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JIŘÍ ADÁMEK

VÁCLAV KOUBEK

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COMPLETIONS OF CONCRETE CATEGORIES

by Jiří ADÁMEK and Václav KUBEK

INTRODUCTION

Given a complete base-category $\mathcal{X}$, we study completions of concrete categories, i.e., categories $\mathcal{K}$ endowed with a faithful (forgetful) functor $U: \mathcal{K} \rightarrow \mathcal{X}$. We prove that each concrete category $\mathcal{K}$ has a universal concrete completion $U^*: \mathcal{K}^* \rightarrow \mathcal{X}$. This means that:

(i) $\mathcal{K}^*$ is a complete category and its limits are concrete (i.e., preserved by U^*),

(ii) $\mathcal{K}$ is a full, concrete subcategory of $\mathcal{K}^*$ closed under all the existing concrete limits, and

(iii) each concrete functor on $\mathcal{K}$, which preserves concrete limits, has a unique such extension to $\mathcal{K}^*$.

It turns out that, moreover, $\mathcal{K}$ is codense in $\mathcal{K}^*$, i.e., each object of $\mathcal{K}^*$ is a limit of some diagram in $\mathcal{K}$.

The category $\mathcal{K}^*$ is constructed by adding formal limits to the objects of $\mathcal{K}$. The same method has already been used by C. Ehresmann [3]. New in our approach is the fact that the addition of limits need not be iterated - hence the codensity. The morphisms of $\mathcal{K}^*$ are defined by a natural transfinite induction. A direct construction of the universal completion will be presented by H. Herrlich in [5].

The completion of concrete categories yields much more satisfactory results than that of «abstract» categories, see for example [6, 7, 8]. V. Trnková even exhibits in [8] a category $\mathcal{K}$ which cannot be fully embedded into any finitely productive category with all the finite products of $\mathcal{K}$ preserved.

1. *Concrete categories* over a base category $\mathcal{X}$ (assumed to be complete throughout the paper) are categories $\mathcal{K}$ together with a functor $U: \mathcal{K} \rightarrow \mathcal{X}$ (denoted by $UA = |A|$ on objects, $Uf = f$ on morphisms) which is faithful and amnesitic, i.e., for each isomorphism $f: A \rightarrow B$ in $\mathcal{K}$ with Uf a unit morphism in $\mathcal{X}$ we have $A = B$. Given concrete categories $\mathcal{K}$ and $\mathcal{L}$ a *concrete functor* is a functor $F: \mathcal{K} \rightarrow \mathcal{L}$ commuting with the forgetful functors (i.e., on objects $|FA| = |A|$; on morphisms $Ff = f$).

A concrete category $\mathcal{K}$ is *concretely complete* if the forgetful functor «detects» limits in the following sense. Let D be a diagram in $\mathcal{K}$. (In the present paper this will always mean a small collection of objects

$$D^0 = \{A_i\}_{i \in I(D)}$$

and sets of morphisms

$$D[i, j] \subset \text{hom}(A_i, A_j) \text{ for } i, j \in I.$$

The forgetful functor *detects* the limit of D if for each limiting cone $\pi_i: X \rightarrow |A_i|$, $i \in I$ of the underlying diagram $|D|$ in $\mathcal{X}$ (with objects $|A_i|$, $i \in I(D)$, and morphisms $|D[i, j]| = D[i, j]$) there exists an *initial lift* A in $\mathcal{K}$. Recall that an object A is an initial lift of a cone $\pi_i: X \rightarrow A_i$ if:

- (i) $|A| = X$ and each $\pi_i: A \rightarrow A_i$ is a morphism in $\mathcal{K}$;
- (ii) given an object B and a map $h: |B| \rightarrow X$ such that each $\pi_i \cdot h: B \rightarrow A_i$ is a morphism in $\mathcal{K}$, then so is $h: B \rightarrow A$.

Now, an initial lift of a limiting cone of $|D|$ is clearly a limit of D . Note that we can speak about *the* initial lift since, due to amnesiticity, it is unique. Note also that a concretely complete category is *transportable*, i.e., for each isomorphism $f: X \rightarrow Y$ in $\mathcal{X}$ and for each object A in $\mathcal{K}$ with $|A| = X$ there exists an object B in $\mathcal{K}$ such that $|B| = Y$ and $f: A \rightarrow B$ is an isomorphism, too. In fact, a concrete category is concretely complete iff it is complete and the forgetful functor

- (i) preserves limits and (ii) is transportable.

Fortunately neither «amnesitic» nor «transportable» are severe restrictions:

2. LEMMA. *For each faithful functor $U: \mathcal{K} \rightarrow \mathcal{X}$ there exists a transportable concrete category $U': \mathcal{K}' \rightarrow \mathcal{X}$ and a concrete equivalence $E: \mathcal{K} \rightarrow \mathcal{K}'$*

with $U = U' \cdot E$.

PROOF. Let $(\mathcal{K}'', U'')$ denote the following category and functor: objects of $\mathcal{K}''$ are triples (X, f, A) with X an object in $\mathcal{X}$, A an object in $\mathcal{K}$ and $f: X \rightarrow |A|$ an isomorphism in $\mathcal{X}$; morphisms $p: (X, f, A) \rightarrow (Y, g, B)$ of $\mathcal{K}''$ are maps $p: X \rightarrow Y$ such that $g \cdot p \cdot f^{-1}: A \rightarrow B$ is a morphism in $\mathcal{K}$;

$$\begin{array}{ccc} X & \xrightleftharpoons[f^{-1}]{f} & A \\ p \downarrow & & \downarrow \\ Y & \xrightarrow{g} & B \end{array}$$

the functor $U'': \mathcal{K}'' \rightarrow \mathcal{X}$ sends (X, f, A) to X and p to p .

Then U'' is transportable but not amnestic. Therefore, we define an equivalence $\approx$ on objects by:

$$(X, f, A) \approx (Y, g, B) \quad \text{iff} \quad X = Y \quad \text{and} \quad id_X: (X, f, A) \rightarrow (Y, g, B) \\ \text{is an isomorphism in } \mathcal{K}''.$$

Denote by $\mathcal{K}'$ any choice class of this equivalence, as a full subcategory of $\mathcal{K}''$, and let $U' = U''/\mathcal{K}'$. Then $(\mathcal{K}', U')$ is clearly a transportable concrete category and the functor $E: \mathcal{K} \rightarrow \mathcal{K}'$, where $E(A)$ is the representant of (A, id_A, A) , is an equivalence functor with $U = U' \cdot E$.

