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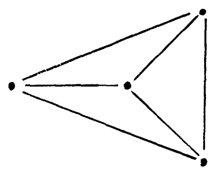
STRUCTURE FUNCTORS

- Compositions of arbitrary right adjoints with topological functors I -

by Manfred B. WISCHNEWSKY

Tout cela est fait pour amuser Ζεῦν
dit le Sphinx).*



INTRODUCTION

In [26] I introduced a new concept for functors which is in fact a class of concepts - called structure functors, resp. more exactly structure functor sequences. Structure functors appear everywhere. Examples are the q-functors in the sense of Ehresmann [6], proclusion functors in the sense of Wyler [29], reflective or coreflective embeddings into categories of sketched structures (A. & C. Ehresmann [7], Gabriel & Ulmer [8]) or into topological categories (Trnkova [24], Wischnewsky [25], Tholen [20]). In [27] I used this concept to characterize completely all full (reflective or) coreflective restrictions of semitopological functors.

In this paper I will use the concept of structure functors in order to characterize all those functors which can be represented as a composition of an arbitrary right adjoint functor with a topological functor. The notion «structure functor» introduced here fills «continuously» the gap between arbitrary topological functors - apparently the best possible type of functor - and arbitrary right adjoint functors. Hence this new notion solves at the same time several long considered open problems. Finally one should mention that the results presented here contain as special instances all fundamental theorems for semitopological functors, given e.g. in Tholen [20] **). In fact more general everything done as yet for semitopological

*) C. Ehresmann, Catégories et Structures (Prélude), Dunod, 1965.
**) The papers [20] and [26] were used as a guideline for this article.

functors has its translation, resp. generalization, for structure functors as it is shown in [28] and several subsequent papers.

0. NOTATIONS.

Let $S: \underline{A} \to \underline{X}$ be a functor. A *S-cone* is a triple $(X, \psi, D(\underline{A}))$, where X is an $\underline{X}$-object, $D(\underline{A}): \underline{D} \to \underline{A}$ is an $\underline{A}$-diagram ($\underline{D}$ may be void or large) and $\psi : \Delta X \to SD(\underline{A})$ is a functorial morphism (Δ denotes the «constant» functor into the functor category). We shall abbreviate often $(X, \psi, D(\underline{A}))$ by ψ. *Cone(S)* denotes the «class» of all S-cones. If $\underline{D}$ is the one point category, then ψ is called a *S-morphism* denoted by (A, a) where A is an $\underline{A}$-object and $a: X \to SA$ is an $\underline{X}$-morphism.

The dual notions are *S-cocone* and *S-comorphism*.

The classes of S-morphisms, S-cocones and S-comorphisms are denoted by *Mor(S)*, *Co-cone(S)*, *Co-Mor(S)*. *Epi(S)* denotes the class of all S-morphisms (A, e) with the property:

for all $\underline{A}$-morphisms $p, q: A \rightrightarrows B$ the equation $(Sp)e = (Sq)e$ implies $q = p$.

The dual notion is *S-monomorphism.*

1. STRUCTURE FUNCTORS. BASIC DEFINITIONS

Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization.

(1.1) DEFINITION *(Semifinal structure functors).*

1) Let $\gamma := (D, \gamma, X)$ be a Q-cocone. A *(P, Q)-extension of* γ (or just an extension) consists of a S-morphism $(A, f: X \to SA)$ and a cocone $\beta : D \to \Delta PA$ such that $Q\beta = (\Delta f)\gamma$. Such an extension will be denoted by (A, f, β).

2) Let (A, f, β) and (A', f', β') be extensions of (D, γ, X). *A morphism between these extensions* $a: (A, f, \beta) \to (A', f', \beta')$ is an $\underline{A}$-morphism $a: A \to A'$ such that

$$f' = (Sa)f \quad \text{and} \quad \beta' = (\Delta Pa)\beta \ .$$

This defines the category of all (P, Q)-extensions of γ.

3) A *semifinal extension of* γ is an initial object in the category of all (P, Q)-extensions of γ.

