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EXTENSIONS OF FACTORIZATION SYSTEMS

by Walter THOLEN and Harvey WOLFF

In this paper we consider the following diagram of categories and functors

$$(*)\qquad \begin{tikzcd} \mathfrak{A} \arrow[r, "G"] \arrow[d, "P"'] \arrow[dr, phantom, "\gamma\Downarrow"] & \mathfrak{B} \arrow[d, "Q"] \\ \mathfrak{X} \arrow[r, "U"'] & \mathfrak{Y} \end{tikzcd}$$

where $\gamma: QG \to UP$ is a natural transformation. Such situations occur quite often, for if $G: \mathfrak{A} \to \mathfrak{B}$ is a functor with a left adjoint L and back adjunction $\epsilon: LG \to 1$ then for any pair of functors U, P we always have the following diagram

$$\begin{tikzcd} \mathfrak{A} \arrow[r, "G"] \arrow[d, "P"'] \arrow[dr, phantom, "\Downarrow UP\epsilon"] & \mathfrak{B} \arrow[d, "UPL"] \\ \mathfrak{X} \arrow[r, "U"'] & \mathfrak{Y} \end{tikzcd}$$

The ordinary extension situation occurs for G and U being embeddings of full subcategories.

We are concerned in $(*)$ in the problem of when factorizations of P-sources can be extended to factorizations of Q-sources of the same type. Our first result is that, under suitable conditions, Q-sources factor in a nice way iff Q-maps factor appropriately (Theorem 1). We then consider the above situation $(*)$ where G and U both have left adjoints. In this adjoint situation we give conditions under which P having a left adjoint implies Q has a left adjoint (cf. Theorem 2). This complements the results in [7] where we dealt with the problem of when adjointness of Q implies adjointness of P. Finally, in the adjoint situation, we prove a sharp version of Theorem 1 (cf. Theorem 3).

In the last section of the paper we discuss a few applications. First we investigate the behavior of the restriction of a functor $P: \mathfrak{A} \to \mathfrak{X}$ to a coreflective subcategory $\mathfrak{B}$ of $\mathfrak{A}$ (cf. Theorem 4). We thereby generalize a result due to Nel [6] on coreflective subcategories of initially structured categories. We then derive a characterization of topological functors due to Hoffmann [5] from Theorem 3 as an easy corollary. Finally we state a sharp version of the Special Adjoint Theorem as a corollary of Theorem 2.

1. THE GENERAL EXTENSION THEOREM

In this section we wish to prove a general theorem about extending factorization structures. Before we do this, we first give some terminology and some basic assumptions which we will use throughout the remainder of the paper.

Let $P: \mathfrak{A} \to \mathfrak{X}$ be a functor, $\mathfrak{E}$ a class of P-maps (i.e., $\mathfrak{X}$-morphisms of type $X \to PA$ with $A \in \mathfrak{A}$), and $\mathfrak{M}$ a class of sources (= discrete cones) in $\mathfrak{A}$.

A *factorization of a P-source* $(x_i: X \to PA_i)_I$ is a pair

$$(e: X \to PA, (m_i: A \to A_i)_I)$$

consisting of a P-map e and a source $(m_i)_I$ in $\mathfrak{A}$ with $Pm_i . e = x_i$ for all $i \in I$. This factorization is *over* $\mathfrak{E}$ if $e \in \mathfrak{E}$, *over* $\mathfrak{M}$ if $(m_i)_I \in \mathfrak{M}$, and *over* $(\mathfrak{E}, \mathfrak{M})$ if both $e \in \mathfrak{E}$ and $(m_i)_I \in \mathfrak{M}$. One says that *P-sources factor over* $\mathfrak{E}$ (over $\mathfrak{M}$, $(\mathfrak{E}, \mathfrak{M})$ resp.) if every P-source admits a factorization over $\mathfrak{E}$ (over $\mathfrak{M}$, $(\mathfrak{E}, \mathfrak{M})$ resp.).

A factorization $(e, (m_i)_I)$ of a P-source is *locally orthogonal with respect to* $\mathfrak{E}$ if for all commutative squares

$$\begin{tikzcd} Z \arrow[rr, "q"] \arrow[d, "z"'] & & PD \arrow[dll, dashed, "Pt"'] \arrow[d, "Pd_i"] \\ X \arrow[r, "e"'] & PA \arrow[r, "Pm_i"'] & PA_i \end{tikzcd}$$

with $q \in \mathfrak{E}$ there is a unique $t: D \to A$ with

$$Pt . q = e . z \quad \text{and} \quad m_i . t = d_i \quad \text{for all } i \in I.$$

The factorization is *orthogonal with respect to* $\mathfrak{E}$ if the factorization $(1_{PA}, (m_i)_I)$ is locally orthogonal with respect to $\mathfrak{E}$. We shall write: $\mathfrak{E} \perp \mathfrak{M}$ if every factorization over $\mathfrak{M}$ is orthogonal with respect to $\mathfrak{E}$. Finally, P-sources *factor (locally) orthogonally over* $\mathfrak{E}$ (over $(\mathfrak{E}, \mathfrak{M})$) if they factor over $\mathfrak{E}$ (over $(\mathfrak{E}, \mathfrak{M})$) such that the factorizations are (locally) orthogonal with respect to $\mathfrak{E}$..