3. DEFINITION. A *universal concrete completion* of a category $\mathcal{K}$ is a concretely complete category $\mathcal{K}^*$, in which $\mathcal{K}$ is a full and concrete subcategory (i.e., the forgetful functor of $\mathcal{K}$ is inherited from $\mathcal{K}^*$) closed to concrete limits and with the following universal property:

Let $\mathcal{Q}$ be a concretely complete category. Then each concrete functor $F: \mathcal{K} \rightarrow \mathcal{Q}$ preserving concrete limits has a concrete continuous extension $F^*: \mathcal{K}^* \rightarrow \mathcal{Q}$, unique up to natural equivalence.

4. MAIN THEOREM. *Every concrete category $\mathcal{K}$ has a universal concrete completion in which $\mathcal{K}$ is codense.*

5. REMARK. «Codense» means that each object of the extension $\mathcal{K}^*$ is a limit of some diagram in $\mathcal{K}$. It then follows that $\mathcal{K}$ is closed under arbitrary colimits in $\mathcal{K}^*$ (see [4]).

6. PROOF OF THE MAIN THEOREM. Let $\mathcal{K}$ be a concrete category. We shall define its concrete completion $\mathcal{K}^*$ of which we shall verify the properties of a universal concrete completion, except transportability. Then we use Lemma 2: there exists a transportable concrete category, say $\mathcal{K}^{**}$, concretely equivalent to $\mathcal{K}^*$, and this is the universal concrete completion of $\mathcal{K}$.

Denote by $\mathcal{D}$ the class of all diagrams in $\mathcal{K}$ which have a concrete limit in $\mathcal{K}$. For each diagram D in $\mathcal{K}$ with $D \notin \mathcal{D}$ choose a limiting cone (in $\mathcal{X}$) of the underlying diagram $|D|$, where $D^0 = \{Q_i\}_{i \in I(D)}$, say

$$\pi_i^D: X^D \rightarrow |Q_i| \quad \text{for } i \in I(D).$$

Define a concrete category $\mathcal{K}^*$. Its objects are:

1) all objects in $\mathcal{K}$, and

2) objects P^D , indexed by all diagrams D in $\mathcal{K}$ with $D \notin \mathcal{D}$ (we assume $P^D \notin \mathcal{K}^0$ and $P^D \neq P^{D'}$ whenever $D \neq D'$).

The forgetful functor of $\mathcal{K}^*$ agrees with that of $\mathcal{K}$ on $\mathcal{K}$ -objects and it sends P^D to X^D . The morphisms of $\mathcal{K}^*$ will be defined by a transfinite induction: for each ordinal k and each pair Q, R of objects in $\mathcal{K}^*$ we define a set of maps $H_k(Q, R) \subset \text{hom}(|Q|, |R|)$ and then a map is a morphism $f: Q \rightarrow R$ in $\mathcal{K}^*$ iff there exists an ordinal k with $f \in H_k(Q, R)$.

H_0 -morphisms are

(i) all $\mathcal{K}$ -morphisms between $\mathcal{K}$ -objects,

(ii) all the connection maps $\pi_i^D: P^D \rightarrow Q_i$ (for a diagram $D \notin \mathcal{D}$ and $i \in I(D)$), and

(iii) the identity maps $id_{X^D}: P^D \rightarrow P^D$ (for a diagram $D \notin \mathcal{D}$).

CONVENTION. A collection of H_0 -morphisms is said to be *distinguished* if either:

(a) it forms a limiting cone (in $\mathcal{K}$) of a diagram $D \notin \mathcal{D}$; or

(b) it is the collection $\{\pi_i^D \mid i \in I(D)\}$ for some diagram $D \notin \mathcal{D}$; or

(c) it is the singleton collection $\{id_Q\}$ for an object Q of $\mathcal{K}^*$.

H_{k+1} -morphisms are «basic» morphisms and their composition. A map $f: |Q| \rightarrow |R|$ (where Q, R are objects in $\mathcal{K}^*$) is a basic H_{k+1} -morphism if there exists a distinguished collection $r_j: R \rightarrow R_j$, $j \in J$, in H_0 such that

$r_j \cdot f \in H_k(Q, R_j)$ for each $j \in I$.

$$H_\gamma = \bigcup_{k < \gamma} H_k \quad \text{for each limit ordinal } \gamma.$$

It is clear that the above defines correctly a concrete category $\mathcal{K}^*$ except that $\mathcal{K}^*$ fails to be amnesic.

(1) $\mathcal{K}$ is a full subcategory of $\mathcal{K}^*$.

PROOF. We shall prove, by induction in k , that for each morphism f in $H_k(Q, R)$ such that Q is an object of $\mathcal{K}$, there exists a distinguished collection $p_i: R \rightarrow R_i$ such that each $p_i \cdot f: Q \rightarrow R_i$ is a morphism in $\mathcal{K}$. It then follows that $\mathcal{K}$ is full in $\mathcal{K}^*$: if also $R \in \mathcal{K}^0$, then the distinguished collection must be a concrete limiting cone of a diagram $D \in \mathcal{D}$. The compatible collection $\{p_i \cdot f\}$ (in $\mathcal{K}!$) factorizes through the collection $\{p_i\}$ —the factorization is necessarily f , thus f is a $\mathcal{K}$ -morphism.

For $k = 0, 1$ the proposition can be proved by a simple inspection.

Assuming the proposition holds for $k \geq 1$, we shall prove it for $k+1$. This is clear for basic H_{k+1} -morphisms: there exists a distinguished collection $p_i: R \rightarrow P_i$ such that each $p_i \cdot f: Q \rightarrow P_i$ is in H_k and, by induction hypothesis, each $p_i \cdot f$ is a $\mathcal{K}$ -morphism.

Let f be a composite of $n+1$ basic morphisms, $f = f_{n+1} \cdot f_n \cdot \dots \cdot f_1$ (with $f_i: R_{i-1} \rightarrow R_i$, where $R = R_0$ and $Q = R_{n+1}$) and assume the proposition holds for compositions of n basic morphisms.

$$\begin{array}{ccccccc} R & \xrightarrow{f_1} & R_1 & \xrightarrow{f_2} & R_2 & \xrightarrow{f_3} & \dots & \xrightarrow{f_n} & R_n & \xrightarrow{f_{n+1}} & Q \\ & & & & & & & & p_i \downarrow & & \downarrow q_j \\ & & & & & & & & P_i & & Q_j \end{array}$$

Particularly, the proposition holds for $g = f_n \cdot \dots \cdot f_1$: there exists a distinguished collection $p_i: R_n \rightarrow P_i$ such that each $p_i \cdot g: R \rightarrow P_i$ is a $\mathcal{K}$ -morphism. There follows $g \in H_1$. Moreover, since f_{n+1} is basic, there exists a distinguished collection $q_j: Q \rightarrow Q_j$ with each $q_j \cdot f_{n+1}$ in H_k . Then also each

$$(q_j \cdot f_{n+1}) \cdot g = q_j \cdot f: R \rightarrow Q_j$$

is in H_k , hence (by the inductive hypothesis) in $\mathcal{K}$.