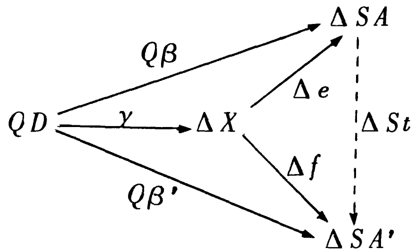


FIGURE 1

4) S is called a *semifinal structure functor* *) with respect to (P, Q) or more exactly a *semifinal topologically-right adjoint structure functor*, with respect to (P, Q)) if for all diagrams D ($\underline{D}$ may be empty or large) and all Q-cocones (D, γ, X) there exists a semifinal extension.

(1.2) REMARKS. 1) If $P = Id$ and $Q = S$ then the semifinal structure functors (with respect to (Id, S)) are just the semitopological functors.

2) A S-morphism $(A, e: X \to SA)$ is called a *S-quotient relative to* (P, Q) or just a S-quotient (if there is no confusion) if there exist a Q-cocone (D, γ, X) and a cocone $\beta: D \to \Delta PA$ such that (A, e, β) is a semifinal extension of (D, γ, X). The class of all S-quotients is denoted by $Quot(S)$.

3) In general $Quot(S)$ does not contain $Iso(S)$, the class of all S-morphisms which are isomorphisms. ($Iso(S) \subset Quot(S)$ if and only if S is semitopological.)

(1.3) DEFINITION *(Semiinitial structure functors).* Let $(D, \phi: \Delta X \to SD)$ be a S-cone where $D: \underline{D} \to \underline{A}$ is a diagram in $\underline{A}$ ($\underline{D}$ may be void or large).

1) A *factorization of* ϕ (along S) consists of a S-morphism $(A, e: X \to SA)$ and an $\underline{A}$-cone $a: \Delta A \to D$ with

$$(*) \qquad \phi = (Sa)(\Delta e).$$

*) Localizable structure functors = multi-structure functors = compositions of arbitrary localizable right adjoints with topological functors are obtained by [26], Appendix A 3; details appear somewhere else.

2) The factorization (A, e, a) is called a *semiinitial coextension* (with respect to (P, Q)) of (X, ϕ, D) iff:

(SI1) For all Q-comorphisms $(B, x: QB \to X)$, and functorial morphisms $\beta: \Delta B \to PD$ with $\phi(\Delta x) = Q\beta$ there exists exactly one P-morphism $(A, t: B \to PA)$ such that

$$Qt = e.x \quad \text{and} \quad \beta = (Pa)(\Delta t).$$

(SI2) If $t: A \to A$ is an $\underline{A}$-morphism with

$$(St)e = e \quad \text{and} \quad a(\Delta t) = a,$$

then $t = id_A$.

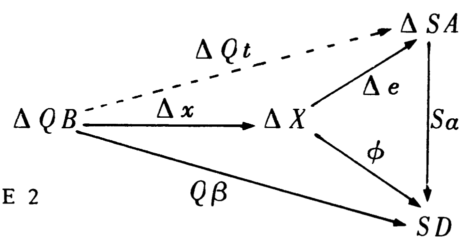


FIGURE 2

3) A functor S is called a *semiinitial structure functor* relative to (P, Q), or more exactly a *semiinitial topologically-right adjoint structure* functor relative to (P, Q), iff for every S-cone (X, ϕ, D) there exists a semiinitial coextension (with respect to (P, Q)).

(1.4) EXAMPLES. The following examples are (semifinal and semiinitial) structure functors:

1) *Topological functors* (take $P = Id$, $Q = S$ and $Quot(S) = Iso(S)$).

2) *Arbitrary right adjoint functors* (take $P = S$, $Q = Id$ and

$$Quot(S) = \text{the class of all units } \eta X: X \to SA,\ X \in \underline{X});$$

in particular arbitrary monadic functors.

3) *The composition of structure functors* (let

$$S_1 = Q_1 P_1: \underline{A}_1 \to \underline{A}_2, \quad S_2 = Q_2 P_2: \underline{A}_2 \to \underline{X}$$

with $Quot(S_1)$, resp. $Quot(S_2)$, structure functors; then $S_2 S_1 = S$ is a structure functor with respect to $Q = Q_2$, $P = P_2 Q_1 P_1$ and

$$Quot(S) := \{ S_1(e_1).e_2 \mid e_1 \in Quot(S_1),\ e_2 \in Quot(S_2)$$

and $S_1(e_1)$ and e_2 composible } ;

in particular all functors which can be decomposed into an arbitrary right adjoint and a topological functor.