Analogous phrases will be used for P-maps as well as for P-sources.

REMARKS. 1. In what follows we often only need weak locally orthogonal factorizations, i.e., the dotted t in the above diagram is not necessarily unique. However, one can prove that if all P-sources factor weakly locally orthogonally over $\mathfrak{E}$ then $\mathfrak{E}$ consists of P-epimorphisms (cf. [8], 6.4 and [1], Lemma 1), hence the factorizations are automatically locally orthogonal.

2. P-sources factor orthogonally over $\mathfrak{E}$ iff they factor locally orthogonally over $\mathfrak{E}$ with $\mathfrak{E}$ being closed under composition)cf. [8], 7.3 and [1], Lemma 3).

A generalized pullback (GP) is a class of commutative diagrams

$$\begin{array}{ccc} D & \xrightarrow{\quad d \quad} & B \\ {\scriptstyle d_i}\downarrow & & \downarrow{\scriptstyle b_i} \\ C_i & \xrightarrow[\quad c_i \quad]{} & B_i \end{array}$$

with the usual universal property: given $f: E \to B$ and $(g_i: E \to C_i)_I$ with $c_i . g_i = b_i . f$ for all i then there is a unique $g: E \to D$ with

$$d.g = f \quad \text{and} \quad d_i . g = g_i \quad \text{for all } i.$$

It can be constructed by forming (pointwise for all i) the pullbacks

$$\begin{array}{ccc} D & \xrightarrow{\quad c_i' \quad} & B \\ {\scriptstyle b_i'}\downarrow & & \downarrow{\scriptstyle b_i} \\ C_i & \xrightarrow[\quad c_i \quad]{} & B_i \end{array}$$

and then the multiple pullback of the c_i''s. So generalized pullbacks exist if ordinary and multiple pullbacks exist.

Throughout Sections 1 and 2 we shall be concerned with the following diagram of categories and functors

$$(*)\qquad \begin{tikzcd} \mathfrak{A} \arrow[r, "G"] \arrow[d, "P"'] \arrow[dr, phantom, "\gamma \Downarrow"] & \mathfrak{B} \arrow[d, "Q"] \\ \mathfrak{X} \arrow[r, "U"'] & \mathfrak{Y} \end{tikzcd}$$

where $\gamma: QG \to UP$ *is a natural transformation. We further assume that there are given classes*

Σ *of maps in* $\mathfrak{B}$,

$\mathfrak{E}$ *of* P*-maps,* $\mathfrak{M}$ *of sources in* $\mathfrak{A}$,

$\mathfrak{F}$ *of* Q*-maps,* $\mathfrak{N}$ *of sources in* $\mathfrak{B}$

which are, as usual, assumed to be closed under composition with isomorphisms. Moreover, $\mathfrak{N}$ *is assumed to be closed under composition, i. e., if* $(n_i: B \to B_i)_I$ *and* $n: A \to B$ *are in* $\mathfrak{N}$ *then* $(n_i . n: A \to B_i)_I$ *is in* $\mathfrak{N}$.

We shall be concerned with the following conditions on the diagram $(*)$:

A. γ *is* Σ*-bounded*, i.e., for every $Y \in \mathfrak{Y}$ there is a U-map $u: Y \to UX$ such that for every Q-map $y: Y \to QB$ there are a P-map $x: X \to PA$ and a map $s: B \to GA$ in Σ so that the following diagram commutes :

$$\begin{tikzcd} Y \arrow[rr, "u"] \arrow[d, "y"'] & & UX \arrow[d, "Ux"] \\ QB \arrow[r, "Qs"'] & QGA \arrow[r, "\gamma A"'] & UPA \end{tikzcd}$$

B. For all $(m_i: A \to A_i)_I$ in $\mathfrak{M}$ the diagrams

$$\begin{tikzcd} QGA \arrow[r, "\gamma A"] \arrow[d, "QGm_i"'] & UPA \arrow[d, "UPm_i"] \\ QGA_i \arrow[r, "\gamma A_i"'] & UPA_i \end{tikzcd}$$

form a generalized pullback.

