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EXTENSIONS OF FACTORIZATION SYSTEMS

by *Walter THOLEN and Harvey WOLFF*

In this paper we consider the following diagram of categories and functors

$$(*) \quad \begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{B} \\ P \downarrow & \gamma \Downarrow & \downarrow Q \\ \mathcal{X} & \xrightarrow{U} & \mathcal{Y} \end{array}$$

where $\gamma: QG \rightarrow UP$ is a natural transformation. Such situations occur quite often, for if $G: \mathcal{A} \rightarrow \mathcal{B}$ is a functor with a left adjoint L and back adjunction $\epsilon: LG \rightarrow 1$ then for any pair of functors U, P we always have the following diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{B} \\ P \downarrow & \Downarrow UP\epsilon & \downarrow UPL \\ \mathcal{X} & \xrightarrow{U} & \mathcal{Y} \end{array}$$

The ordinary extension situation occurs for G and U being embeddings of full subcategories.

We are concerned in (*) in the problem of when factorizations of P -sources can be extended to factorizations of Q -sources of the same type. Our first result is that, under suitable conditions, Q -sources factor in a nice way iff Q -maps factor appropriately (Theorem 1). We then consider the above situation (*) where G and U both have left adjoints. In this adjoint situation we give conditions under which P having a left adjoint implies Q has a left adjoint (cf. Theorem 2). This complements the results in [7] where we dealt with the problem of when adjointness of Q implies adjointness of P . Finally, in the adjoint situation, we prove a sharp version of Theorem 1 (cf. Theorem 3).

In the last section of the paper we discuss a few applications. First we investigate the behavior of the restriction of a functor $P: \mathfrak{A} \rightarrow \mathfrak{X}$ to a coreflective subcategory \mathfrak{B} of \mathfrak{A} (cf. Theorem 4). We thereby generalize a result due to Nel [6] on coreflective subcategories of initially structured categories. We then derive a characterization of topological functors due to Hoffmann [5] from Theorem 3 as an easy corollary. Finally we state a sharp version of the Special Adjoint Theorem as a corollary of Theorem 2.

1. THE GENERAL EXTENSION THEOREM

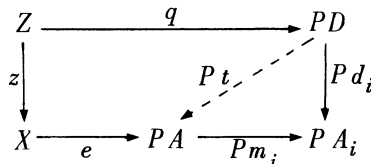
In this section we wish to prove a general theorem about extending factorization structures. Before we do this, we first give some terminology and some basic assumptions which we will use throughout the remainder of the paper.

Let $P: \mathfrak{A} \rightarrow \mathfrak{X}$ be a functor, \mathfrak{E} a class of P -maps (i.e., \mathfrak{X} -morphisms of type $X \rightarrow PA$ with $A \in \mathfrak{A}$), and \mathfrak{M} a class of sources (= discrete cones) in \mathfrak{A} .

A factorization of a P -source $(x_i: X \rightarrow PA_i)_I$ is a pair
 $(e: X \rightarrow PA, (m_i: A \rightarrow A_i)_I)$

consisting of a P -map e and a source $(m_i)_I$ in \mathfrak{A} with $Pm_i \cdot e = x_i$ for all $i \in I$. This factorization is *over* \mathfrak{E} if $e \in \mathfrak{E}$, *over* \mathfrak{M} if $(m_i)_I \in \mathfrak{M}$, and *over* $(\mathfrak{E}, \mathfrak{M})$ if both $e \in \mathfrak{E}$ and $(m_i)_I \in \mathfrak{M}$. One says that P -sources factor *over* \mathfrak{E} (*over* \mathfrak{M} , $(\mathfrak{E}, \mathfrak{M})$ resp.) if every P -source admits a factorization *over* \mathfrak{E} (*over* \mathfrak{M} , $(\mathfrak{E}, \mathfrak{M})$ resp.).

A factorization $(e, (m_i)_I)$ of a P -source is *locally orthogonal with respect to* \mathfrak{E} if for all commutative squares



with $q \in \mathfrak{E}$ there is a unique $t: D \rightarrow A$ with

$$Pt \cdot q = e \cdot z \quad \text{and} \quad m_i \cdot t = d_i \quad \text{for all } i \in I.$$

The factorization is *orthogonal with respect to* \mathcal{E} if the factorization $(I_{PA}, (m_i)_I)$ is locally orthogonal with respect to \mathcal{E} . We shall write: $\mathcal{E} \perp \mathfrak{M}$ if every factorization over \mathfrak{M} is orthogonal with respect to \mathcal{E} . Finally, P -sources *factor (locally) orthogonally over* \mathcal{E} (over $(\mathcal{E}, \mathfrak{M})$) if they factor over \mathcal{E} (over $(\mathcal{E}, \mathfrak{M})$) such that the factorizations are (locally) orthogonal with respect to \mathcal{E} .