2. THE DUALITY THEOREM FOR STRUCTURE FUNCTORS

The duality for (topologically-right adjoint) structure functors is a special instance of a much more general duality theorem for structure functor sequences (Wischnewsky [26]). First we need the following

(2.1) LEMMA. *Let*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a semifinal or semiinitial structure functor with respect to (P,Q). Then Q is faithful with respect to P-morphisms.

PROOF. 1) Let S be a semifinal structure functor with respect to (P,Q). Let $f, g: B \rightrightarrows PA'$, $A' \in \underline{A}$, be two P-morphisms with $Qf = Qg$ and $f \neq g$. Let I be the class of all $\underline{A}$-morphisms. Let

$$B_i = B,\ \gamma_i = Qf: QB_i \to SA' \text{ for all } i \in I.$$

This defines an I-indexed discrete Q-cocone

$$(B_i, \gamma_i: QB_i \to QPA')_{i \in I}.$$

Take the (P,Q)-semifinal extension of $(B_i, \gamma_i)_{i\in I}$ denoted by $(A, e, b_i)_{i\in I}$. Let $b'_i := f: B_i \to PA'$, $i \in I$. Then $Qb'_i = \gamma_i$ for all $i \in I$. Hence there exists a unique morphism $t: A \to A$ such that

$$b'_i = (Pt)b_i,\ i \in I, \text{ and } (St)e = id.$$

This implies that the class

$$J := \{ t \in \underline{A}(A, A') \mid \text{for all } i \in I,\ (Pt)b_i \in \{f, g\} \}$$

is nonvoid. Hence there exists a surjective mapping $\sigma: I \to J$. Now define

$$b'_i := \begin{cases} f & \text{if } (P\sigma(i))b_i = g \\ g & \text{if } (P\sigma(i))b_i = f. \end{cases}$$

Then $\gamma_i = Qb'_i$ for all $i \in I$. The universal property delivers an $\underline{A}$-morphism $t: A \to A'$ such that $(Pt)b_i = b'_i$ for all $i \in I$. Let $i_0 \in I$ with $\sigma(i_0) = t$.

Then

$$(P\sigma(i_0))b_i = g \Leftrightarrow (P\sigma(i_0))b_i = f.$$

But this is a contradiction. Hence $f = g$.

2) The proof for semiinitial structure functors is done in the same way.

(2.2) COROLLARY (Tholen [20]). *Every semitopological functor is faithful.*

(2.3) THEOREM *(Duality for structure functors - Wischnewsky [26]). Notation as in Section 1. For a factorization*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

there are equivalent:

(i) S is a semiinitial structure functor (for (P, Q)).

(ii) S is a semifinal structure functor (for (P, Q)).

PROOF (Outline [26]). (i)→(ii) Let $(D, \phi: QD \to \Delta X)$ be a Q-cocone. Let C be the full subcategory of the comma-category $(X{\downarrow}S)$ consisting of all those $A, x: X \to SA)$ for which there exists a cone $\beta: D \to \Delta PA$ with $Q\beta = (\Delta x)\phi$. β is uniquely determined by (A, x) (Lemma 2.1). Let

$$C: \underline{C} \to \underline{A}: (A, x) \to A, \quad \psi: \Delta X \to SC: \underline{C} \to \underline{X}: \psi(A, x) := x: X \to SA,$$
$$\beta_d(A, x) := \beta(d): QDd \to SA \quad \text{for } d \in Ob\underline{D}.$$

Let

$$\psi = (Sa)(\Delta e): \Delta X \to \Delta SA \to SC$$

be a semiinitial factorization. Since $Q\beta_d = \psi(\Delta\phi(d))$ there exists a unique $\gamma(d): Dd \to PA$. This defines a functorial morphism $\gamma: D \to \Delta PA$. Then $Q\gamma = (\Delta e)\phi$ is a semifinal extension for ϕ.

(ii) → (i) is proved in a similar way.

(2.4) COROLLARY *(Duality for semitopological functors - Tholen [20]). The following assertions are equivalent for a functor $S: \underline{A} \to \underline{X}$:*

(i) S is a semifinal functor.

(ii) S is a semiinitial functor.

3. THE LOCALLY-ORTHOGONAL, RESP. THE LEFT-EXTENSION CHARACTERIZATION OF STRUCTURE FUNCTORS

Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization of S and $\Pi \subset Mor(S)$ be a class of S-morphisms.