(2) $\mathcal{K}$ is closed to limits of $\mathcal{D}$ -diagrams in $\mathcal{K}^*$.

PROOF. Let D be a diagram in $\mathcal{D}$; denote its limiting cone by $\pi_i: P \rightarrow Q_i$, $i \in I$. We are to show that for each compatible cone $\pi'_i: P' \rightarrow Q_i$, $i \in I$ in $\mathcal{K}^*$, there is a unique morphism

$$p: P' \rightarrow P \text{ with } \pi'_i = \pi_i \cdot p \text{ for each } i.$$

Since the limit of D is concrete, we have a limiting cone $\pi_i: |P| \rightarrow |Q_i|$ for the diagram $|D|$ in $\mathcal{X}$. And the cone $\pi'_i: |P'| \rightarrow |Q_i|$ is compatible for $|D|$, hence there exists a unique map $p: |P'| \rightarrow |P|$ with the required property. It remains to show that $p: P' \rightarrow P$ is a morphism in $\mathcal{K}^*$. Since each π'_i is a morphism in $\mathcal{K}^*$, there exists an ordinal γ such that

$$\pi'_i \in H_\gamma(P', Q_i) \text{ for each } i \in I.$$

Now, the collection $\{\pi_i\}_{i \in I}$ is distinguished (it is of the first type of distinguished collections), thus $\pi'_i = \pi_i \cdot p \in H_\gamma$ (for each $i \in I$) implies $p \in H_{\gamma+1}$.

(3) Each diagram $D \notin \mathcal{D}$ in $\mathcal{K}$ has a concrete limit in $\mathcal{K}^*$, viz,

$$\pi_i^D: P^D \rightarrow Q_i, \quad i \in I(D).$$

The proof is analogous to (2) above: the collection $\{\pi_i^D\}$ is distinguished and it forms a limit of the underlying diagram.

COROLLARIES. Every diagram in $\mathcal{K}$ has a concrete limit in $\mathcal{K}^*$;

$\mathcal{K}$ is codense in $\mathcal{K}^*$.

(4) $\mathcal{K}^*$ has limits, preserved by U .

We shall prove it in two steps: first, with each diagram D in H_0 (more precisely, each diagram in $\mathcal{K}^*$, all morphisms of which belong to H_0) we associate a diagram D^+ in $\mathcal{K}$ (which has a concrete limit by (2) and (3)) such that $\lim D^+ = \lim D$. Second, with each diagram D in $\mathcal{K}^*$ we associate a diagram $\hat{D}$ in H_0 with $\lim \hat{D} = \lim D$.

(4A) Let D be a diagram in H_0 , say on objects Q_j , $j \in J$. Put

$$J' = \{j \in J \mid Q_j \text{ is not an object of } \mathcal{K}\};$$

thus, for each $j \in J'$ we have a diagram $D_j \notin \mathcal{D}$ in $\mathcal{K}$ (say, on objects Q_{ji} for $i \in I_j$) with $Q_j = P^{D_j}$. Assuming the index sets I_j are pairwise disjoint

and disjoint from J (by which we do not lose generality, of course) we define a diagram D^+ in $\mathcal{K}$ as follows: Its objects are

$$\{Q_j\}_{j \in J-J'} \cup \{Q_{ji}\}_{j \in J', i \in I_j}.$$

Its morphisms are:

(i) all D -morphisms in $\mathcal{K}$:

$$D^+[j_1, j_2] = D[j_1, j_2] \quad \text{for each } j_1, j_2 \in J-J';$$

(ii) for $j \in J'$, all morphisms in D_j :

$$D^+[ji_1, ji_2] = D_j[ji_1, ji_2] \quad \text{for each } j \in J' \text{ and } i_1, i_2 \in I_j;$$

(iii) for each limiting-cone morphism in D

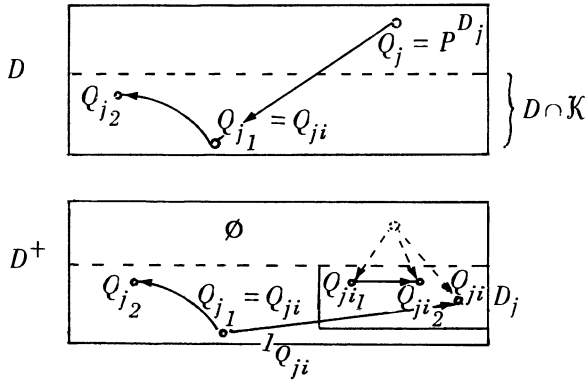
$$\pi_i^j: P^D j \rightarrow Q_{ji} = Q_{ji} \quad \text{with } j_1 \in J-J' \text{ and } j \in J', i \in I_j,$$

(thus $P^D j = Q_j$), we add the unit morphism to D^+ :

$$D^+[j_1, ji] = \{1_{Q_{j_1}}\} \quad \text{if } \pi_i^j \in D[j, j_1], \text{ where } j_1 \in J-J', j \in J',$$

$$i \in I_j \text{ with } Q_{ji} = Q_{j_1},$$

$$D^+[j_1, ji] = \emptyset \quad \text{else.}$$



We shall prove that $\lim D^+ = \lim D$. More precisely, given the (concrete) limit $S = \lim D^+$ with the limiting cone

$$\phi^j: S \rightarrow Q_j \quad \text{for } j \in J-J' \quad \text{and} \quad \phi_{ji}: S \rightarrow Q_{ji} \quad \text{for } j \in J' \text{ and } i \in I_j,$$

define for $j \in J'$ a morphism $\phi^j: S \rightarrow Q_j$ by

$$\pi_i^j \cdot \phi^j = \phi_{ji} \quad \text{for each } i \in I_j.$$

(This is correct: $\{\pi_i^{Dj}\}_{i \in I_j}$ is a limit of D_j and, by (ii) above, $\{\phi_{ji}\}_{i \in I_j}$ is a compatible cone.) Then the cone $\{\phi^j\}_{j \in J}$ is a concrete limit of D .