Analogous phrases will be used for P -maps as well as for P -sources.

REMARKS. 1. In what follows we often only need weak locally orthogonal factorizations, i. e., the dotted t in the above diagram is not necessarily unique. However, one can prove that if all P -sources factor weakly locally orthogonally over \mathcal{E} then \mathcal{E} consists of P -epimorphisms (cf. [8], 6.4 and [1], Lemma 1), hence the factorizations are automatically locally orthogonal.

2. P -sources factor orthogonally over \mathcal{E} iff they factor locally orthogonally over \mathcal{E} with \mathcal{E} being closed under composition) cf. [8], 7.3 and [1], Lemma 3).

A *generalized pullback* (GP) is a class of commutative diagrams

$$\begin{array}{ccc} D & \xrightarrow{d} & B \\ d_i \downarrow & & \downarrow b_i \\ C_i & \xrightarrow{c_i} & B_i \end{array}$$

with the usual universal property: given $f: E \rightarrow B$ and $(g_i: E \rightarrow C_i)_I$ with $c_i \cdot g_i = b_i \cdot f$ for all i then there is a unique $g: E \rightarrow D$ with

$$d \cdot g = f \text{ and } d_i \cdot g = g_i \text{ for all } i.$$

It can be constructed by forming (pointwise for all i) the pullbacks

$$\begin{array}{ccc} D & \xrightarrow{c'_i} & B \\ b'_i \downarrow & & \downarrow b_i \\ C_i & \xrightarrow{c_i} & B_i \end{array}$$

and then the multiple pullback of the c'_i 's. So generalized pullbacks exist if ordinary and multiple pullbacks exist.

Throughout Sections 1 and 2 we shall be concerned with the following diagram of categories and functors

$$(*) \quad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{G} & \mathfrak{B} \\ P \downarrow & \gamma \Downarrow & \downarrow Q \\ \mathfrak{X} & \xrightarrow{U} & \mathfrak{Y} \end{array}$$

where $\gamma : QG \rightarrow UP$ is a natural transformation. We further assume that there are given classes

- Σ of maps in \mathfrak{B} ,
- \mathfrak{E} of P -maps, \mathfrak{M} of sources in \mathfrak{A} ,
- \mathfrak{F} of Q -maps, \mathfrak{N} of sources in \mathfrak{B}

which are, as usual, assumed to be closed under composition with isomorphisms. Moreover, \mathfrak{N} is assumed to be closed under composition, i. e., if $(n_i : B \rightarrow B_i)_I$ and $n : A \rightarrow B$ are in \mathfrak{N} then $(n_i \cdot n : A \rightarrow B_i)_I$ is in \mathfrak{N} .

We shall be concerned with the following conditions on the diagram $(*)$:

A. γ is Σ -bounded, i. e., for every $Y \in \mathfrak{Y}$ there is a U -map $u : Y \rightarrow UX$ such that for every Q -map $y : Y \rightarrow QB$ there are a P -map $x : X \rightarrow PA$ and a map $s : B \rightarrow GA$ in Σ so that the following diagram commutes:

$$\begin{array}{ccccc} Y & \xrightarrow{u} & UX & & \\ y \downarrow & & \downarrow Ux & & \\ QB & \xrightarrow{Qs} & QGA & \xrightarrow{\gamma A} & UPA \end{array}$$

B. For all $(m_i : A \rightarrow A_i)_I$ in \mathfrak{M} the diagrams

$$\begin{array}{ccc} QGA & \xrightarrow{\gamma A} & UPA \\ QGm_i \downarrow & & \downarrow UPm_i \\ QGA_i & \xrightarrow{\gamma A_i} & UPA_i \end{array}$$

form a generalized pullback.

C. For all $(m_i : A \rightarrow A_i)_I$ in \mathfrak{M} and $(s_i : B \rightarrow GA_i)_I$ with $s_i \in \Sigma$ for all $i \in I$, there exists the following generalized pullback with $(n_i)_I$ in \mathfrak{N} , which is preserved by Q .

$$\begin{array}{ccc}
 C & \xrightarrow{b} & GA \\
 n_i \downarrow & & \downarrow Gm_i \\
 B_i & \xrightarrow{s_i} & GA_i
 \end{array}$$

REMARKS. The above conditions are often trivial:

1. Condition A is automatic if U is weakly right adjoint and if G is weakly right adjoint with weak units in Σ .

2. Condition B is automatic for $\gamma = I$, i.e., $QG = UP$.

3. Condition C is automatic if $\Sigma \subset Iso B$ and $G\mathcal{M} \subset \mathcal{N}$.

4. For \mathcal{N} being all sources, condition C holds if \mathcal{B} is Σ -quasi-complete (a and b below) and Q is Σ -continuous (c below), i.e.,

a) For all $s: C \rightarrow B$ in Σ and $f: D \rightarrow B$ there exists a pullback

$$\begin{array}{ccc}
 E & \xrightarrow{s'} & D \\
 f' \downarrow & & \downarrow f \\
 C & \xrightarrow{s} & B
 \end{array}$$

with $s' \in \Sigma$,

b) For all $(s_i: C_i \rightarrow B)_I$ with $s_i \in \Sigma$ for all $i \in I$ the multiple pullback $s: D \rightarrow B$ of $(s_i)_I$ exists,

c) Q preserves the limits of a and b.

5. If in 4-b the multiple pullback s is assumed to be in Σ , conditions a and b mean Σ -completeness as defined in [2]; Σ -completeness can be equivalently described by the property that, in \mathcal{B}^{op} , all sources factor over Σ , and the factorizations are locally orthogonal with respect to Σ (cf. [8], 6.3).

THEOREM 1. *Assume that conditions A, B, C hold in diagram (*). If P-sources factor over \mathcal{M} , then Q-sources factor over $(\mathcal{F}, \mathcal{N})$ iff Q-maps do. If moreover $\mathcal{F} \perp G\mathcal{M}$, then the factorizations of Q-sources are (locally) orthogonal with respect to \mathcal{F} iff the factorizations of Q-maps are.*

PROOF. Let $(\gamma_i: Y \rightarrow QB_i)_I$ be a Q-source. For each $i \in I$ we have the following commutative diagram with $s_i \in \Sigma$. Since P-sources factor over \mathcal{M} , the source $(x_i)_I$ factors as

$$\begin{array}{ccc}
 Y & \xrightarrow{u} & UX \\
 y_i \downarrow & & \downarrow Ux_i \\
 QB_i & \xrightarrow{Qs_i} & QGA_i \xrightarrow{\gamma A_i} UPA_i
 \end{array}$$

$(e: X \rightarrow PA, (m_i: A \rightarrow A_i)_I)$ with $(m_i)_I \in \mathfrak{M}$.

Successively we get the two GP's described in Conditions B and C.

Since

$$UPm_i \cdot Ue \cdot u = \gamma A_i \cdot Qs_i \cdot y_i$$

there is a (unique)

$$t: Y \rightarrow QGA \quad \text{with} \quad \gamma A \cdot t = Ue \cdot u \quad \text{and} \quad QGm_i \cdot t = Qs_i \cdot y_i.$$

Since

$$\begin{array}{ccc}
 QC & \xrightarrow{Qb} & QGA \\
 Qn_i \downarrow & & \downarrow QGm_i \\
 QB_i & \xrightarrow{Qs_i} & QGA_i
 \end{array}$$

is a GP there is a (unique) $y: Y \rightarrow QC$ with

$$Qb \cdot y = t \quad \text{and} \quad Qn_i \cdot y = y_i \quad \text{for all } i.$$

Finally, since Q -maps have $(\mathcal{F}, \mathfrak{N})$ -factorizations we get

$$y = Qn \cdot j \quad \text{with} \quad j: Y \rightarrow QB \text{ in } \mathcal{F} \quad \text{and} \quad n: B \rightarrow C \text{ in } \mathfrak{N}.$$

Therefore $(j, (n_i \cdot n)_I)$ is the desired $(\mathcal{F}, \mathfrak{N})$ -factorization of $(y_i)_I$.

Now assume that the factorizations of Q -maps are locally orthogonal with respect to \mathcal{F} , $\mathcal{F} \perp G\mathfrak{M}$, and let $h \in \mathcal{F}$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
 Z & \xrightarrow{h} & & & & & QD \\
 z \downarrow & & & & & & \downarrow Qd_i \\
 Y & \xrightarrow{j} & QB & \xrightarrow{Qn} & QC & \xrightarrow{Qn_i} & QB_i \\
 & \searrow t & & & \downarrow Qb & & \downarrow Qs_i \\
 & & & & QGA & \xrightarrow{QGm_i} & QGA_i
 \end{array}$$

Since $\mathcal{F} \perp G\mathfrak{M}$ there exists a unique $d: D \rightarrow GA$ with

$$Qd.h = t.z \text{ and } Gm_i.d = s_i.d_i \text{ for all } i \in I.$$

Because of C there exists a unique $c: C \rightarrow C$ with $b.c = d$ and $n_i.c = d_i$ for all $i \in I$. Hence $Qb.Qc.h = t.z$ and thus $Qc.h = Qn.j.z$. Since the factorizations of Q -maps are locally orthogonal, there is a unique

$$f: D \rightarrow B \text{ with } Qf.h = j.z \text{ and } n.f = c.$$

So $(n_i.n).f = d_i$ for all $i \in I$. The uniqueness of f follows from the uniqueness of the constructions involved.

