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3^{e} COLLOQUE SUR LES CATEGORIES DEDIE A CHARLES EHRESMANN
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RESTORATION OF STRUCTURE *

by Peter HILTON and Joseph ROITBERG

INTRODUCTION

The contributions of Charles Ehresmann in the domains of differential geometry and topology are so vast and deep that they will stand for a long time as a lasting testimony to our indebtedness to his great mathematical talent. Let us only instance his pioneering work on foliation theory in the early 1940's, which laid the foundation of what is today one of the most active areas of differential topology; his fundamental work pointing out the crucial importance of fibre bundles; and his work on Lie groups where he, Pontryagin and Richard Brauer independently calculated the rational homology of the classical groups of the four principal types.

When Charles Ehresmann turned to category theory, it was with the intention always of providing an appropriate analytical framework for the pursuit of his basic interests in geometry; he did not, we believe, ever lose sight of that fundamental purpose. He wished to abstract the *structure* present in a given concrete mathematical situation and study it removed from its adventitious context (see [0]).

In view of the emphasis which Charles Ehresmann placed on structure, we have thought it appropriate that this article dedicated to his memory should be concerned with the role of structure. We have taken two situations, one in group theory, the other in homotopy theory, where a natural construction leads to the loss of group structure. As a result, a property known to be possessed by group structures is thrown into jeopardy. We show that the property continues to be enjoyed by restoring the group structure.

* Talk delivered by Peter Hilton at the Amiens Category Theory Colloquium in honor of Charles Ehresmann, July 12, 1980.

This we do by enriching the original structure so that the construction, while involving a loss of structure, no longer destroys the group-theoretical basis of our arguments.

Let us sketch the first application, to group theory - the second application is, conceptually, of the same kind, though the details are, of course, very different. If the group Q acts on the group N, then we may construct the orbit set $N\,|\,Q$; however, the passage from N to $N\,|\,Q$ involves a loss of group structure. Now, if we put on suitable nilpotency hypotheses, we know that localization induces an injection

$$N \to \check{N}, \quad \text{where} \quad \check{N} = \Pi_p N_p\,.$$

We would like to prove similarly that localization induces an injection $N\,|\,Q \to \check{N}\,|\,\check{Q}$, but we have lost group structure, so the conclusion is not readily available. We enrich N by regarding it as an extension of N/Γ by Γ where $\Gamma = \Gamma_Q^{c-1}N$, given that $nil_Q N = c$. Then $N\,|\,Q$ is shown to be representable as the disjoint union of homomorphic images of Γ. We have thus restored group structure to $N\,|\,Q$ and may use our group-theoretical results to justify the conclusion that $N\,|\,Q \to \check{N}\,|\,\check{Q}$ is, indeed, injective.

The technique we use in Section 1 is based on an idea due to Steiner [5]. We are also happy to record that the exact sequences derived in Sections 1 and 2 have been generalised by Heath and Kamps [1] using groupoid techniques. Indeed, Klaus Heiner Kamps talked about this in his contribution to this colloquium.

1. GROUPS WITH OPERATORS

Let the group Q operate on the group N and let $N\,|\,Q$ stand for the set of orbits under the Q-action. In the passage from N to $N\,|\,Q$ we have lost group structure. To regain it we must enrich the structure of N.

In the first instance we replace N by a short exact sequence of Q-groups

$$(1.1) \qquad \Gamma \rightarrowtail N \overset{k}{\twoheadrightarrow} M$$

on which Q acts. We make the two simplifying assumptions:

(1.2) *Γ is commutative and Q acts initially on Γ.*

Choose an element $a \epsilon N$ and let

$$Q(a) = \{ x \epsilon Q \mid xa = a \}.$$

Define $Q(ka)$ similarly. Let $[a]$ be the orbit of a; similarly $[ka]$.

