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# ENRICHED CATEGORIES AND ENRICHED MODULES

*by Harald LINDNER*

Our purpose is to show that most of the results on categories enriched over a symmetric monoidal closed category $\underline{V}$ can be formulated and proved in the merely monoidal case. This permits to apply the theory of enriched categories to further examples, to gain a better understanding of the basic notions of (enriched) category theory, and to present enriched category theory more concisely.

An important tool is the notion of enriched modules (Bénabou: «actions of multiplicative categories»), i.e., categories on which a monoidal category acts. We hope to show that the two notions of enriched categories and enriched modules are equally important. These two kinds of objects are the 0-cells of two well-known 2-categories. We have described in previous papers how these two 2-categories can be embedded into a 2-category $\mathcal{O}$ by introducing 1-cells (and 2-cells) from $\underline{V}$-categories to $\underline{V}$-modules, and vice versa. Our examples prove that such 1-cells and 2-cells occur naturally even in the familiar symmetric monoidal closed case.

The key result (1.9) is a characterization of tensored $\underline{V}$-categories in terms of isomorphisms between enriched categories and enriched modules. We discuss duality, limits and Kan-extensions in our context. Details on further topics such as functor categories will be considered elsewhere. Proofs are usually omitted.

## 1. THE 2-CATEGORY $\mathcal{O}$ OF ENRICHED CATEGORIES AND ENRICHED MODULES.

We recall the definition of the 2-category $\mathcal{O}$ (cf. [15, 17]). Let $\underline{V} = (\underline{V}_0, \otimes, I, a, \lambda, \rho)$ be a monoidal category, i.e., $\otimes : \underline{V}_0 \times \underline{V}_0 \to \underline{V}_0$ is a functor (written between its arguments), $I$ is an object of $\underline{V}_0$, and

$$a_{X,Y,Z}: X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z,\quad \lambda_X: X\to I\otimes X,\quad \rho_X: X\to X\otimes I$$

are compatible natural transformations.

1.1. DEFINITION. A $\underline{V}$- *(left-) module* $\underline{A}=(\underline{A}_o,\otimes^{\underline{A}},a^{\underline{A}},\lambda^{\underline{A}})$ consists: of a category $\underline{A}_o$, a functor $\otimes^{\underline{A}}: \underline{V}_o\times\underline{A}_o\to\underline{A}_o$, and two natural transformations $a^{\underline{A}}$, $\lambda^{\underline{A}}$

$$a^{\underline{A}}_{X,Y,A}: X\otimes^{\underline{A}}(Y\otimes^{\underline{A}}A)\to(X\otimes^{\underline{V}}Y)\otimes^{\underline{A}}A,\quad \lambda^{\underline{A}}_A: A\to I\otimes^{\underline{A}}A$$

such that three evident diagrams commute. $\underline{A}$ is called *normal* if $a^{\underline{A}}$ and $\lambda^{\underline{A}}$ are both isomorphic; their inverses are then denoted by $\beta^{\underline{A}}$ and $\nu^{\underline{A}}$, respectively.

(Cf. [1], 2.3 («actions of multiplicative categories»); [2], 3, Section 1; [15], 5.1; [16], 2; [17], 5.1.)

$(\underline{V}_o,\otimes^{\underline{V}},a^{\underline{V}},\lambda^{\underline{V}})$ is an example of a normal module which we usually denote by $\underline{V}$, if there is no danger of confusion. Also, we often drop the indices $\underline{A}$, $\underline{V}$, $X$, $Y$, $A$ of $\otimes^{\underline{A}}$, $\otimes^{\underline{V}}$, $a^{\underline{A}}_{X,Y,A}$, etc..., if the context seems to exclude any danger of confusion. We often write $|\underline{A}|$ instead of $|\underline{A}_o|$ for the class of objects of a $\underline{V}$-module $\underline{A}$. If $|\underline{A}|$ is a set, $\underline{A}$ is called *small*. If $\underline{A}$ is a tensored $\underline{V}$-category, $\underline{A}$ is canonically equipped with the structure of a normal $\underline{V}$-module (cp. 1.9 below).