For the non-local case the proof is similar.

REMARK. The first part of the above proof shows that it suffices to have *weak* generalized pullbacks in Conditions B and C. But the corresponding weak version of Theorem 1 is not used in the following.

2. THE ADJOINT CASE

In this section we consider the diagram () where both U and G have left adjoints. We assume throughout this section that F is left adjoint to U with unit δ and that L is left adjoint to G with unit η .*

For every Q -map $j: Y \rightarrow QB$ let $\bar{j}: FY \rightarrow PLB$ be the P -map which corresponds by adjointness of U to

$$Y \xrightarrow{j} QB \xrightarrow{Q\eta B} QGLB \xrightarrow{\gamma LB} UPLB.$$

One then has:

LEMMA 1. 1. *If η is a pointwise monomorphism and \bar{j} is a P -epimorphism, then j is a Q -epimorphism.*

2. *If γ is a pointwise monomorphism and j is a Q -epimorphism, then \bar{j} is a P -epimorphism.*

PROOF. 1. Suppose $Qf.j = Qg.j$ where $f, g: B \rightarrow C$. Then we have the following diagram (cf. next page). We have

$$UPLg.U\bar{j}.\delta Y = UPLf.U\bar{j}.\delta Y.$$

Hence $PLg.\bar{j} = PLf.\bar{j}$. Consequently $Lf = Lg$. Since ηG is monic, we get $f = g$.

The proof of 2 is similar.

$$\begin{array}{ccccc}
 Y & \xrightarrow{\delta Y} & & & U F Y \\
 j \downarrow & & & & \downarrow U \bar{j} \\
 Q B & \xrightarrow{Q \eta B} & Q G L B & \xrightarrow{\gamma L B} & U P L B \\
 Q f \downarrow & \downarrow Q g & Q G L f \downarrow & \downarrow Q G L g & U P L f \downarrow & \downarrow U P L g \\
 Q C & \xrightarrow{Q \eta C} & Q G L C & \xrightarrow{\gamma L C} & U P L C
 \end{array}$$

Recall that under the assumptions of this section if η is pointwise in Σ then Condition A is automatic (cf. the remarks before Theorem 1). Choosing $\mathfrak{N} =$ all sources, we then have :

THEOREM 2. *Suppose that the unit η of G is a pointwise monomorphism in Σ and that P -sources factor over $(\mathfrak{E}, \mathfrak{M})$ for \mathfrak{E} consisting of P -epimorphisms. If Conditions B and C hold, then Q has a left adjoint.*

PROOF. It suffices to show the source of all Q -maps $(y_i: Y \rightarrow Q B_i)_I$ with domain Y factors over a Q -epimorphism. To this end we proceed as in the proof of Theorem 1 by factoring the corresponding source $(\bar{y}_i: F Y \rightarrow P L B_i)$ as

$$(e: F Y \rightarrow P A, (m_i: A \rightarrow L B_i)_I), \quad e \in \mathfrak{E} \text{ and } (m_i)_I \in \mathfrak{M}.$$

As in that proof we get a factorization as $(\gamma: Y \rightarrow Q C, (n_i: X \rightarrow B_i)_I)$ and a commuting diagram

$$\begin{array}{ccccc}
 Y & \xrightarrow{\delta Y} & & & U F Y \\
 y \downarrow & & & & \downarrow U e \\
 Q C & \xrightarrow{Q b} & Q G A & \xrightarrow{\gamma A} & U P A \\
 Q n_i \downarrow & \quad 1 \quad & \downarrow Q G m_i & \quad 2 \quad & \downarrow U P m_i \\
 Q B_i & \xrightarrow{Q \eta B_i} & Q G L B_i & \xrightarrow{\gamma A_i} & U P L B_i
 \end{array}$$

with diagrams 1 and 2 being GP's.