(3.1) DEFINITION. S is called a Π-*locally-orthogonal structure functor* with respect to (P, Q) iff every S-cone $(D, \phi: \Delta X \to SD)$ where $D: \underline{D} \to \underline{A}$ may be void or large has a factorization

$$\phi = (Sa)(\Delta e^*) \quad \text{where} \quad (A^*, e^*: X \to SA^*) \in \Pi$$

and $a: \Delta A^* \to D$ is an $\underline{A}$-cone such that for every

$$x: Y \to X \text{ in } \underline{X}, \quad (A, e: Y \to SA) \in \Pi \text{ and}$$
$$\beta: \Delta PA \to PD \quad \text{with} \quad (Q\beta)(\Delta e) = \phi(\Delta x)$$

there exists exactly one $\underline{A}$-morphism $t: A \to A^*$ such that

$$(\Delta e^*)(\Delta x) = (\Delta St)(\Delta e) \quad \text{and} \quad \beta = (Pa)(\Delta Pt).$$

The factorization (A^*, e^*, a) itself is called Π-locally orthogonal.

(3.2) DEFINITION. Let $D(\underline{X}): \underline{D} \to \underline{X}$ and $D(\underline{A}): \underline{D} \to \underline{A}$ be diagrams ($\underline{D}$ may be void or large) and $\phi: D(\underline{X}) \to \Delta X$ and $\pi: D(\underline{X}) \to SD(\underline{A})$ be functorial morphisms with

$$(D(\underline{A})d, \pi d: D(\underline{X})d \to SD(\underline{A})d) \in \Pi \quad \text{for all} \quad d \in \underline{D}.$$

A Π-*left-extension of* $(D(\underline{A}), \pi, \phi)$ consists of an S-morphism

$$(A^*, e^*: X \to SA^*) \in \Pi$$

and a functorial morphism

$$a: D(\underline{A}) \to \Delta A^* \quad \text{with} \quad (Sa)\pi = (\Delta e^*)\phi$$

such that for all S-morphisms $(A, x: X \to SA)$ and cones

$$\beta: PD(\underline{A}) \to \Delta PA \quad \text{with} \quad (Q\beta)\pi = (\Delta x)\phi$$

there exists exactly one $\underline{A}$-morphism $t: A^* \to A$ with

$$(St)e^* = x \text{ and } (\Delta Pt)(Pa) = \beta.$$

S is called a *Π-left-extension structure functor* (for (P, Q)) iff, for all «double cones» $(D(\underline{A}), \pi, \phi)$ there exists a Π-left-extension.

Applying again the generalized duality Theorem in [26] we obtain the following

(3.3) THEOREM *(Duality theorem for left-extension structure functors). Let*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization and $\Pi \subset Mor(S)$ *a class of S-morphisms. Then the following assertions are equivalent:*

(i) S is a Π-locally orthogonal structure functor (for (P, Q)).

(ii) S is a Π-left-extension structure functor (for (P, Q)).

In the following we will always assume the following situation:

(1) The factorizations

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

in question have the property that P has a left-adjoint with unit η.

(2) The subclasses $\Pi \subset Mor(S)$ in question are assumed to contain all morphisms $Q\eta(B)$, $B \in \underline{B}$. [1)]

(3.4) THEOREM. *Let*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization. Then the following assertions are equivalent:

(i) S is a structure functor (for (P, Q)).

(ii) S is a Quot(S)-left-extension functor for (P, Q).

(iii) S is a Π-left-extension functor for $\Pi \subset Mor(S)$.

(iv) S is a Π-left extension functor for $\Pi \subset Epi(S)$.