THEOREM 1.1 (see Theorem 2.5 of [4]). *There is an exact 5-term sequence*

$$\Sigma(a): \qquad Q(a) \rightarrowtail Q(ka) \xrightarrow{\partial} \Gamma \xrightarrow{\rho} N \mid Q \xrightarrow{k_*} M \mid Q$$

where (i) $\partial x = a(xa)^{-1}$, $x \epsilon Q(ka)$; $\rho b = [ba]$, $b \epsilon \Gamma$,

(ii) the sequence is exact at Γ in the sense that

$$\rho b_1 = \rho b_2 \iff \exists x \epsilon Q(ka) \text{ with } b_1 = b_2 \partial x,$$

(iii) the sequence is exact at $N \mid Q$ in the sense that

$$k_*^{-1} [ka] = \rho\Gamma .$$

In the light of property (ii) we see that $\rho\Gamma$ may be given a unique group structure such that $\rho : \Gamma \twoheadrightarrow \rho\Gamma$ is a homomorphism. Moreover, Γ acts on $N \mid Q$ by the rule

(1.3) $$b.[a] = [ba] .$$

Let us write ρ_a for ρ to emphasize the dependence on a. A *selection S* of elements of N is obtained by first taking a set of elements of $N \mid Q$, one for each Γ-orbit under the action (1.3), and then taking, within each selected element of $N \mid Q$, one representative element of N. It then follows from Theorem 1.1 and (1.3) that

THEOREM 1.2. *We may represent $N \mid Q$ as the disjoint union*

$$N \mid Q = \coprod_{a \epsilon S} \rho_a \Gamma$$

of commutative groups, each a homomorphic image of Γ.

Thus group structure is restored to $N \mid Q$. The following special case is of interest. Suppose that Q acts *nilpotently* on N so that $nil_Q N = d$ and let $\Gamma = \Gamma_Q^{d-1} N$ (see [3]). Then (1.2) is satisfied (Γ is, in fact, central in N) and $nil_Q M = d-1$. Thus if the action of Q is nilpotent of class d, then the orbit set $N \mid Q$ may be represented as a disjoint union of homomorphic images of $\Gamma_Q^{d-1}N$.

Let us demonstrate how this may be used to prove results on orbit sets. Let P be a family of primes and let N_P be the P-localization of N. Assume further that Q is also nilpotent so that the nilpotent action of Q on N induces a nilpotent action of Q_P on N_P. Since

$$(\Gamma_Q^i N)_P = \Gamma_{Q_P}^i N_P ,$$

it follows that for each $a \in N$, we have a map of exact sequences

$$(1.4) \qquad \begin{array}{ccccccccc} Q(a) & \rightarrowtail & Q(ka) & \xrightarrow{\partial} & \Gamma & \xrightarrow{\rho} & N \mid Q & \xrightarrow{k*} & M \mid Q \\ \downarrow{\scriptstyle e_1} & & \downarrow{\scriptstyle e_2} & & \downarrow{\scriptstyle e_3} & & \downarrow{\scriptstyle e_4} & & \downarrow{\scriptstyle e_5} \\ Q_P(ea) & \rightarrowtail & Q_P(eka) & \xrightarrow{\partial_P} & \Gamma_P & \xrightarrow{\rho_P} & N_P \mid Q_P & \xrightarrow{k_{P*}} & M_P \mid Q_P \end{array}$$

induced by

$$\begin{array}{ccccccc} \Gamma & \rightarrowtail & N & \twoheadrightarrow & M & \qquad & Q \\ \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle e} \\ \Gamma_P & \rightarrowtail & N_P & \twoheadrightarrow & M_P & & Q_P \end{array}$$

e = localization.

It follows easily by induction on d that e_1, e_2 are localization (since e_3 is localization and localization is exact); and hence that e_3 induces $e_0 : \rho\Gamma \to \rho_P \Gamma_P$ which is also localization. We may thus prove a *Hasse principle* for localization of orbit-sets.