1.2. DEFINITION. A *1-cell* $F:\underline{A}\to\underline{B}$ in $\mathfrak{C}$ consists of a functor

$F_o:\underline{A}_o\to\underline{B}_o$ (we often omit the index «$o$»),

together with a natural family of morphisms in $\underline{V}_o$ or $\underline{B}_o$, indexed by pairs of objects $A$, $B\in|\underline{A}|$ or $X\in\underline{V}$, $A\in|\underline{A}|$, resp.

a) $F_{A,B}:\underline{A}(A,B)\to\underline{B}(FA,FB)$ if $\underline{A}$, $\underline{B}$ are $\underline{V}$-categories,

b) $F_{A,B}:\underline{A}(A,B)\otimes FA\to FB$ if $\underline{A}$ is a $\underline{V}$-category, $\underline{B}$ is a $\underline{V}$-module,

c) $F_{X,A}: X\to\underline{B}(FA,F(X\otimes A))$ if $\underline{A}$ is a $\underline{V}$-module, $\underline{B}$ is a $\underline{V}$-category,

d) $F_{X,A}: X\otimes FA\to F(X\otimes A)$ if $\underline{A}$, $\underline{B}$ are $\underline{V}$-modules,

such that two evident corresponding diagrams commute, e.g. in case c:

c) (i)
$$\begin{tikzcd} X\otimes Y \arrow[r, "F_{X,Y\otimes A}\otimes F_{Y,A}"] \arrow[d, "F_{X\otimes Y,A}"'] & \underline{B}(F(Y\otimes A),F(X\otimes(Y\otimes A)))\otimes\underline{B}(FA,F(Y\otimes A)) \arrow[d, "\mu^{\underline{B}}"] \\ \underline{B}(FA,F((X\otimes Y)\otimes A)) & \underline{B}(FA,F(X\otimes(Y\otimes A))) \arrow[l, "\underline{B}(1,Fa^{\underline{A}})"'] \end{tikzcd}$$

(ii)
$$\begin{tikzcd} I \arrow[rr, "F_{I,A}"] \arrow[dr, "\iota^{\underline{B}}_{FA}"'] & & \underline{B}(FA, F(I\otimes A)) \\ & \underline{B}(FA, FA) \arrow[ur, "\underline{B}(1, F\lambda^{\underline{A}}_{A})"'] & \end{tikzcd}$$

(cf. e.g., [12], 1; [15], 5.2; [17], 5).

1.3. EXAMPLES. (i) Let $C$ be an object of a $\underline{V}$-category $\underline{A}$. The hom functor $\underline{A}_0(C,-): \underline{A}_0 \to \underline{V}_0$, together with the family

$$\underline{A}(C,-)_{A,B} := \mu^{\underline{A}}_{C,A,B}: \underline{A}(A,B)\otimes \underline{A}(C,A) \to \underline{A}(C,B),$$

is a 1-cell in the sense of 1.2 (b). (Cf. [19]; [17], 5.7.)

(ii) Let $C$ be an object of a $\underline{V}$-module $\underline{B}$. The functor $(-\otimes C): \underline{V}_0 \to \underline{B}_0$ together with the family

$$(-\otimes C)_{X,Y} := \alpha^{\underline{B}}_{X,Y,X}: X\otimes(Y\otimes C) \to (X\otimes Y)\otimes C$$

is a 1-cell from $\underline{V}$ to $\underline{B}$ in the sense of 1.2 (d).

1.4. DEFINITION. The composition of 1-cells $F: \underline{A} \to \underline{B}$ and $G: \underline{B} \to \underline{C}$ in $\mathfrak{V}$ is defined by composing the underlying functors $F_0$ and $G_0$ and by, e.g.,

$$\underline{A}(A,B)\otimes GFA \xrightarrow{G_{\underline{A}(A,B),FA}} G(\underline{A}(A,B)\otimes FA) \xrightarrow{G(F_{A,B})} GFB$$

if $\underline{A}$ is a $\underline{V}$-category and $\underline{B}$, $\underline{C}$ are $\underline{V}$-modules.

1.5. DEFINITION. A 2-cell $\theta: F \to H: \underline{A} \to \underline{B}$ in $\mathfrak{V}$ is a natural transformation $\theta: F_0 \to H_0$ such that an evident diagram commutes, e.g. in case c:

c)
$$\begin{tikzcd} X \arrow[r, "F_{X,A}"] \arrow[d, "H_{X,A}"'] & \underline{B}(FA, F(X\otimes A)) \arrow[d, "\underline{B}(1, \theta_{X\otimes A})"] \\ \underline{B}(HA, H(X\otimes A)) \arrow[r, "\underline{B}(\theta_A, 1)"] & \underline{B}(FA, H(X\otimes A)) \end{tikzcd}$$

The composition of 2-cells is evident. We leave to the reader the straightforward proof that these definitions yield a 2-category $\mathfrak{V}$ (cf. [15], 5).