PROOF. (a) The cone $\{\phi^j\}$ is compatible for D , i. e., for each morphism $f \in D[j, j_1]$ we have $\phi^{j_1} = f \cdot \phi^j$. This is clear if f is a morphism of $\mathcal{K}$ (then it belongs to D^+). If f is in $H_0\text{-}\mathcal{K}^m$ then either f is a unit morphism (and the compatibility is clear) or $f = \pi_i^{Dj}$ for some $j \in J'$ and $i \in I_j$ such that $Q_j = P^{Dj}$ and $Q_{j_1} = Q_{ji}$. In that case

$$1_{Q_{j_1}} \in D^+[j_1, j_1], \text{ hence } \phi^{j_1} = \phi_{ji}.$$

Since

$$f \cdot \phi^j = \pi_i^{Dj} \cdot \phi^j = \phi_{ji} = \phi^{j_1},$$

compatibility is proved.

(b) The cone $\{\phi^j\}$ is universal. Let $\psi^j: T \rightarrow Q_j$, $j \in J$, be a compatible cone for D . Define a compatible cone for D^+ by putting

$$\psi_{ji} = \pi_i^{Dj} \cdot \psi^j: T \rightarrow Q_{ji} \text{ for each } j \in J' \text{ and } i \in I_j.$$

Then there exists a unique morphism $\psi: T \rightarrow S$ with

$$\text{I) } \psi^j = \phi^j \cdot \psi \text{ for } j \in J - J',$$

$$\text{II) } \psi_{ji} = \phi_{ji} \cdot \psi \text{ for } j \in J', i \in I_j.$$

For each $j \in J'$, the condition II is equivalent to $\psi^j = \phi^j \cdot \psi$ (because $\{\pi_i^{Dj}\}$ is a limiting cone for D_j and we have

$$\pi_i^{Dj} \cdot \psi^j = \psi_{ji} = \phi_{ji} \cdot \psi = \pi_i^{Dj} \cdot (\phi^j \cdot \psi).$$

Thus, the cone $\{\psi^j\}$ factorizes uniquely through the cone $\{\phi^j\}$.

(c) This limit of D is concrete. More generally: given an arbitrary functor $F: \mathcal{K}^* \rightarrow \mathcal{Q}$ (e. g., the forgetful functor) which preserves limits of all diagrams in $\mathcal{K}$, then F preserves the limit of D . The proof is analogous to (b): Given a compatible cone $\psi^j: T \rightarrow FQ_j$ for $F(D)$ in $\mathcal{Q}$, put

$$\psi_{ji} = (F\pi_i^{Dj}) \cdot \psi^j \text{ for } j \in J' \text{ and } i \in I_j.$$

This yields a compatible cone for $F(D^+)$. By hypothesis, F preserves the limit of D^+ , hence there exists a unique $\psi: T \rightarrow FS$ with

$$\text{I) } \psi^j = F\phi^j \cdot \psi \text{ for } j \in J - J',$$

$$\text{II) } \psi_{ji} = F\phi_{ji} \cdot \psi \text{ for } j \in J', i \in I_j.$$

For each $j \in J'$, the condition II is equivalent to $\psi^j = F\phi^j \cdot \psi$.

(4B) For each diagram D in $\mathbb{K}^*$ we shall construct a diagram $\hat{D}$ in H_0 such that each D -object is a $\hat{D}$ -object, and we shall prove:

(i) $\lim D = \lim \hat{D}$ and the restriction of the limiting cone of $\hat{D}$ to the objects of D is the limiting cone of D , and

(ii) each functor, preserving limits of diagrams in H_0 , preserves the limit of D .

The method is first to construct $\hat{D}$ in case D consists of a single morphism f (then $\hat{D}$ is denoted by $\hat{D}(f)$) and then, given an arbitrary diagram D , to obtain $\hat{D}$ by merging the diagrams $\hat{D}(f)$ with f ranging through the morphisms of D .

Thus, we first define a diagram $\hat{D}(f)$ for each morphism $f: P \rightarrow Q$ in $\mathbb{K}^*$. We shall proceed by induction in k where $f \in H_k(P, Q)$. The objects of $\hat{D}(f)$ will form a collection R_{fi} , $i \in I(f)$, with two distinguished ones: R_{fd_f} (the domain object) and R_{fc_f} (the codomain object) for

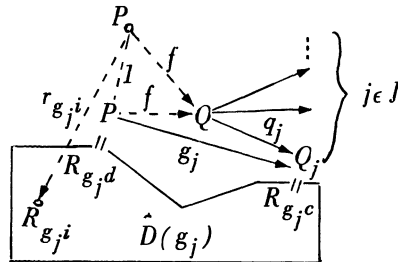
$$d_f, c_f \in I(f) \text{ such that } P = R_{fd_f} \text{ and } Q = R_{fc_f}$$

(we write also just R_{fd} and R_{fc}). And we shall also observe that there exist morphisms $r_{fi}: P \rightarrow R_{fi}$, $i \in I(f)$, forming a limiting cone of $\hat{D}(f)$ such that $r_{fd} = id_P$ and $r_{fc} = f$.

I. For $k = 0$ we let $\hat{D}(f)$ have just two objects $P = R_{fd}$, $Q = R_{fc}$, and just one morphism f . The limit is $id_P: P \rightarrow R_{fd}$ and $f: P \rightarrow R_{fc}$, of course.

II. Let $f \in H_{k+1}$ be a basic morphism. We fix a distinguished collection $q_j: Q \rightarrow Q_j$, $j \in J$, such that $g_j \stackrel{\text{def}}{=} q_j \cdot f$ is in H_k for each $j \in J$. Thus we have diagrams $\hat{D}(g_j)$. The diagram $\hat{D}(f)$ is obtained as follows:

(i) Form the disjoint union of the diagrams $\hat{D}(g_j)$, $j \in J$;



(ii) Merge all their domain objects $R_{g_j d}$ ($= P$): the merged object will be the domain object R_{fd} of $\hat{D}(f)$;

(iii) Add Q as a new object; this is the codomain object R_{fc} of $\hat{D}(f)$;

(iv) Add a new morphism $q_j: Q \rightarrow R_{g_j c}$ for each $j \in J$.

Thus, we obtain a diagram $\hat{D}(f)$ in H_0 . We claim that its limiting cone is

$$l_P: P \rightarrow R_{fd}, \quad f: P \rightarrow R_{fc} \quad \text{and} \quad r_{g_j i}: P \rightarrow R_{g_j i} \quad \text{for } j \in J, \quad i \in I(g_j).$$

First, this cone is compatible for $\hat{D}(f)$: for each $j \in J$ we have

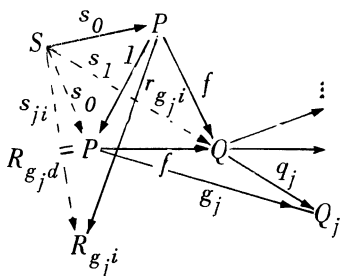
$$g_j \cdot l_P = g_j = q_j \cdot f;$$

and the compatibility with each morphism inside $\hat{D}(g_j)$ is clear. Second, given another compatible cone

$$s_0: S \rightarrow R_{fd}, \quad s_1: S \rightarrow R_{fc} \quad \text{and} \quad s_{ji}: S \rightarrow R_{g_j i} \quad (j \in J \text{ and } i \in I(g_j)),$$

we shall show that its unique factorization through the given cone is s_0 .