THEOREM 1.3. *The function* $e_* : N \mid Q \to \Pi_p N_p \mid Q_p$ *is injective.*

PROOF. We know that, for any nilpotent group N, $e_*: N \to \Pi_p N_p$ is injective. Thus we prove Theorem 1.3 by *restoring group structure.* Specifically we argue by induction on d. If $d = 1$ we simply use the group-theoretical result. If the assertion is true for $(d-1)$, then $e_*: M \mid Q \to \Pi_p M_p \mid Q_p$ is injective. Thus if

$$[a],\ [a'] \in N \mid Q \quad \text{with} \quad e_*[a] = e_*[a']$$

then it follows from looking at the right-hand square of (1.4) that $k_*[a] = k_*[a']$, so that, by Theorem 1.1 (iii), $[a'] \in \rho_a \Gamma$. We thus have two elements $[a]$, $[a']$ in $\rho\Gamma$ whose e_*-images, in $\Pi_p \rho_p \Gamma_p$, coincide. It follows, again from the group-theoretical result, that $[a] = [a']$.

A similar argument may be used to prove a Hasse principle for profinite completion (see [3]).

2. FREE HOMOTOPY THEORY

We consider maps $f\colon W \to X$ from a finite [1] CW-complex W to a nilpotent space X. Let WX be the function space of such free maps and let (W, X) be the set of free homotopy classes. If X is a loop-space, then (W, X) has a natural group structure. In general, in computing (W, X), we may assume that X is the limit of a finite tower of principal fibrations whose fibers are Eilenberg-MacLane spaces. Thus the role of the sequence $\Gamma \to N \twoheadrightarrow M$ in Section 1 is here played by a principal fibration

$$K(G, n) \longrightarrow X \xrightarrow{\ q\ } Y \tag{2.1}$$

and we argue by induction on the height of the (refined) principal Postnikov tower. Let

$$f\colon W \to X, \quad \bar{f} = qf\colon W \to Y.$$

There is then an exact sequence, for each such f (with $\rho = \rho_f$)

$$\ldots \to \pi_1(WX, f) \to \pi_1(WY, \bar{f}) \xrightarrow{\delta} H^n(W, G) \xrightarrow{\rho} (W, X) \xrightarrow{q_*} (W, Y) \to \ldots \tag{2.2}$$

Moreover, $H^n(W, G)$ acts on (W, X) by the rule

$$a\{f\} = \rho_f(a), \quad a \in H^n(W, G), \tag{2.3}$$

where $\{f\}$ is the (free) homotopy class of f. The exactness of the sequence at $H^n(W, G)$ asserts that

$$\rho(a_1) = \rho(a_2) \iff \exists\, \xi \in \pi_1(WY, \bar{f}) \text{ with } a_1 = a_2(\delta\xi)\ ;$$

and the exactness at (W, X) asserts that

$$q_*^{-1}\{\bar{f}\} = \rho H^n(W, G).$$

A *selection* S of elements of WX is obtained by first taking one element of (W, X) from each $H^n(W, G)$-orbit, and then taking one map f from each

1) It would suffice that W be homologically finite.

selected homotopy classes. We then have

THEOREM 2.1. *We may represent (W, X) as the disjoint union*

$$(W, X) = \coprod_{f \in S} \rho_f H^n(W, G)$$

of commutative groups, each a homomorphic image of $H^n(W, G)$.

Thus we have restored group structure to (W, X) by enriching the structure of X by means of the principal fibration (2.1).

We will not go into the same detail here as we did in the previous section to show how this restoration of group structure may be exploited, but we will state one interesting consequence.

Let X be a nilpotent space of finite type. Then, following Sullivan [6], we may associate with X its *completion* $c: X \to \hat{X}$. We may characterize $\hat{X}$ as the inverse limit of all nilpotent spaces Z with finite homotopy groups which «approximate» X, in the sense that there are given maps $u: X \to Z$; indeed $\hat{X}$ is, more strictly, the limit $\hat{X} = \varprojlim_u Z$. Then we may infer from Theorem 2.1

COROLLARY 2.2. *The map c induces an injection $c_*: (W, X) \to (W, \hat{X})$.*

We note that, for any nilpotent group G of finite type, $c: G \to \hat{G}$ is injective, since G is residually finite. Thus Corollary 2.2 is proved by restoring group structure to (W, X) in accordance with Theorem 2.1.

Finally we remark that we may argue similarly using based homotopy instead of free homotopy; the result is, in the based case (with W connected), due to Sullivan. We may also use the technique of restoring group structure to prove a similar result involving localization instead of completion (and referring to free or based homotopy).

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