We now show that $\gamma: Y \rightarrow Q C$ is Q -epimorphic. By the Lemma it suf-

fices to show that the corresponding \bar{y} is P -epimorphic. First note that there exists a unique $d: LC \rightarrow A$ with $Gd \cdot \eta C = b$. Also, since the original source consists of all Q -maps with domain Y , there is a $c: C \rightarrow C$ (namely one to the n_i 's) with

$$Qc \cdot y = y \quad \text{and} \quad \eta C \cdot c = Gm \cdot b,$$

for m being in $(m_i)_I$. Now

$$\begin{aligned} UPd \cdot UPm \cdot Ue \cdot \delta Y &= UPd \cdot \gamma LC \cdot Q\eta C \cdot y = \\ &= \gamma A \cdot QGd \cdot Q\eta C \cdot y = \gamma A \cdot Qb \cdot y = Ue \cdot \delta Y. \end{aligned}$$

Hence $Pd \cdot Pm \cdot e = e$ and consequently $d \cdot m = 1$. Now

$$b = Gd \cdot Gm \cdot b = Gd \cdot \eta C \cdot c = b \cdot c.$$

Furthermore, for each $i \in I$,

$$\eta B_i \cdot n_i \cdot c = Gm_i \cdot b \cdot c = Gm_i \cdot b = \eta B_i \cdot n_i.$$

Since ηB_i is a monomorphism, we have $n_i \cdot c = n_i$ for all $i \in I$. Consequently, since 1 is a GP, we get $c = 1$. So

$$Gm \cdot Gd \cdot \eta C = Gm \cdot b = \eta C \cdot c = \eta C.$$

Hence $m \cdot d = 1$.

Because $Pd \cdot \bar{y} = e$ we now have $\bar{y} \approx e$ which is P -epimorphic.

The next corollary generalizes Theorem 1.8 of [2]; this is gotten by taking $\gamma = 1$.

COROLLARY 1. *In (*), let $P = 1$ and let condition B be satisfied with $\mathfrak{M} =$ all sources. Suppose that \mathfrak{B} is Σ -complete and that the units of G are pointwise in Σ . Then Q has a left adjoint iff Q is Σ -continuous.*

COROLLARY 2. *For any right adjoint functor $G: \mathfrak{A} \rightarrow \mathfrak{B}$ with units in Σ and \mathfrak{B} being Σ -complete one has: A functor $Q: \mathfrak{B} \rightarrow \mathfrak{Y}$ is right adjoint iff QG is right adjoint and Q is Σ -continuous.*

In the adjoint situation as described at the beginning of this section we take up again the question of when does Q admit orthogonal factorizations. We shall prove a sharpened version of Theorem 1 in which Condition B appears as a necessary condition. We first identify in our situa-

tion the maps orthogonal to $G\mathfrak{M}$ (cf. Theorem 1).

LEMMA 2. *Suppose that, in the situation of this section, Condition B holds. Then, for every Q -map $j: Y \rightarrow QB$, $\{j\} \perp G\mathfrak{M}$ iff $\{\bar{j}\} \perp \mathfrak{M}$.*

PROOF. Suppose $\{j\} \perp G\mathfrak{M}$ and consider the diagram

$$\begin{array}{ccc} FY & \xrightarrow{\bar{j}} & PLB \\ f \downarrow & & \downarrow Pd_i \\ PA & \xrightarrow{Pm_i} & PA_i \end{array}$$

with $(m_i)_I \in \mathfrak{M}$. The source $(d_i: LB \rightarrow A_i)_I$ corresponds, by adjointness of G , to $(\bar{d}_i: B \rightarrow GA_i)_I$, and $f: FY \rightarrow PA$ corresponds to $\bar{f}: Y \rightarrow UPA$ by adjointness of U . We get the following diagram in \mathcal{Y} :

$$\begin{array}{ccc} Y & \xrightarrow{j} & QB \\ \bar{f} \downarrow & & \downarrow Q\bar{d}_i \\ UPA & \xrightarrow{UPm_i} & UPA_i \\ & & \downarrow \gamma A_i \\ & & QGA_i \end{array}$$

By B there exists a unique $h: Y \rightarrow QGA$ with

$$\gamma A_i \cdot h = \bar{f} \text{ and } QGm_i \cdot h = Q\bar{d}_i \cdot j \text{ for all } i \in I.$$

Since $\{j\} \perp G\mathfrak{M}$ there is a unique $l: B \rightarrow GA$ with

$$Ql \cdot j = h \text{ and } Gm_i \cdot l = \bar{d}_i \text{ for all } i \in I.$$

Then l corresponds by adjunction to

$$\bar{l}: LB \rightarrow A \text{ with } P\bar{l} \cdot \bar{j} = f \text{ and } m_i \cdot \bar{l} = d_i.$$

Uniqueness of l follows from the uniqueness of the constructions involved. We therefore have $\{\bar{j}\} \perp \mathfrak{M}$.

The converse assertion is proved similarly.