C. For all $(m_i: A \to A_i)_I$ in $\mathfrak{M}$ and $(s_i: B \to GA_i)_I$ with $s_i \in \Sigma$ for all $i \in I$, there exists the following generalized pullback with $(n_i)_I$ in $\mathfrak{N}$, which is preserved by Q.

$$\begin{array}{ccc} C & \xrightarrow{b} & GA \\ {\scriptstyle n_i}\downarrow & & \downarrow{\scriptstyle Gm_i} \\ B_i & \xrightarrow{s_i} & GA_i \end{array}$$

REMARKS. The above conditions are often trivial:

1. Condition A is automatic if U is weakly right adjoint and if G is weakly right adjoint with weak units in Σ.

2. Condition B is automatic for $\gamma = 1$, i.e , $QG = UP$.

3. Condition C is automatic if $\Sigma \subset Iso\, B$ and $G\mathfrak{M} \subset \mathfrak{N}$.

4. For $\mathfrak{N}$ being all sources, condition C holds if $\mathfrak{B}$ is Σ-*quasi-complete* (a and b below) and Q is Σ-*continuous* (c below), i.e.,

a) For all $s: C \to B$ in Σ and $f: D \to B$ there exists a pullback

$$\begin{array}{ccc} E & \xrightarrow{s'} & D \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{s} & B \end{array}$$

with $s' \in \Sigma$,

b) For all $(s_i: C_i \to B)_I$ with $s_i \in \Sigma$ for all $i \in I$ the multiple pullback $s: D \to B$ of $(s_i)_I$ exists,

c) Q preserves the limits of a and b.

5. If in 4-b the multiple pullback s is assumed to be in Σ, conditions a and b mean Σ-completeness as defined in [2]; Σ-completeness can be equivalently described by the property that, in $\mathfrak{B}^{op}$, all sources factor over Σ, and the factorizations are locally orthogonal with respect to Σ (cf. [8], 6.3).

THEOREM 1. *Assume that conditions A, B, C hold in diagram* (*). *If P-sources factor over* $\mathfrak{M}$, *then Q-sources factor over* $(\mathfrak{F}, \mathfrak{N})$ *iff Q-maps do. If moreover* $\mathfrak{F} \perp G\mathfrak{M}$, *then the factorizations of Q-sources are (locally) orthogonal with respect to* $\mathfrak{F}$ *iff the factorizations of Q-maps are.*

PROOF. Let $(y_i: Y \to QB_i)_I$ be a Q-source. For each $i \in I$ we have the following commutative diagram with $s_i \in \Sigma$. Since P-sources factor over $\mathfrak{M}$, the source $(x_i)_I$ factors as

$$\begin{tikzcd}
Y \arrow[rr, "u"] \arrow[d, "y_i"'] & & UX \arrow[d, "Ux_i"] \\
QB_i \arrow[r, "Qs_i"'] & QGA_i \arrow[r, "\gamma A_i"'] & UPA_i
\end{tikzcd}$$

$(e: X \to PA, (m_i: A \to A_i)_I)$ with $(m_i)_I \in \mathfrak{M}$.

Successively we get the two GP's described in Conditions B and C. Since

$$UPm_i . Ue . u = \gamma A_i . Q s_i . y_i$$

there is a (unique)

$$t: Y \to QGA \quad \text{with } \gamma A . t = Ue . u \text{ and } QGm_i . t = Qs_i . y.$$

Since

$$\begin{tikzcd}
QC \arrow[r, "Qb"] \arrow[d, "Qn_i"'] & QGA \arrow[d, "QGm_i"] \\
QB_i \arrow[r, "Qs_i"'] & QGA_i
\end{tikzcd}$$

is a GP there is a (unique) $y: Y \to QC$ with

$$Qb . y = t \text{ and } Qn_i . y = y_i \text{ for all } i.$$

Finally, since Q-maps have $(\mathfrak{F}, \mathfrak{N})$-factorizations we get

$$y = Qn . j \text{ with } j: Y \to QB \text{ in } \mathfrak{F} \text{ and } n: B \to C \text{ in } \mathfrak{N}.$$

Therefore $(j, (n_i . n)_I)$ is the desired $(\mathfrak{F}, \mathfrak{N})$-factorization of $(y_i)_I$.