PROOF. (i) → (ii). Let

$$(D(\underline{A}), SD(\underline{A}) \xleftarrow{\pi} D(\underline{X}) \xrightarrow{\phi} \Delta X)$$

be a double cone with π being pointwise in Π where $D(\underline{A}): \underline{D} \to \underline{A}$ and $D(\underline{X}): \underline{D} \to \underline{X}$. Since $\pi d \in \Pi = Quot(S)$ for all $d \in \underline{D}$ there exist

1) If S is a structure functor, then P has a left adjoint and $\Pi = Quot(S)$ always fulfills condition (2).

$$D_d\colon \underline{D} \to \underline{B},\ \beta_d\colon D_d \to \Delta P D(\underline{A})d \ \text{ and } \ \phi_d\colon Q D_d \to \Delta D(\underline{X})d$$

defining a semifinal extension with $Q\beta_d = (\Delta \pi d)\phi d$. The Q-cocone $Q D_d \to \Delta X$ has a semifinal extension

$$(A_d,\ e_d\colon X \to SA_d),\quad a_d\colon D_d \to \Delta P A\ .$$

This defines a discrete S-cone $(e_d\colon \Delta X \to SA_d)_{d \in D}$. The semifinal factorization delivers a $Quot(S)$-morphism $(A,\ e\colon X \to SA)$, and an $\underline{A}$-cone $a\colon D(\underline{A}) \to \Delta A$ being the $Quot(S)$-left-extension looked for. The other implications are left to the reader.

4. THE REPRESENTATION THEOREM FOR STRUCTURE FUNCTORS

(4.1) *The « Canonical Factorization »*. Let

$$(\underline{A} \xrightarrow{\ S\ } \underline{X}) = (\underline{A} \xrightarrow{\ P\ } \underline{B} \xrightarrow{\ Q\ } \underline{X})$$

be a structure functor. I will construct a factorization of S:

$$(\underline{A} \xrightarrow{\ S\ } \underline{X}) = (\underline{A} \xrightarrow{\ R\ } \underline{C} \xrightarrow{\ T\ } \underline{X})$$

such that

(i) $R\colon \underline{A} \to \underline{C}$ is a right-adjoint functor, and

(ii) $T\colon \underline{C} \to \underline{X}$ is a topological functor.

The construction is given by the following datas:

(1) The objects of $\underline{C}$ are all S-morphisms $(A,\ e\colon X \to SA)$ in $Quot(S)$.

(2) A morphism $(x, f)\colon (A, e) \to (A', e')$ in $\underline{C}$ is a pair consisting of an $\underline{X}$-morphism $x\colon X \to X'$ and an $\underline{A}$-morphism $f\colon A \to A'$ such that

$$e'x = (Sf)e.$$

(3) The functor $T\colon \underline{C} \to \underline{X}$ is given by the assignments:

$$(A,\ e\colon X \to SA) \mapsto X,\quad (x, f) \mapsto x.$$

(4) The functor $R\colon \underline{A} \to \underline{C}$ is given by the assignments

$A \mapsto (A^*,\ e^*\colon SA \to SA^*)$, where $e^*\colon SA \to SA^*$ is the S-quotient of the identity $id\colon SA \to SA$,

$$\begin{array}{ccccccc} A & & SA & \xrightarrow{\ e^*\ } & SA^* & & A^* \\ {\scriptstyle f}\downarrow & \mapsto & {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Sf^*} & & \downarrow{\scriptstyle f^*} \\ A' & & SA' & \xrightarrow{\ e'^*\ } & SA'^* & & A'^* \end{array}$$

FIGURE 3

where $f^*: A^* \to A'^*$ is uniquely determined by the semifinal property of (A^*, e^*).

(4.2) THEOREM *(Representation Theorem for structure functors). Let $S: \underline{A} \to \underline{X}$ be a functor. Then the following assertions are equivalent:*

(i) S is a structure functor (with respect to a factorization

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})).$$

(ii) There exists a factorization

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{R} \underline{C} \xrightarrow{T} \underline{X})$$

where R = right adjoint functor and T = topological functor.

(iii) The canonical factorization

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{R} \underline{C} \xrightarrow{T} \underline{X})$$

fulfills (ii).

PROOF. (i) → (iii) Take the factorization $S = TR$ in (4.1).