THEOREM 3. *Suppose that the unit of G belongs to Σ and that Condition C holds, with $\mathfrak{N} =$ all sources. Suppose further that P -sources factor (locally) orthogonally over $(\mathfrak{E}, \mathfrak{M})$, and that $\mathcal{F} = \{j \mid \bar{j} \in \mathfrak{E}\}$. Then, for*

the statements :

(i) Q -sources factor (locally) orthogonally over \mathcal{F} ,

(ii) Q -maps factor (locally) orthogonally over \mathcal{F} and condition B holds one has (ii) \Rightarrow (i), whereas (i) \Rightarrow (ii) holds for $\mathcal{E} \perp \mathcal{M}$.

PROOF. (ii) \Rightarrow (i): The non-local case follows immediately from Theorem 1 and Lemma 2. For the local case we look to the second part of the proof of Theorem 1. We again assume the factorizations of Q -maps to be locally orthogonal with respect to \mathcal{F} , but we cannot assume $\mathcal{F} \perp G\mathcal{M}$. Nevertheless in the situation of the last diagram of that proof, one gets also a unique

$$d: D \rightarrow GA \text{ with } Qd \cdot h = t \cdot z \text{ and } Gm_i \cdot d = \eta B_i \cdot d_i.$$

This is easily proved by taking $d = Gf \cdot \eta D$, where $f: LD \rightarrow A$ is the unique diagonal of the commutative diagram :

$$\begin{array}{ccc} FZ & \xrightarrow{\bar{h}} & PLD \\ Fz \downarrow & \nearrow Pf & \downarrow PLd_i \\ FY & \xrightarrow{e} PA & \xrightarrow{Pm_i} PLB_i \end{array}$$

Now the proof can be completed as in Theorem 1.

(i) \Rightarrow (ii): From Condition (i) we have that Q has a left adjoint S with unit π pointwise in \mathcal{F} , because the source of all Q -maps factors over \mathcal{F} , and \mathcal{F} necessarily consists of Q -epimorphisms only (see remarks at the beginning of Section 1). Now consider the following commutative diagram :

$$\begin{array}{ccc} Y & \xrightarrow{g_i} & QGA_i \\ f \downarrow & & \downarrow \gamma A_i \\ UPA & \xrightarrow{UPm_i} & UPA_i \end{array}$$

with $(m_i: A \rightarrow A_i)_I$ in \mathcal{M} . For each $i \in I$, there exists a unique

$$d_i: SY \rightarrow GA_i \text{ with } Qd_i \cdot \pi Y = g_i.$$

By adjunction of G and U , d_i corresponds to $\bar{d}_i: LSY \rightarrow A_i$ and f corresponds to $\bar{f}: FY \rightarrow PA$. Since $\bar{\pi} \bar{Y} \in \mathcal{E} \perp \mathcal{M}$ from the commutative diagram

$$\begin{array}{ccc}
 FY & \xrightarrow{\overline{\pi Y}} & PLSY \\
 \tilde{f} \downarrow & & \downarrow P \tilde{d}_i \\
 PA & \xrightarrow{Pm_i} & PA_i
 \end{array}$$

we get a unique $t: LSY \rightarrow A$ with

$$Pt \cdot \overline{\pi Y} = \tilde{f} \quad \text{and} \quad m_i \cdot t = \tilde{d}_i \quad \text{for all } i \in I.$$

Then t corresponds to

$$\tilde{t}: SY \rightarrow GA \quad \text{with} \quad Gm_i \cdot \tilde{t} = d_i.$$

So we get

$$Q\tilde{t} \cdot \pi Y: Y \rightarrow QGA \quad \text{with} \quad QGm_i \cdot Q\tilde{t} \cdot \pi Y = g_i \quad \text{for all } i \in I.$$

Since $\gamma A \cdot Q\tilde{t} \cdot \pi Y$ corresponds by adjunction to \tilde{f} we have $\gamma A \cdot Q\tilde{t} \cdot \pi Y = f$.

If $h: Y \rightarrow QGA$ is a map with $\gamma A \cdot h = f$ and $QGm_i \cdot h = g_i$ for all $i \in I$, then we get a unique

$$l: SY \rightarrow GA \quad \text{with} \quad Ql \cdot \pi Y = h.$$

One sees that l corresponds to $\tilde{l}: LSY \rightarrow A$ with

$$P\tilde{l} \cdot \overline{\pi Y} = \tilde{f} \quad \text{and} \quad m_i \cdot \tilde{l} = \tilde{d}_i.$$

Thus $\tilde{l} = t$ and so $l = \tilde{t}$.