Now assume that the factorizations of Q-maps are locally orthogonal with respect to $\mathfrak{F}$, $\mathfrak{F} \perp G\mathfrak{M}$, and let $h \in \mathfrak{F}$. Consider the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[rrr, "h"] \arrow[d, "z"'] & & & QD \arrow[d, "Qd_i"] \\
Y \arrow[r, "j"] \arrow[rrd, "t"'] & QB \arrow[r, "Qn"] & QC \arrow[r, "Qn_i"] \arrow[d, "Qb"] & QB_i \arrow[d, "Qs_i"] \\
& & QGA \arrow[r, "QGm_i"] & QGA_i
\end{tikzcd}$$

Since $\mathfrak{F} \perp G\mathfrak{M}$ there exists a unique $d: D \to GA$ with

$$Qd.h = t.z \quad \text{and} \quad Gm_i.d = s_i.d_i \quad \text{for all } i \in I.$$

Because of C there exists a unique $c: C \to C$ with $b.c = d$ and $n_i.c = d_i$ for all $i \in I$. Hence $Qb.Qc.h = t.z$ and thus $Qc.h = Qn.j.z$. Since the factorizations of Q-maps are locally orthogonal, there is a unique

$$f: D \to B \quad \text{with} \quad Qf.h = j.z \quad \text{and} \quad n.f = c.$$

So $(n_i.n).f = d_i$ for all $i \in I$. The uniqueness of f follows from the uniqueness of the constructions involved.

For the non-local case the proof is similar.

REMARK. The first part of the above proof shows that it suffices to have *weak* generalized pullbacks in Conditions B and C. But the corresponding weak version of Theorem 1 is not used in the following.

2. THE ADJOINT CASE

In this section we consider the diagram $(*)$ *where both* U *and* G *have left adjoints. We assume throughout this section that* F *is left adjoint to* U *with unit* δ *and that* L *is left adjoint to* G *with unit* η.

For every Q-map $j: Y \to QB$ let $\bar{j}: FY \to PLB$ be the P-map which corresponds by adjointness of U to

$$Y \xrightarrow{\;j\;} QB \xrightarrow{\;Q\eta B\;} QGLB \xrightarrow{\;\gamma LB\;} UPLB.$$

One then has:

LEMMA 1. *1. If* η *is a pointwise monomorphism and* $\bar{j}$ *is a P-epimorphism, then* j *is a Q-epimorphism.*

2. If γ *is a pointwise monomorphism and* j *is a Q-epimorphism, then* $\bar{j}$ *is a P-epimorphism.*

PROOF. 1. Suppose $Qf.j = Qg.j$ where $f, g: B \to C$. Then we have the following diagram (cf. next page). We have

$$UPLg.U\bar{j}.\delta Y = UPLf.U\bar{j}.\delta Y.$$

Hence $PLg.\bar{j} = PLf.\bar{j}$. Consequently $Lf = Lg$. Since ηG is monic, we get $f = g$.

The proof of 2 is similar.

$$\begin{tikzcd}
Y \arrow[rr, "\delta Y"] \arrow[d, "j"'] & & UFY \arrow[d, "U\bar{j}"] \\
QB \arrow[r, "Q\eta B"] \arrow[d, shift right, "Qf"'] \arrow[d, shift left, "Qg"] & QGLB \arrow[r, "\gamma LB"] \arrow[d, shift right, "QGLf"'] \arrow[d, shift left, "QGLg"] & UPLB \arrow[d, shift right, "UPLf"'] \arrow[d, shift left, "UPLg"] \\
QC \arrow[r, "Q\eta C"] & QGLC \arrow[r, "\gamma LC"] & UPLC
\end{tikzcd}$$

Recall that under the assumptions of this section if η is pointwise in Σ then Condition A is automatic (cf. the remarks before Theorem 1). Choosing $\mathfrak{N}$ = all sources, we then have :

THEOREM 2. *Suppose that the unit η of G is a pointwise monomorphism in Σ and that P-sources factor over $(\mathfrak{S}, \mathfrak{M})$ for $\mathfrak{S}$ consisting of P-epimorphisms. If Conditions B and C hold, then Q has a left adjoint.*

PROOF. It suffices to show the source of *all* Q-maps $(y_i : Y \to QB_i)_I$ with domain Y factors over a Q-epimorphism. To this end we proceed as in the proof of Theorem 1 by factoring the corresponding source $(\bar{y}_i : FY \to PLB_i)$ as

$$(e : FY \to PA, (m_i : A \to LB_i)_I), \quad e \in \mathfrak{S} \text{ and } (m_i)_I \in \mathfrak{M}.$$

As in that proof we get a factorization as $(y : Y \to QC, (n_i : X \to B_i)_I)$ and a commuting diagram

$$\begin{tikzcd}
Y \arrow[rr, "\delta Y"] \arrow[d, "y"'] & & UFY \arrow[d, "Ue"] \\
QC \arrow[r, "Qb"] \arrow[d, "Qn_i"'] \arrow[dr, phantom, "1"] & QGA \arrow[r, "\gamma A"] \arrow[d, "QGm_i"] \arrow[dr, phantom, "2"] & UPA \arrow[d, "UPm_i"] \\
QB_i \arrow[r, "Q\eta B_i"] & QGLB_i \arrow[r, "\gamma A_i"] & UPLB_i
\end{tikzcd}$$

with diagrams 1 and 2 being GP's.