The uniqueness is evident, since $l_P: P \rightarrow R_{fd}$ is in the given cone. Further, for each $j \in J$ we have a compatible cone $\{s_{ji}\}_{i \in I(g_j)}$ for $\hat{D}(g_j)$,



which factorizes through the limiting cone $\{r_{g_j i}\}$ of $\hat{D}(g_j)$ - and the factorizing morphism must be s_0 again, thus

$$s_{ji} = r_{g_j i} \cdot s_0 \quad \text{for } j \in J' \text{ and } i \in I_j.$$

Finally, there follows $s_1 = f \cdot s_0$ because, for each $j \in J$, we have

$$r_{g_j c} = g_j = q_j \cdot f$$

(by the inductive hypothesis), hence

$$q_j \cdot s_1 = s_{jc} = r_{g_j c} \cdot s_0 = q_j \cdot (f \cdot s_0) \quad \text{for } j \in J.$$

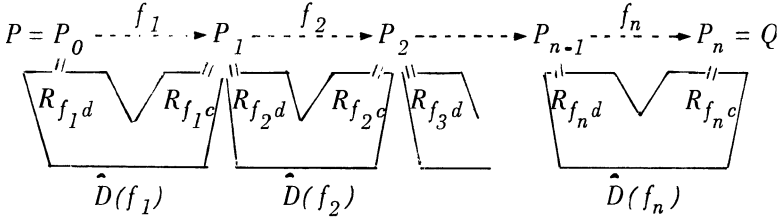
Now, $\{q_j\}_{j \in J}$ is a distinguished family, hence a limiting cone for some diagram (see (2) and (3) above), thus $s_1 = f \cdot s_0$.

III. Let $f = f_n \cdot \dots \cdot f_1$ be a composite of basic morphisms in H_{k+1} . We have diagrams $\hat{D}(f_1), \dots, \hat{D}(f_n)$ and we define $\hat{D}(f)$ as follows:

- (i) Form the disjoint union of the diagrams $\hat{D}(f_1), \dots, \hat{D}(f_n)$;
- (ii) Merge the codomain object of $\hat{D}(f_t)$ with the domain object of $\hat{D}(f_{t+1})$; the domain object of $\hat{D}(f_1)$ will be R_{fd} and the codomain object of $\hat{D}(f_n)$ will be R_{fc} .

We claim that the limiting cone of $\hat{D}(f)$ is:

$$\begin{aligned} r_{f_1 i} &: P \rightarrow R_{f_1 i} \text{ for } i \in I(f_1), \\ r_{f_2 i \cdot f_1} &: P \rightarrow R_{f_2 i} \text{ for } i \in I(f_2), \\ &\vdots \\ r_{f_n i \cdot (f_{n-1} \cdot \dots \cdot f_1)} &: P \rightarrow R_{f_n i} \text{ for } i \in I(f_n). \end{aligned}$$



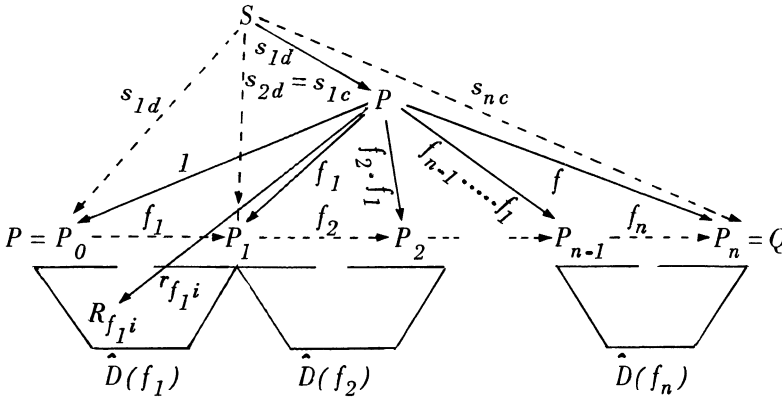
(In particular,

$$r_{f_1 d} = id_P : P \rightarrow R_{f_1 d} \text{ and } r_{f_n c} = f_n \cdot (f_{n-1} \cdot \dots \cdot f_1) = f : P \rightarrow R_{f_n c}.)$$

The compatibility of this cone is evident. Given another cone

$$s_{ti} : S \rightarrow R_{f_t i}, \text{ for } t = 1, \dots, n \text{ and } i \in I(f_t)$$

compatible with $\hat{D}(f)$, for each t we have a cone $\{s_{ti}\}$ compatible with



$\hat{D}(f)$. Thus, there exists a unique $s^t: S \rightarrow P_t$ with $s_{ti} = r_{f_i} \cdot s^t$ - in particular $s_{td} = s^t$ (because $r_{f_d} = id_{P_{t-1}}$ by the inductive hypothesis); further $s_{tc} = f_t \cdot s^t$ (because $r_{f_c} = f_t$), hence

$$s^{t+1} = s_{(t+1)d} = s_{tc} = f_t \cdot s^t.$$

Thus, the cone $\{s_{ti}\}$ factorizes (uniquely) through the above cone

$$\{r_{f_i} \cdot (f_{t-1} \cdot \dots \cdot f_1)\};$$

viz., by: $s_{1d}: S \rightarrow P$,

$$s_{1i} = r_{f_1 i} \cdot s^1 = r_{f_1 i} \cdot s_{1d} \quad \text{for } i \in I(f_1),$$

$$s_{2i} = r_{f_2 i} \cdot s^2 = (r_{f_2 i} \cdot f_1) \cdot s_{1d} \quad \text{for } i \in I(f_2),$$

$$\vdots$$

$$s_{ni} = r_{f_n i} \cdot s^n = (r_{f_n i} \cdot f_{n-1} \cdot \dots \cdot f_1) \cdot s \quad \text{for } i \in I(f_n).$$

IV. Let γ be a limit ordinal. If $\hat{D}(f)$ is constructed for all $f \in H_k$ with $k < \gamma$, then $\hat{D}(f)$ is constructed for all $f \in H_\gamma$.

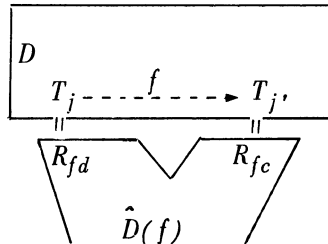
Thus we have constructed $\hat{D}(f)$ for each morphism in $\mathcal{K}^*$.

