}_1 .$$

In our case, $\underline{T}_1 = Top$ and $\underline{T}_2 = Unif$. If $S \in Born$, $X \in Top$ and $Y \in Unif$, then writing

$$S \otimes' X = T(S, X) \quad \text{and} \quad S \pitchfork' Y = C(S, Y),$$

the required natural isomorphisms are

$$Unif(S \otimes' X, Y) \approx Born(S, H'(X, Y)) \approx Top(X, S \pitchfork' Y).$$

This can be interpreted either at the level of sets or in *Born* providing a meaning is given to the last term.

In Section 2, we described such a situation for *Top* replaced by *Unif*. There

$$H(X, Y) = Unif(X, Y), \quad S \pitchfork Y = \underline{F}_\beta(S, Y)$$

and $S \otimes X$ was a suitable structure on $S \times X$. Let $|\cdot| : Unif \to Top$ be the forgetful functor with left adjoint $F: Top \to Unif$. (F for fine, cf. [8].) In [3], Proposition 1.2, let $G_1 = G_3 = id$, and $G_2 = |\cdot|$. Then

$$S \otimes' X = S \otimes FX, \quad H'(X, Y) = Unif(FX, Y)$$
$$\text{and } S \pitchfork' Y = |S \pitchfork Y| = |\underline{F}_\beta(S, Y)|$$

is a THC-situation for *Born*, *Top* and *Unif*.

Classically, $H'(X, Y) = Top(X, |Y|)$ with the bornology given by equicontinuous sets. However, the cotensor which is contructed (e.g., in [1]) is easily seen to be $|\underline{F}_\beta(S, Y)|$. Since the cotensors agree, so do the other functors and hence

$$Top(X, |Y|) \approx Unif(FX, Y)$$

holds in *Born* (i.e., an equicontinuous family of maps from X to Y is the same as a uniformly equicontinuous family of maps from FX to Y).

We cannot take $G(X) = Cp(X)$ to be the compact subsets of X since they don't form a bornology. However, they generate a bornology; namely

$$G(X) = Sc(X) = \{ A \subset X \mid \exists \text{ compact } A', A \subset A' \subset X \}.$$

$Sc(X)$ is called the subcompact subsets of X. It clearly defines a functor $Sc: Top \to Born$ over *Sets*. Since it does not preserve equalizers, it has no left adjoint, but it does preserve products so Axiom Bi is satisfied. An Sc-space is a compact space while an Sc-closed subspace is a closed subspace. For Axiom A, it is sufficient to show that a compact set of continuous functions for the topology of uniform convergence is equicontinuous. Here we have

$$Cp(Top_u(X, |Y|)) = Cp(|Unif_u(FX, Y)|) \leq Tb(Unif_u(FX, Y))$$
$$\leq Unif(FX, Y) = Top(X, |Y|).$$

Axiom B ii is the statement that continuous functions form a closed set with respect to the topology of uniform convergence on sets in β_X. In Axiom B iii, one takes H' to be the closure of H, which is equicontinuous if H is. Axiom C is a standard result and then Theorem 1.6 yields the other case of [1], X, 2.5, Theorem 2.

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