We now show that $y : Y \to QC$ is Q-epimorphic. By the Lemma it suffices to show that the corresponding $\bar{y}$ is P-epimorphic. First note that there exists a unique $d: LC \to A$ with $Gd.\eta C = b$. Also, since the original source consists of all Q-maps with domain Y, there is a $c: C \to C$ (namely one to the n_i's) with

$$Qc.y = y \quad \text{and} \quad \eta C.c = Gm.b,$$

for m being in $(m_i)_I$. Now

$$UPd.UPm.Ue.\delta Y = UPd.\gamma LC.Q\eta C.y =$$
$$= \gamma A.QGd.Q\eta C.y = \gamma A.Qb.y = Ue.\delta Y.$$

Hence $Pd.Pm.e = e$ and consequently $d.m = 1$. Now

$$b = Gd.Gm.b = Gd.\eta C.c = b.c.$$

Furthermore, for each $i \in I$,

$$\eta B_i.n_i.c = Gm_i.b.c = Gm_i.b = \eta B_i.n_i.$$

Since ηB_i is a monomorphism, we have $n_i.c = n_i$ for all $i \in I$. Consequently, since 1 is a GP, we get $c = 1$. So

$$Gm.Gd.\eta C = Gm.b = \eta C.c = \eta C.$$

Hence $m.d = 1$.

Because $Pd.\bar{y} = e$ we now have $\bar{y} \approx e$ which is P-epimorphic.

The next corollary generalizes Theorem 1.8 of [2]; this is gotten by taking $\gamma = 1$.

COROLLARY 1. *In* $(*)$, *let* $P = 1$ *and let condition B be satisfied with* $\mathfrak{M}$ = *all sources. Suppose that* $\mathfrak{B}$ *is* Σ-*complete and that the units of* G *are pointwise in* Σ. *Then* Q *has a left adjoint iff* Q *is* Σ-*continuous.*

COROLLARY 2. *For any right adjoint functor* $G: \mathfrak{A} \to \mathfrak{B}$ *with units in* Σ *and* $\mathfrak{B}$ *being* Σ-*complete one has: A functor* $Q: \mathfrak{B} \to \mathfrak{Y}$ *is right adjoint iff* QG *is right adjoint and* Q *is* Σ-*continuous.*

In the adjoint situation as described at the beginning of this section we take up again the question of when does Q admit orthogonal factorizations. We shall prove a sharpened version of Theorem 1 in which Condition B appears as a necessary condition. We first identify in our situation the maps orthogonal to $G\mathfrak{M}$ (cf. Theorem 1).

LEMMA 2. *Suppose that, in the situation of this section, Condition B holds. Then, for every Q-map $j: Y \to QB$, $\{j\} \perp G\mathfrak{M}$ iff $\{\bar{j}\} \perp \mathfrak{M}$.*

PROOF. Suppose $\{j\} \perp G\mathfrak{M}$ and consider the diagram

$$\begin{tikzcd} FY \arrow[r, "\bar{j}"] \arrow[d, "f"'] & PLB \arrow[d, "Pd_i"] \\ PA \arrow[r, "Pm_i"] & PA_i \end{tikzcd}$$

with $(m_i)_I \in \mathfrak{M}$. The source $(d_i: LB \to A_i)_I$ corresponds, by adjointness of G, to $(\tilde{d}_i: B \to GA_i)_I$, and $f: FY \to PA$ corresponds to $\tilde{f}: Y \to UPA$ by adjointness of U. We get the following diagram in $\mathcal{Y}$:

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[dd, "\tilde{f}"'] & QB \arrow[d, "Q\tilde{d}_i"] \\ & QGA_i \arrow[d, "\gamma A_i"] \\ UPA \arrow[r, "UPm_i"] & UPA_i \end{tikzcd}$$

By B there exists a unique $h: Y \to QGA$ with

$$\gamma A.h = \tilde{f} \quad \text{and} \quad QGm_i.h = Q\tilde{d}_i.j \quad \text{for all } i \in I.$$

Since $\{j\} \perp G\mathfrak{M}$ there is a unique $l: B \to GA$ with

$$Ql.j = h \quad \text{and} \quad Gm_i.l = \tilde{d}_i \quad \text{for all } i \in I.$$

Then l corresponds by adjunction to

$$\tilde{l}: LB \to A \quad \text{with} \quad P\tilde{l}.\bar{j} = f \quad \text{and} \quad m_i.\tilde{l} = d_i.$$

Uniqueness of l follows from the uniqueness of the constructions involved. We therefore have $\{\bar{j}\} \perp \mathfrak{M}$.