V. Given an arbitrary diagram D in $\mathcal{K}^*$ on objects T_j , $j \in J$, define a diagram $\hat{D}$ as follows:

(i) Form the disjoint union of diagrams $\hat{D}(f)$, with f ranging over all morphisms of the diagram D ;

(ii) Add the objects of D as new objects;

(iii) For each $f \in D[j, j']$ merge T_j with the domain object of $\hat{D}(f)$ and merge $T_{j'}$ with the codomain object of $\hat{D}(f)$.



The diagram $\hat{D}$ lies in H_0 , hence it has a concrete limit (by (4A)), say

$$t_j: T \rightarrow T_j \quad (j \in J),$$

$$t_{fi}: T \rightarrow R_{fi} \quad (f \text{ a morphism of } D, i \in I(f)).$$

We claim that the former part $\{t_j\}_{j \in J}$ is a concrete limiting cone for $\hat{D}$.

(a) The cone $\{t_j\}$ is compatible for D , i. e.,

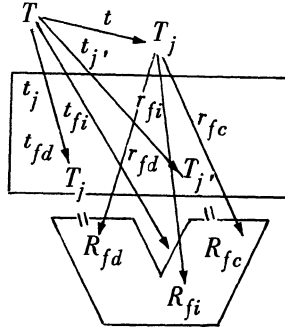
$$t_{j'} = f \cdot t_j \text{ for each } f \in D[j, j'].$$

Morphisms t_{fi} , $i \in I(f)$, form a compatible cone for $\hat{D}(f)$. This cone factorizes through the limiting cone $\{r_{fi}\}$: there is a

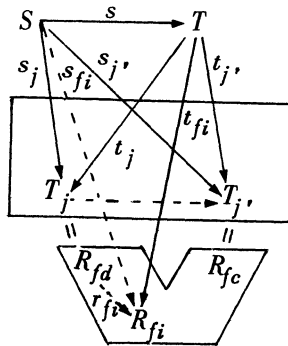
$$t: T \rightarrow T_j \text{ with } t_{fi} = r_{fi} \cdot t.$$

Necessarily $t = t_{fd} = t_j$ (since $r_{fd} = id_{T_j}$), hence (since $r_{fc} = f$):

$$t_{j'} = t_{fc} = r_{fc} \cdot t_j = f \cdot t_j.$$



(b) The cone $\{t_j\}$ is universal. Proof: Given another compatible cone $\{s_j\}$ with $s_j: S \rightarrow T_j$, define $s_{fi} = r_{fi} \cdot s_j$ for each morphism $f: T_i \rightarrow T_{j'}$ in D and each $i \in I(f)$. This clearly yields a compatible cone for $\hat{D}$ (the compatibility of $\{s_j\}$ guarantees that the definition of s_{fi} is correct, i. e., $s_j = s_{fd}$ and $s_{j'} = s_{fc}$: recall $r_{fd} = id$ and $r_{fc} = f$). Thus, there exists a unique morphism $s: S \rightarrow T$ with



$$s_j = t_j \cdot s \text{ and } s_{fi} = t_{fi} \cdot s \text{ for all } j, f \text{ and } i.$$

Since the latter follows from the former ($s_{fi} = r_{fi} \cdot s_j$ and $t_{fi} = r_{fi} \cdot t_j$ imply

$$s_{fi} = r_{fi} \cdot s_j = r_{fi} \cdot t_j \cdot s = t_{fi} \cdot s)$$

the unicity holds also for D .

(c) This limit of D is concrete. More generally: given an arbitrary functor $F: \mathcal{K}^* \rightarrow \mathcal{Q}$ which preserves limits of all diagrams in H_0 , then F preserves the limit of D . (E.g., the forgetful functor can be taken as F , see (4Ac).) The proof is analogous to (b): Given a compatible cone $s_j: S \rightarrow FT_j$ for $F(D)$ in $\mathcal{Q}$, define $s_{fi} = Fr_{fi} \cdot s_j$ to obtain a compatible cone for $F(\hat{D})$. Since $\{Ft_j\} \cup \{Ft_{fi}\}$ is a limiting cone for $F(\hat{D})$, the cone $\{s_j\}$ factorizes uniquely through $\{Ft_j\}$.

(5) *The conclusion of the proof.* Let $\mathcal{K}^*$ be a transportable category, concretely equivalent to $\mathcal{K}^*$. We shall verify that $\mathcal{K}^*$ is a universal concrete completion of $\mathcal{K}$. Without loss of generality we assume that $\mathcal{K}$ is a full concrete subcategory of $\mathcal{K}^*$.

Since $\mathcal{K}^*$ has limits preserved by the forgetful functor, so does $\mathcal{K}^*$ - recall that equivalences preserve limits. This implies that $\mathcal{K}^*$ is concretely complete; also, since $\mathcal{K}$ is closed to concrete limits in $\mathcal{K}^*$, so it is in $\mathcal{K}^*$.

Given a concretely complete category $\mathcal{Q}$ and a concrete functor $F: \mathcal{K} \rightarrow \mathcal{Q}$ preserving concrete limits, we are to find a concrete continuous extension of F to $\mathcal{K}^*$. We shall verify that F has a unique concrete, continuous extension to $\mathcal{K}^*$; then it has such an extension to $\mathcal{K}^*$, unique up to a natural equivalence. For each diagram D in $\mathcal{K}$, $D \nmid \mathcal{D}$, we have a diagram $F(D)$ in $\mathcal{Q}$ such that $|D| = |FD|$ (since F is a concrete functor). We have chosen a limit $\pi_i^D: X^D \rightarrow |Q_i|$ in $\mathcal{X}$ for the diagram $|F(D)|$. Since $\mathcal{Q}$ is a transportable concretely complete category, there exists an object R^D in $\mathcal{Q}$ with $|R^D| = X^D$ and such that $\pi_i^D: R^D \rightarrow FQ_i$ is a limiting cone for $F(D)$ (since $\mathcal{Q}$ is amnestic, R^D is unique). There is no other choice of a concrete, continuous extension F^* of F than

$$F^*(P^D) = R^D \text{ on objects, } F^*f = Ff \text{ on morphisms.}$$

We must verify that, on the other hand, this defines a concrete continuous functor $F^*: \mathcal{K}^* \rightarrow \mathcal{Q}$. First, F^* is indeed a functor, i. e., given a morphism $f: P^D \rightarrow P^{D'}$ in $\mathcal{K}^*$ then also $f: R^D \rightarrow R^{D'}$ is a morphism in $\mathcal{Q}$. This is easy to see (by induction in i with $f \in H_i$). Second, F^* is concrete by its very definition:

$$|F^*(P^D)| = X^D = |P^D|.$$

Finally, given a diagram D in $\mathcal{K}^*$, we shall verify that F^* preserves its limit. This is clear if D is a diagram in $\mathcal{K}$: either $D \in \mathcal{D}$ and then F^* ($= F$ on $\mathcal{K}$) preserves its limit by hypothesis; or $D \notin \mathcal{D}$, in which case the limiting cone is $\pi_i^D: P^D \rightarrow Q_i$ (see (3)). This is mapped by F^* to the cone $\pi_i^D: R^D \rightarrow FQ_i$, which has been chosen as the limiting cone for $F(D)$. Further, if D is a diagram in H_0 then, by (4Ac) above, F^* preserves its limit, too. Hence, if D is an arbitrary diagram in $\mathcal{K}^*$, then, by (4Bc), F^* preserves its limit again.

This concludes the proof of the theorem.

7. Without any change in the proof, the completion theorem can be generalized to $\mathcal{D}$ -universal completions. Let $\mathcal{K}$ be a concrete category and let $\mathcal{D}$ be a class of diagrams in $\mathcal{K}$, each having a concrete limit in $\mathcal{K}$. Then a $\mathcal{D}$ -universal concrete completion of $\mathcal{K}$ is its concrete completion $\mathcal{K}^*$, in which $\mathcal{K}$ is closed to limits of diagrams in $\mathcal{D}$, and which has the following universal property:

Let $\mathcal{Q}$ be a concretely complete category; then each concrete functor $F: \mathcal{K} \rightarrow \mathcal{Q}$, preserving limits of $\mathcal{D}$ -diagrams, has a unique concrete, continuous extension $F^*: \mathcal{K}^* \rightarrow \mathcal{Q}$.

In the proof of the Main Theorem, let $\mathcal{D}$ denote the given class (and not, as before, the class of *all* diagrams with concrete limits). Then the proof of the following theorem is obtained:

8. THEOREM. *Let $\mathcal{D}$ be a class of diagrams in a concrete category $\mathcal{K}$, each having a concrete limit in $\mathcal{K}$. Then $\mathcal{K}$ has a $\mathcal{D}$ -universal completion, in which $\mathcal{K}$ is codense.*

9. We shall use this theorem to prove the existence of universal bicompletions. First, we observe that the completion theorems above can be dualized: if $\mathcal{K}$ is concrete over $\mathcal{X}$, then $\mathcal{K}^{op}$ is concrete over $\mathcal{X}^{op}$. Hence for a cocomplete base-category, we see that each concrete category has a universal concrete cocompletion. (The generalization to $\mathcal{D}$ -universality is obvious.)

Now, let *bicomplete* stand for complete plus cocomplete. Let $\mathcal{X}$ be a bicomplete base-category. Then a *universal concrete bicompletion* of a concrete category $\mathcal{K}$ is a full, concrete and concretely bicomplete extension $\mathcal{K}^*$ of $\mathcal{K}$ in which $\mathcal{K}$ is closed to concrete limits and concrete colimits with the following universal property:

Let $\mathcal{Q}$ be a concretely bicomplete category; then each functor $F: \mathcal{K} \rightarrow \mathcal{Q}$ preserving concrete limits and concrete colimits has a unique bicontinuous extension $F^*: \mathcal{K}^* \rightarrow \mathcal{Q}$, unique up to natural equivalence.

10. THEOREM. *Every concrete category over a bicomplete base-category has a universal concrete bicompletion.*

PROOF. We shall define a transfinite sequence $\mathcal{K}^{(i)}$ of concrete categories the union of which will be the universal concrete bicompletion ¹⁾.

First, $\mathcal{K}^{(0)} = \mathcal{K}$ and $\mathcal{K}^{(1)}$ is the universal (concrete) completion of $\mathcal{K}$ (we omit the word concrete for shortness); second, $\mathcal{K}^{(2)}$ is the $\mathcal{D}^{(2)}$ -universal cocompletion of $\mathcal{K}^{(1)}$, where $\mathcal{D}^{(2)}$ is the class of all diagrams in $\mathcal{K}^{(1)}$ which lie in $\mathcal{K}^{(0)}$ and have a concrete colimit in $\mathcal{K}^{(0)}$.

Generally, given a limit ordinal γ , then:

$\mathcal{K}^{(\gamma)}$ is the $\mathcal{D}^{(\gamma)}$ -universal completion of $\bigcup_{i < \gamma} \mathcal{K}^{(i)}$, where

$$\mathcal{D}^{(\gamma)} = \bigcup_{i < \gamma} \mathcal{D}^{(i)};$$

$\mathcal{K}^{(\gamma+1)}$ is the $\mathcal{D}^{(\gamma+1)}$ -universal cocompletion of $\mathcal{K}^{(\gamma)}$, where

$$\mathcal{D}^{(\gamma+1)} = \mathcal{D}^{(\gamma)};$$

$\mathcal{K}^{(\gamma+2)}$ is the $\mathcal{D}^{(\gamma+2)}$ -universal completion of $\mathcal{K}^{(\gamma+1)}$, where $\mathcal{D}^{(\gamma+2)}$

1) This union is set-theoretically legitimate: the transfinite induction defines a relation ρ of all pairs (x, i) where i is an ordinal and $x \in \mathcal{K}^{(i)}$; the domain of ρ is then the union.

is the class of all diagrams in $\mathcal{K}^{(\gamma)}$ with a concrete limit;

$\mathcal{K}^{(\gamma+3)}$ is the $\mathcal{D}^{(\gamma+3)}$ -universal cocompletion of $\mathcal{K}^{(\gamma+2)}$, where $\mathcal{D}^{(\gamma+3)}$ is the class of all diagrams in $\mathcal{K}^{(\gamma+1)}$ with a concrete colimit,

etc...

Then the concrete category $\mathcal{K}^* = \bigcup_{i \in \text{Ord}} \mathcal{K}^{(i)}$ is concretely bicomplete and it has $\mathcal{K}$ as its full, concrete subcategory, closed by concrete limits and concrete colimits. All this easily follows from the fact that every concrete category is closed to concrete $\mathcal{D}$ -limits as well as concrete (in fact, all) colimits in its $\mathcal{D}$ -universal completion; analogously for cocompletions. And every diagram in $\mathcal{K}^*$, being small, it lies in some $\mathcal{K}^{(i)}$ and so it has a concrete limit and a concrete colimit in $\mathcal{K}^{(i+1)}$.