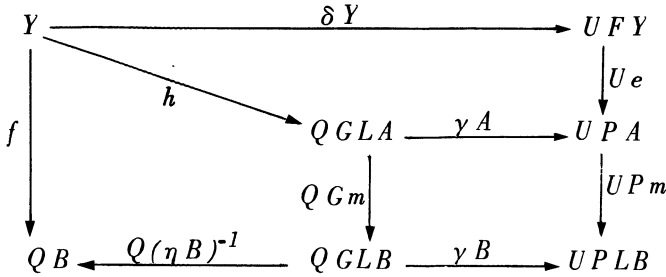
If the left adjoint of G is full and faithful, the unit of G is an isomorphism. Then Σ can be taken to be the class of all isomorphisms, and Condition C is automatic. So we get

COROLLARY 3. *Let P -sources factor (locally) orthogonally over \mathfrak{E} . Assume that G has a full and faithful left adjoint. Then, for the statements*

- (i) *Q -sources factor (locally) orthogonally over $\mathfrak{F} = \{ \bar{j} \mid j \in \mathfrak{E} \}$,*
- (ii) *Condition B holds,*

one has (ii) \Rightarrow (i), whereas (i) \Rightarrow (ii) holds in the non-local case.

PROOF. We need to verify that Q -maps factor (locally) orthogonally over \mathfrak{F} , if B holds. Let $f: Y \rightarrow QB$ be a Q -map and let \tilde{f} factor as $\tilde{f} = Pm \cdot e$, where $e: FY \rightarrow PA$ in \mathfrak{E} . Recalling that the unit η is an isomorphism, we get the following commutative diagram



For $\bar{h}: F Y \rightarrow P L G A$ and the counit $\epsilon: L G \rightarrow I$ of G one now has

$$\begin{aligned}
 U P \epsilon A \cdot U \bar{h} \cdot \delta Y &= U P \epsilon A \cdot \gamma L G A \cdot Q \eta G A \cdot h = \\
 &= \gamma A \cdot Q G \epsilon A \cdot Q \eta G A \cdot h = U e \cdot \delta Y.
 \end{aligned}$$

Hence $e = P \epsilon A \cdot \bar{h} \in \mathfrak{E}$. Therefore, by factoring \bar{h} over \mathfrak{E} , one easily gets $\bar{h} \in \mathfrak{E}$ and so $h \in \mathfrak{F}$. Orthogonality of the factorization

$$f = Q((\eta B)^{-1} \cdot G m) \cdot h$$

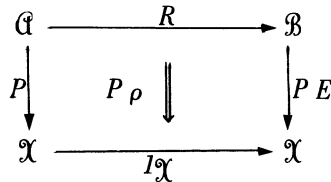
follows by Lemma 2, whereas the local case is treated as in Theorem 2, (ii) \Rightarrow (i).

Note that for γ being an isomorphism, Condition B is automatic. Hence assertion (i) of Corollary 3 holds in this case.

3. APPLICATIONS.

In this section we give some applications of the above results. Many others can be added by specializing the data of (*).

3.1. *Coreflective subcategories.* Let $P: \mathfrak{A} \rightarrow \mathfrak{X}$ be a functor and let $E: \mathfrak{B} \rightarrow \mathfrak{A}$ be the embedding of a full coreflective subcategory with coreflector R and coreflection ρ . Finally, let \mathfrak{F} be the class of PE -maps $e: X \rightarrow PEB$ such that $e: X \rightarrow P(EB)$ belongs to a given class \mathfrak{E} of P -maps. Applying Corollary 3 to the diagram



we get:

THEOREM 4. *Let P -sources factor (locally) orthogonally over $(\mathfrak{E}, \mathfrak{M})$. Then PE -sources factor (locally) orthogonally over \mathfrak{F} if the diagrams*

$$\begin{array}{ccc}
 PRA & \xrightarrow{P\rho A} & PA \\
 \downarrow P R m_i & & \downarrow P m_i \\
 PRA_i & \xrightarrow{P\rho A_i} & PA_i
 \end{array}$$

form a GP in \mathfrak{X} for each source $(m_i: A \rightarrow A_i)_I$ in \mathfrak{M} . This condition is necessary in the non-local case.

All the generalized pullbacks are trivial for $P\rho$ being an isomorphism. Therefore, considering the canonical factorization structures for P , by Theorem 4 we get immediately:

COROLLARY 4. *If $P: \mathfrak{A} \rightarrow \mathfrak{X}$ belongs to one of the following classes of functors (of which each is contained in the next one), so does every restriction of P to a full coreflective subcategory of \mathfrak{A} such that the P -images of the coreflection maps are isomorphisms:*

- topological functors (cf. [8]),*
- $(\mathfrak{E}, \mathfrak{M})$ -topological functors (cf. [3]),*
- topologically-algebraic functors (cf. [1, 4]),*
- semitopological functors (cf. [8]),*
- right adjoint functors.*

The assertion of Corollary 4 for $(\mathfrak{E}, \mathfrak{M})$ -topological functors contains in particular Nel's corresponding result on «initially structured» categories (cf. [6], Theorem 1.3). For various applications we refer to his paper.