The converse assertion is proved similarly.

THEOREM 3. *Suppose that the unit of G belongs to Σ and that Condition C holds, with $\mathfrak{N}$ = all sources. Suppose further that P-sources factor (locally) orthogonally over $(\mathfrak{E}, \mathfrak{M})$, and that $\mathfrak{F} = \{ j \mid \bar{j} \in \mathfrak{E} \}$. Then, for* *the statements:*

(i) Q-sources factor (locally) orthogonally over $\mathfrak{F}$,

(ii) Q-maps factor (locally) orthogonally over $\mathfrak{F}$ and condition B holds one has (ii) $\Rightarrow$ (i), whereas (i) $\Rightarrow$ (ii) holds for $\mathcal{E} \perp \mathfrak{M}$.

PROOF. (ii) $\Rightarrow$ (i): The non-local case follows immediately from Theorem 1 and Lemma 2. For the local case we look to the second part of the proof of Theorem 1. We again assume the factorizations of Q-maps to be locally orthogonal with respect to $\mathfrak{F}$, but we cannot assume $\mathfrak{F} \perp G\mathfrak{M}$. Nevertheless in the situation of the last diagram of that proof, one gets also a unique

$$d: D \to GA \quad \text{with} \quad Qd.h = t.z \quad \text{and} \quad Gm_i.d = \eta B_i.d_i.$$

This is easily proved by taking $d = Gf.\eta D$, where $f: LD \to A$ is the unique diagonal of the commutative diagram:

$$\begin{tikzcd}
FZ \arrow[rr, "\bar{h}"] \arrow[d, "Fz"'] & & PLD \arrow[dl, "Pf"'] \arrow[d, "PLd_i"] \\
FY \arrow[r, "e"] & PA \arrow[r, "Pm_i"] & PLB_i
\end{tikzcd}$$

Now the proof can be completed as in Theorem 1.

(i) $\Rightarrow$ (ii): From Condition (i) we have that Q has a left adjoint S with unit π pointwise in $\mathfrak{F}$, because the source of all Q-maps factors over $\mathfrak{F}$, and $\mathfrak{F}$ necessarily consists of Q-epimorphisms only (see remarks at the beginning of Section 1). Now consider the following commutative diagram:

$$\begin{tikzcd}
Y \arrow[r, "g_i"] \arrow[d, "f"'] & QGA_i \arrow[d, "\gamma A_i"] \\
UPA \arrow[r, "UPm_i"] & UPA_i
\end{tikzcd}$$

with $(m_i: A \to A_i)_I$ in $\mathfrak{M}$. For each $i \in I$, there exists a unique

$$d_i: SY \to GA_i \quad \text{with} \quad Qd_i.\pi Y = g_i.$$

By adjunction of G and U, d_i corresponds to $\bar{d}_i: LSY \to A_i$ and f corresponds to $\bar{f}: FY \to PA$. Since $\overline{\pi Y} \in \mathcal{E} \perp \mathfrak{M}$ from the commutative diagram

$$\begin{tikzcd} FY \arrow[r, "\overline{\pi Y}"] \arrow[d, "\tilde{f}"'] & PLSY \arrow[d, "P\tilde{d}_i"] \\ PA \arrow[r, "Pm_i"] & PA_i \end{tikzcd}$$

we get a unique $t: LSY \to A$ with

$$Pt.\overline{\pi Y} = \tilde{f} \text{ and } m_i.t = \tilde{d}_i \text{ for all } i \in I.$$

Then t corresponds to

$$\bar{t}: SY \to GA \text{ with } Gm_i.\bar{t} = d_i.$$

So we get

$$Q\bar{t}.\pi Y: Y \to QGA \text{ with } QGm_i.Q\bar{t}.\pi Y = g_i \text{ for all } i \in I.$$

Since $\gamma A.Q\bar{t}.\pi Y$ corresponds by adjunction to $\tilde{f}$ we have $\gamma A.Q\bar{t}.\pi Y = f$.

If $h: Y \to QGA$ is a map with $\gamma A.h = f$ and $QGm_i.h = g_i$ for all $i \in I$, then we get a unique

$$l: SY \to GA \text{ with } Ql.\pi Y = h.$$

One sees that l corresponds to $\tilde{l}: LSY \to A$ with

$$P\tilde{l}.\overline{\pi Y} = \tilde{f} \text{ and } m_i.\tilde{l} = \tilde{d}_i.$$

Thus $\tilde{l} = t$ and so $l = \bar{t}$.