1) R has a left adjoint $L: \underline{C} \to \underline{A}$. L is given by the assignments

$$L: \underline{C} \to \underline{A}: (A, e) \mapsto A, \quad (x, f) \mapsto f.$$

Let (A, e) be a (P, Q)-quotient. Then there exists a P-morphism $(A^*, b: PA \to PA^*)$ with

$$RL(A, e) = RA = (A^*, Qb: SA \to SA^*).$$

The semifinal property of (A, e) induces a unique $\underline{A}$-morphism

$$a: A \to A^* \text{ such that } (Qb)e = (Sa)e.$$

Now the $\underline{C}$-morphism $(e, a): (A, e) \to (A^*, Qb)$ is universal with respect to R. Let $A' \in \underline{A}$ and $(x, f): (A, e) \to RA'$ be a $\underline{C}$-morphism. The universal property of $RA' = (A'^*, Qb')$ implies an $\underline{A}$-morphism $a^*: A'^* \to A'$ such that $(Sa^*)(Qb') = id$. Then $a^*f: L(A, e) = A \to A'$ is the unique morphism with $(R(a^*f))(e, a) = (x, f)$.

2) T is topological. Let $D: \underline{D} \to \underline{C}$ be a diagram and $\phi: TD \to \Delta X$ be a T-cone over $X \in \underline{X}$. Let

$$D(\underline{X}) := TD: \underline{D} \to \underline{X}, \text{ resp. } D(\underline{A}) := LD: \underline{D} \to \underline{A}.$$

Then

$$\pi d := e: D(\underline{X})d = TDd = T(A, e: X \to SA) \to SD(\underline{A})d = SA$$

defines a functorial morphism $\pi: D(\underline{X}) \to SD(\underline{A})$ being pointwise in $Quot(S)$. Hence there exist a functorial morphism $a: D(\underline{A}) \to \Delta A^*$ and $(A, e^*: X \to SA^*) \in Quot(S)$ forming a $Quot(S)$-left-extension of the double cone (π, ϕ). Then $(a, \phi): D \to \Delta(A, e)$ is the final structure induced by ϕ.

(ii) → (i) follows immediately from (1.4). (iii) → (ii) is trivial.

If $P = Id$ then obviously R is a full embedding. Hence we obtain the important

(4.3) COROLLARY *(Tholen & Wischnewsky [20, 25], Representation Theorem for semitopological functors). Let $S: \underline{A} \to \underline{X}$ be a functor. Then there are equivalent:*

(i) S is semitopological.

(ii) S is a full reflective restriction of a topological functor.

By dualizing we obtain

(4.4) THEOREM *(Representation Theorem for co-structure functors). Let $S: \underline{A} \to \underline{X}$ be a functor. Then there are equivalent:*

(i) S is a co-structure functor.

(ii) There exists a factorization

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{L} \underline{C} \xrightarrow{T} \underline{X})$$

where L = left adjoint functor and T = topological functor.

(iii) The « dual» canonical factorization of S fulfills (ii).

5. CHARACTERIZATION AND EXISTENCE THEOREMS FOR STRUCTURE FUNCTORS

The results of this paragraph show that a functor is a structure functor iff certain colimits or limits exist. This is in some sense astonishing for the special instance of monadic functors since Adamek gave an example of a non-cocomplete Eilenberg-Moore category over a cocomplete base category [1]. The theorems here show that nevertheless certain (even large) colimits do exist in every Eilenberg-Moore category. The characterization of structure functors by certain «limits» was not known even in the case of semitopological functors. Finally one should mention that these results here give new formal existence criteria for left-adjoints.

Let

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization of the functor S and $\Pi \subset Mor(S)$.

(5.1) DEFINITION. 1) Given

$$(A, SA \xleftarrow{e} Y \xrightarrow{x} X) \text{ with } (A, e: Y \to SA) \in \Pi,$$

a Π-left-extension of (A, e, x) is called a Π-*semi-pushout of* (A, e, x) (relative to (P, Q)).

2) Given a discrete functorial morphism

$$(A_i, e_i: X \to SA_i)_{i \in I}$$

being pointwise in Π, a Π-left-extension of $(A_i, e_i)_{i\in I}$ is called a Π-*multiple pushout of* $(A_i, e_i)_{i\in I}$ (relative to (P, Q)).

3) S is called Π-*cocomplete* (with respect to (P, Q)) if every chain (A, e, x) (as in 1) has a Π-semi-pushout and if every discrete functorial chain $(A_i, e_i)_{i\in I}$ being pointwise in Π has a Π-multiple pushout.

(5.2) DEFINITION. S is called Π-*locally orthogonal complete* (or just Π-*complete* if there is no confusion) if:

1) Every S-morphism $(A, x: X \to SA)$ has a Π-locally orthogonal factorization, and

2) Every discrete S-cone $(A_i, e_i: X \to SA_i)_{i\in I}$ being pointwise in Π has a Π-locally orthogonal factorization.