3.2. *Characterization of topological functors.* As a further consequence of Theorem 3 we obtain a characterization of topological functors due to Hoffmann [5]:

COROLLARY 5. *A functor $P: \mathfrak{A} \rightarrow \mathfrak{X}$ is topological iff \mathfrak{A} is Σ -complete for $\Sigma = P^{-1}(\text{iso } \mathfrak{X})$ and P has a full and faithful right adjoint.*

PROOF. We only need to show that the condition is sufficient for topologicity. We apply Theorem 3 to the diagram

$$\begin{array}{ccc}
 \mathcal{X} & \xrightarrow{R} & \mathcal{A} \\
 I\mathcal{X} \downarrow & \epsilon \Downarrow & \downarrow P \\
 \mathcal{X} & \xrightarrow{I\mathcal{X}} & \mathcal{X}
 \end{array}$$

where R is the full and faithful right adjoint and ϵ the isomorphic counit. With $\mathcal{E} = iso \mathcal{X}$ one obtains for \mathcal{F} the class of all P -maps which are isomorphisms in \mathcal{X} . By definition of Σ , P is trivially Σ -continuous, and the unit η of P belongs to Σ . In order to get orthogonal factorizations of P -sources over \mathcal{F} it suffices therefore to have those for P -maps. But given a P -map $x: X \rightarrow PB$ one obtains this factorization by considering the P -image of the pullback

$$\begin{array}{ccc}
 A & \xrightarrow{s} & RX \\
 f \downarrow & & \downarrow Rx \\
 B & \xrightarrow{\eta B} & RP B
 \end{array}$$

which exists by Σ -completeness.

REMARK. The functor

$$P: \mathcal{Cat} \rightarrow \mathcal{Set}, K \mapsto Ob K,$$

has a full and faithful right adjoint, and \mathcal{Cat} is, of course, small Σ -complete with $\Sigma = P^{-1}(iso \mathcal{Set})$, i. e., pullbacks and small-indexed intersections of Σ -maps exist and belong to Σ . Nevertheless, the non-faithful functor P is not topological. With respect to Corollary 5 the reason for this is that \mathcal{A} fails to be Σ -complete: For each cardinal k consider a category \mathcal{K}_k having two objects $0, 1$ and k arrows $0 \rightarrow 1$. Identifying these arrows one gets a family of functors $\mathcal{K}_k \rightarrow \{0 \rightarrow 1\}$ (indexed by all cardinals) which fails to admit an intersection.

3.3. *The Special Adjoint Functor Theorem.* We give a slight generalization of a theorem stated in [2] by application of Theorem 2 in the following situation. Let $Q: \mathcal{B} \rightarrow \mathcal{Y}$ be a functor whose right adjointness shall be proved. Let \mathcal{G} be a subset of the objects of \mathcal{B} such that all products

$$GX = \prod_{C \in \mathcal{C}} \prod_{X_C} C \quad \text{and} \quad UX = \prod_{C \in \mathcal{C}} \prod_{X_C} QC$$

exist in \mathcal{B} and \mathcal{Y} where $X = (X_C)_{\mathcal{C}}$ is any object in $(\text{Set}^{\mathcal{C}})^{op} = \mathcal{A}$. The functors $G: \mathcal{A} \rightarrow \mathcal{B}$ and $U: \mathcal{A} \rightarrow \mathcal{Y}$ have left adjoints given by

$$LB = (\mathcal{B}(B, C))_{\mathcal{C}} \quad \text{and} \quad FY = (\mathcal{Y}(Y, QC))_{\mathcal{C}}.$$

There is a natural transformation $\gamma: QG \rightarrow U$ which is an isomorphism iff Q preserves the products GX .

COROLLARY 6. *Let the category \mathcal{B} be Σ -complete and let \mathcal{C} be a Σ -cogenerating set in \mathcal{B} (i. e., the units $\eta_B: B \rightarrow GLB$ belong to Σ). The functor $Q: \mathcal{B} \rightarrow \mathcal{Y}$ then has a left adjoint iff*

- (1) Q est Σ -continuous,
- (2) there is a pair $(\mathcal{E}, \mathcal{M})$ such that sources in $\mathcal{A} = (\text{Set}^{\mathcal{C}})^{op}$ factor over $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E} \subset \text{Epi } \mathcal{A}$ and Condition B (depending on \mathcal{M} and γ) holds.

In particular condition (2) holds if Q preserves products. Therefore, for \mathcal{B} being complete and Σ -wellpowered, (1) and (2) are fulfilled for Q preserving all small limits; this is the usual version of the Special Adjoint Functor Theorem.

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