If the left adjoint of G is full and faithful, the unit of G is an isomorphism. Then Σ can be taken to be the class of all isomorphisms, and Condition C is automatic. So we get

COROLLARY 3. *Let P-sources factor (locally) orthogonally over $\mathcal{E}$. Assume that G has a full and faithful left adjoint. Then, for the statements*

(i) Q-sources factor (locally) orthogonally over $\mathcal{F} = \{\bar{j} \mid j \in \mathcal{E}\}$,

(ii) Condition B holds,

one has (ii) ⇒ (i), whereas (i) ⇒ (ii) holds in the non-local case.

PROOF. We need to verify that Q-maps factor (locally) orthogonally over $\mathcal{F}$, if B holds. Let $f: Y \to QB$ be a Q-map and let $\bar{f}$ factor as $\bar{f} = Pm.e$, where $e: FY \to PA$ in $\mathcal{E}$. Recalling that the unit η is an isomorphism, we get the following commutative diagram

$$\begin{tikzcd}
Y \arrow[rrr, "\delta Y"] \arrow[drr, "h"] \arrow[dd, "f"'] & & & UFY \arrow[d, "Ue"] \\
& & QGLA \arrow[r, "\gamma A"] \arrow[d, "QGm"'] & UPA \arrow[d, "UPm"] \\
QB & & QGLB \arrow[ll, "Q(\eta B)^{-1}"'] \arrow[r, "\gamma B"] & UPLB
\end{tikzcd}$$

For $\bar{h}: FY \to PLGA$ and the counit $\epsilon: LG \to 1$ of G one now has

$$UP\epsilon A.\, U\bar{h}.\, \delta Y = UP\epsilon A.\, \gamma LGA.Q\eta GA.h =$$
$$= \gamma A.QG\epsilon A.Q\eta\, GA.h = Ue.\delta Y.$$

Hence $e = P\epsilon A.\bar{h} \in \mathcal{E}$. Therefore, by factoring $\bar{h}$ over $\mathcal{E}$, one easily gets $\bar{h} \in \mathcal{E}$ and so $h \in \mathcal{F}$. Orthogonality of the factorization

$$f = Q((\eta B)^{-1}.\,Gm).h$$

follows by Lemma 2, whereas the local case is treated as in Theorem 2, (ii) ⇒ (i).

Note that for γ being an isomorphism, Condition B is automatic. Hence assertion (i) of Corollary 3 holds in this case.

3. APPLICATIONS.

In this section we give some applications of the above results. Many others can be added by specializing the data of $(*)$.

3.1. *Coreflective subcategories.* Let $P: \mathfrak{A} \to \mathfrak{X}$ be a functor and let $E: \mathfrak{B} \to \mathfrak{A}$ be the embedding of a full coreflective subcategory with coreflector R and coreflection ρ. Finally, let $\mathcal{F}$ be the class of PE-maps $e: X \to PEB$ such that $e: X \to P(EB)$ belongs to a given class $\mathcal{E}$ of P-maps. Applying Corollary 3 to the diagram

$$\begin{tikzcd}
\mathfrak{A} \arrow[r, "R"] \arrow[d, "P"'] \arrow[dr, phantom, "P\rho \Downarrow"] & \mathfrak{B} \arrow[d, "PE"] \\
\mathfrak{X} \arrow[r, "1_{\mathfrak{X}}"'] & \mathfrak{X}
\end{tikzcd}$$

we get:

THEOREM 4. *Let P-sources factor (locally) orthogonally over $(\mathfrak{S}, \mathfrak{M})$. Then PE-sources factor (locally) orthogonally over $\mathfrak{F}$ if the diagrams*

$$\begin{array}{ccc} PRA & \xrightarrow{P\rho A} & PA \\ {\scriptstyle PRm_i}\downarrow & & \downarrow{\scriptstyle Pm_i} \\ PRA_i & \xrightarrow{P\rho A_i} & PA_i \end{array}$$

form a GP in $\mathfrak{X}$ for each source $(m_i : A \to A_i)_I$ in $\mathfrak{M}$. This condition is necessary in the non-local case.

All the generalized pullbacks are trivial for $P\rho$ being an isomorphism. Therefore, considering the canonical factorization structures for P, by Theorem 4 we get immediately:

COROLLARY 4. *If $P : \mathfrak{A} \to \mathfrak{X}$ belongs to one of the following classes of functors (of which each is contained in the next one), so does every restriction of P to a full coreflective subcategory of $\mathfrak{A}$ such that the P-images of the coreflection maps are isomorphisms:*

topological functors (cf. [8]),
$(\mathfrak{S}, \mathfrak{M})$-topological functors (cf. [3]),
topologically-algebraic functors (cf. [1, 4]),
semitopological functors (cf. [8]),
right adjoint functors.

The assertion of Corollary 4 for $(\mathfrak{S}, \mathfrak{M})$-topological functors contains in particular Nel's corresponding result on «initially structured» categories (cf. [6], Theorem 1.3). For various applications we refer to his paper.