(5.3) LEMMA. *If S is Π-complete or Π-cocomplete, then* $\Pi \subset Epi(S)$.

PROOF. This is proved in the usual way with the Cantor's diagonalization principle [4].

(5.4) THEOREM (*Characterization and existence Theorem for structure functors*). *Let*

$$(\underline{A} \xrightarrow{S} \underline{X}) = (\underline{A} \xrightarrow{P} \underline{B} \xrightarrow{Q} \underline{X})$$

be a factorization and $\Pi \subset Mor(S)$. *The following assertions are equivalent:*

(i) S is a structure functor.

(ii) S is Π*-complete.*

(iii) S is Π*-cocomplete.*

PROOF. From the results in Section 3 follows that we have only to prove the implications (ii) → (i) and (iii) → (i).

1) (ii) → (i). Let

$$D(\underline{A}): \underline{D} \to \underline{A}, \ D(\underline{X}): \underline{D} \to \underline{X}, \ \phi: D(\underline{X}) \to \Delta X \text{ and } \pi: D(\underline{X}) \to SD(\underline{A}),$$

where π is pointwise in Π, be given. Consider the class of all factorizations of the double cone (π, ϕ), i.e., all

$$(A_i, e_i: X \to SA_i) \in \Pi \text{ and } a_i: D \to \Delta A_i \text{ with } (Sa_i)\pi = (\Delta e_i)\phi .$$

The Π-locally orthogonal factorization of the cone $(A_i, e_i: X \to SA_i)_{i \in I}$ delivers a Π-object $(A^*, e^*: X \to SA^*)$ and $\underline{A}$-morphisms $a_i: A^* \to A_i$ with $e_i = (Sa_i)e^*$. Hence we obtain a unique functorial morphism

$$\beta: D \to \Delta A^* \text{ with } (S\beta)\pi = (\Delta e^*)\phi .$$

Let now $(A, x: X \to SA)$ be an arbitrary S-morphism and $\gamma: PD(\underline{A}) \to \Delta PA$ be a functorial morphism with $(\Delta x)\phi = (Q\gamma)\pi$. Let $x = (Sf)e$ be the Π-locally orthogonal factorization of x where $(A', e: X \to SA') \in \Pi$. This induces a functorial morphism

$$\delta: D(\underline{A}) \to \Delta A' \quad \text{with} \quad (S\delta)\pi = (\Delta e')\phi .$$

Hence there exists an index i with

$$e' = e_i \text{ and } a_i = \delta .$$

Then $t = f.a_i: A^* \to A$ is the unique morphism looked for. Hence S is a Π-left extension structure functor and thus a structure functor.

2) (iii) → (i). This is proved in an analogous way. One starts with the class of all factorizations

$$\phi = (Sa_i)\Delta e_i \text{ with } (A_i, e_i: X \to SA\ ) \in \Pi \text{ and } a_i: \Delta A_i \to D,$$

$D: \underline{D} \to \underline{A}$, of a given functorial morphism $\phi: \Delta X \to SD$. Then take the Π-multiple pushout of $(A_i, e_i: X \to SA_i)$. This induces a factorization of

$\phi = (Sa)(\Delta e)$ which turns out to be Π-locally orthogonal.

Theorem (5.4) has several important corollaries of which I will state here just two of them.

(5.5) COROLLARY. *Let $S: \underline{A} \to \underline{X}$ be a functor. Then there are equivalent:*

(i) S is semitopological.

(ii) S is Π-complete (for (Id, S)).

(iii) S is Π-cocomplete (for (Id, S)).

The equivalence (i) <=> (iii) was first proved by Tholen [20]. The equivalence (i) <=> (ii) is new.

(5.6) COROLLARY *(Tholen [20]). The following assertions are equivalent for a category $\underline{A}$:*

(i) $\underline{A}$ is a locally orthogonal (Π, Mono-Cone($\underline{A}$))-category.

(ii) a) $\underline{A}$ is Π-cocomplete or Π-complete.

b) $\underline{A}$ has coequalizers.

c) Π is closed under composition with extremal epimorphisms from the left.

If Π is closed under composition, then $\underline{A}$ is an orthogonal (Π, Mono-Cone($\underline{A}$))-category.

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