What remains to verify is the universality. Let $\mathcal{Q}$ be a concretely bicomplete category and let $F: \mathcal{K} \rightarrow \mathcal{Q}$ be a concrete functor, preserving concrete limits and colimits. Then F can be uniquely extended into a concrete functor $F^{(1)}: \mathcal{K}^{(1)} \rightarrow \mathcal{Q}$, and $F^{(1)}$ preserves concrete colimits of diagrams in $\mathcal{K}$ (i.e., of $\mathcal{D}^{(2)}$ -diagrams), hence it has a unique cocontinuous concrete extension $F^{(2)}: \mathcal{K}^{(2)} \rightarrow \mathcal{Q}$, preserving concrete limits of diagrams in $\mathcal{K}^{(1)}$ (i.e., of $\mathcal{D}^{(3)}$ -diagrams), etc... Given functors $F^{(i)}$ for all $i < \gamma$, where γ is a limit ordinal, then their joint extension to $\mathcal{K}^{(\gamma)} = \bigcup_{i < \gamma} \mathcal{K}^{(i)}$ preserves concrete colimits and concrete limits of the diagrams lying in some $\mathcal{K}^{(i_0)}$ (e.g., of $\mathcal{D}^{(\gamma)}$ -diagrams). Then there is a unique continuous concrete extension to $F^{(\gamma)}: \mathcal{K}^{(\gamma)} \rightarrow \mathcal{Q}$. Etc. This concludes the proof.

11. REMARK. A closely related problem to concrete completions is that of initial completions. Let $\mathcal{C}$ be a conglomerate of cones in $\mathcal{X}$, i.e., of (possibly large) collections

$$\langle f_i: X \rightarrow X_i \mid i \in I \rangle$$

of maps with a joint domain. A concrete category $\mathcal{K}$ is *initially $\mathcal{C}$ -complete* if for each cone $\langle f_i: X \rightarrow X_i \rangle$ in $\mathcal{C}$ and each collection $\{A_i\}$ of objects of $\mathcal{K}$ with $X_i = |A_i|$ there exists an initial lift (see Introduction). A concrete functor $F: \mathcal{K} \rightarrow \mathcal{Q}$ *preserves $\mathcal{C}$ -initial lifts* if, given an initial lift A of a cone $\langle f_i: X \rightarrow |A_i| \rangle$ in $\mathcal{C}$, then FA is an initial lift of the cone

$\langle f_i: X \rightarrow |FA_i| \rangle$ in $\mathcal{L}$.

A *universal initial $\mathcal{C}$ -completion* of a concrete category $\mathcal{K}$ is its full, initially $\mathcal{C}$ -complete extension $\mathcal{K}^*$, in which $\mathcal{K}$ is closed to $\mathcal{C}$ -initial lifts (i.e., the embedding $\mathcal{K} \rightarrow \mathcal{K}^*$ preserves $\mathcal{C}$ -initial lifts) and which has the obvious universal property with respect to functors preserving $\mathcal{C}$ -initial lifts. The existence of a universal initial completion is investigated in [1] for $\mathcal{C}$ = all cones in $\mathcal{X}$: a possibly non-legitimate concrete category $\hat{\mathcal{K}}$ is constructed such that either $\hat{\mathcal{K}}$ is legitimate and then it is the universal initial completion, or $\hat{\mathcal{K}}$ fails to be legitimate, in which case the universal completion fails to exist.

In case $\mathcal{C}$ is a class of *small* cones in $\mathcal{X}$, the universal initial $\mathcal{C}$ -completion always exists: we have $\mathcal{K} = \mathcal{K}_0$ as a subcategory of the (possibly non-legitimate) category $\hat{\mathcal{K}}$ and we denote by

$\mathcal{K}_1$ the closure of $\mathcal{K}_0$ for initial lifts of $\mathcal{C}$ -sources in $\hat{\mathcal{K}}$,

$\mathcal{K}_2$ the closure of $\mathcal{K}_1$, etc...

$\mathcal{K}_\omega = \bigcup_{i < \omega} \mathcal{K}_i$,

$\mathcal{K}_{\omega+1}$ the closure of $\mathcal{K}_\omega$ for initial lifts of $\mathcal{C}$ -sources, etc...

Then the category $\mathcal{K}^* = \bigcup_{i \in \text{Ord}} \mathcal{K}_i$ is always legitimate and it is evidently the universal $\mathcal{C}$ -initial completion of $\mathcal{K}$.

Starting with $\mathcal{C}$ = all limiting cones for diagrams in $\mathcal{X}$, we obtain the universal concrete completions. But in this way we cannot verify that a concrete category is codense in its universal concrete completion. This is why we had to prove our theorem in a much more complicated manner.

The proof of the Main Theorem above can also be modified for this situation of initial $\mathcal{C}$ -completion but, again, an iteration would be used generally. This would lose the codensity, but not the closedness for colimits.

12. EXAMPLE. Let $\mathcal{X}$ be a finitely productive base-category. For each concrete category $\mathcal{K}$ there exists a universal CFP-extension $\mathcal{K}^*$. This is a CFP-category (= concrete category with Concrete Finite Products) in which $\mathcal{K}$ is a full CFP-subcategory such that, given a CFP-category $\mathcal{L}$, then each CFP-functor $F: \mathcal{K} \rightarrow \mathcal{L}$ (= concrete functor preserving concrete

finite products) has a «unique» CFP-extension $F^*: \mathcal{K}^* \rightarrow \mathcal{L}$. Proof: let $\mathcal{C}$ be the class of all limiting cones for finite discrete diagrams, then a universal $\mathcal{C}$ -initial completion is precisely a universal CFP-extension.

13. REMARK. In a subsequent paper [2] on cartesian closed extensions we shall need a generalization of the previous example: Given a concrete category $\mathcal{K}$ and a class $\mathcal{D}$ of finite collections of its objects, there exists a $\mathcal{D}$ -universal CFP-extension of $\mathcal{K}$. (This is a CFP-category $\mathcal{K}^*$, in which $\mathcal{K}$ is closed to concrete products of $\mathcal{D}$ -collections, which has the obvious universal property.) The proof of this statement is an easy modification of the proof of the Main Theorem above: the objects of $\mathcal{K}^*$ will be the objects of $\mathcal{K}$ and objects P^D , where D is a finite collection of objects of $\mathcal{K}$ with $D \notin \mathcal{D}$; morphisms are defined transfinitely in a natural way.

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Faculty of Electrical Engineering
 Technical University Prague
 Suchbátarova 2,
 166 27 PRAGUE 6

and

Faculty of Mathematics and Physics
 Charles University Prague
 Malostranské nám. 25
 PRAGUE 1, TCHECOSLOVAQUIE