3.2. *Characterization of topological functors.* As a further consequence of Theorem 3 we obtain a characterization of topological functors due to Hoffmann [5]:

COROLLARY 5. *A functor $P : \mathfrak{A} \to \mathfrak{X}$ is topological iff $\mathfrak{A}$ is Σ-complete for $\Sigma = P^{-1}(iso\, \mathfrak{X})$ and P has a full and faithful right adjoint.*

PROOF. We only need to show that the condition is sufficient for topologicity. We apply Theorem 3 to the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "R"] \arrow[d, "1_{\mathfrak{X}}"'] \arrow[dr, phantom, "\epsilon \Downarrow"] & \mathfrak{A} \arrow[d, "P"] \\ \mathfrak{X} \arrow[r, "1_{\mathfrak{X}}"'] & \mathfrak{X} \end{tikzcd}$$

where R is the full and faithful right adjoint and ϵ the isomorphic counit. With $\mathfrak{E} = iso\,\mathfrak{X}$ one obtains for $\mathfrak{F}$ the class of all P-maps which are isomorphisms in $\mathfrak{X}$. By definition of Σ, P is trivially Σ-continuous, and the unit η of P belongs to Σ. In order to get orthogonal factorizations of P-sources over $\mathfrak{F}$ it suffices therefore to have those for P-maps. But given a P-map $x: X \to PB$ one obtains this factorization by considering the P-image of the pullback

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "f"'] & RX \arrow[d, "Rx"] \\ B \arrow[r, "\eta B"] & RPB \end{tikzcd}$$

which exists by Σ-completeness.

REMARK. The functor

$$P: \mathcal{C}at \to \mathcal{S}et, \quad K \mapsto Ob\,K,$$

has a full and faithful right adjoint, and $\mathcal{C}at$ is, of course, small Σ-complete with $\Sigma = P^{-1}(iso\,\mathcal{S}et)$, i.e., pullbacks and small-indexed intersections of Σ-maps exist and belong to Σ. Nevertheless, the non-faithful functor P is not topological. With respect to Corollary 5 the reason for this is that $\mathfrak{A}$ fails to be Σ-complete: For each cardinal k consider a category K_k having two objects 0, 1 and k arrows $0 \to 1$. Identifying these arrows one gets a family of functors $\mathrm{K}_k \to \{0 \to 1\}$ (indexed by all cardinals) which fails to admit an intersection.

3.3. *The Special Adjoint Functor Theorem.* We give a slight generalization of a theorem stated in [2] by application of Theorem 2 in the following situation. Let $Q: \mathfrak{B} \to \mathfrak{Y}$ be a functor whose right adjointness shall be proved. Let $\mathfrak{G}$ be a subset of the objects of $\mathfrak{B}$ such that all products

$$GX = \prod_{C \in \mathcal{G}} \prod_{X_C} C \quad \text{and} \quad UX = \prod_{C \in \mathcal{G}} \prod_{X_C} QC$$

exist in $\mathfrak{B}$ and $\mathfrak{Y}$ where $X = (X_C)_{\mathcal{G}}$ is any object in $(\mathcal{S}et^{\mathcal{G}})^{op} = \mathfrak{A}$. The functors $G: \mathfrak{A} \to \mathfrak{B}$ and $U: \mathfrak{A} \to \mathfrak{Y}$ have left adjoints given by

$$LB = (\mathfrak{B}(B, C))_{\mathcal{G}} \quad \text{and} \quad FY = (\mathfrak{Y}(Y, QC))_{\mathcal{G}}.$$

There is a natural transfomation $\gamma: QG \to U$ which is an isomorphism iff Q preserves the products GX.

COROLLARY 6. *Let the category $\mathfrak{B}$ be Σ-complete and let $\mathcal{G}$ be a Σ-cogenerating set in $\mathfrak{B}$ (i.e., the units $\eta B: B \to GLB$ belong to Σ). The functor $Q: \mathfrak{B} \to \mathfrak{Y}$ then has a left adjoint iff*

(1) Q est Σ-continous,

(2) there is a pair $(\mathcal{E}, \mathfrak{M})$ such that sources in $\mathfrak{A} = (\mathcal{S}et^{\mathcal{G}})^{op}$ factor over $(\mathcal{E}, \mathfrak{M})$ with $\mathcal{E} \subset Epi\, \mathfrak{A}$ and Condition B (depending on $\mathfrak{M}$ and γ) holds.

In particular condition (2) holds if Q preserves products. Therefore, for $\mathfrak{B}$ being complete and Σ-wellpowered, (1) and (2) are fulfilled for Q preserving all small limits; this is the usual version of the Special Adjoint Functor Theorem.

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