1.6. EXAMPLES OF 2-CELLS IN $\mathfrak{V}$. Let $F: \underline{A} \to \underline{B}$ be a 1-cell in $\mathfrak{V}$ and let $A \in |\underline{A}|$. We consider the four cases a-d in 1.2:

a) $F_{A,-}: \underline{A}(A,-) \to \underline{B}(FA,-)\circ F$ (cf. (1)),

b) $F_{A,-} : (-\otimes FA) \circ \underline{A}(A,-) \to F$ (cf. (2)),

c) $F_{-,A} : 1_{\underline{V}} \to \underline{B}(FA,-) \circ F \circ (-\otimes A)$ (cf. (3)),

d) $F_{-,A} : -\otimes FA \to F \circ (-\otimes A)$ (cf. (4)),

e) $\mu^{A}_{A,B,-} : (-\otimes \underline{A}(A,B)) \circ \underline{A}(B,-) \to \underline{A}(A,-)$ is a 2-cell. This is a specialization of b (cp. 1.3 (i)).

f) $a_{-,Y,A} : (-\otimes(Y\otimes A)) \to (-\otimes A) \circ (-\otimes Y)$ is a 2-cell. This is a specialisation of d (cp. 1.3 (ii)).

(1)
$$\begin{tikzcd} \underline{A} \arrow[r, "F"] \arrow[dr, "\underline{A}(A,-)"'] & \underline{B} \arrow[d, "\underline{B}(FA,-)"] \arrow[dl, phantom, "F_{A,-}\nearrow"] \\ & \underline{V} \end{tikzcd}$$

(2)
$$\begin{tikzcd} \underline{A} \arrow[rr, "F"] \arrow[dr, "\underline{A}(A,-)"'] & \arrow[d, phantom, "\Uparrow F_{A,-}"] & \underline{B} \\ & \underline{V} \arrow[ur, "-\otimes FA"'] & \end{tikzcd}$$

(3)
$$\begin{tikzcd} \underline{A} \arrow[r, "F"] & \underline{B} \arrow[d, "\underline{B}(FA,-)"] \\ \underline{V} \arrow[u, "-\otimes A"] \arrow[r, "1"'] \arrow[ur, phantom, "\Uparrow F_{-,A}"] & \underline{V} \end{tikzcd}$$

(4)
$$\begin{tikzcd} \underline{A} \arrow[r, "F"] & \underline{B} \\ \underline{V} \arrow[u, "-\otimes A"] \arrow[ur, "-\otimes FA"'] \arrow[ur, phantom, "\nwarrow F_{-,A}", near end] & \end{tikzcd}$$

In this setup we are able to extend the usual definition of tensored $\underline{V}$-categories (cf. [8], 4), in which $\underline{V}$ had to be symmetric monoidal closed, to the case of a merely monoidal category (cp. [10], 9).

1.7. DEFINITION. A *tensored $\underline{V}$-category* consists of a $\underline{V}$-category $\underline{C}$ together with an adjunction (5) in $\mathfrak{V}$ for every $A \in |\underline{C}|$ (cf. 1.3 (i)):

(5)
$$(-\otimes A) \underset{i_{-,A}}{\overset{e_{A,-}}{\dashv}} \underline{C}(A,-) : \underline{C} \to \underline{V}.$$

Although a tensored $\underline{V}$-category consists of a $\underline{V}$-category $\underline{C}$ together with additional data, rather than a specific property of $\underline{C}$, it is customary to denote a tensored $\underline{V}$-category by the same symbol as the «underlying» $\underline{V}$-category $\underline{C}$. This is of course justified to some extent, since (co-)adjoints are determined uniquely up to isomorphism. The reader is invited to draw the commutative diagrams, provided by 1.7, for later reference.

As an example we list the adjunction equations:

(6) $\underline{C}(A,B) \xrightarrow{i_{\underline{C}(A,B),A}} \underline{C}(A, C(A,B)\otimes A) \xrightarrow{\underline{C}(A,e_{A,B})} \underline{C}(A,B) = 1_{\underline{C}(A,B)}$,

$$(7) \qquad X\otimes A \xrightarrow{i_{X,A}\otimes A} \underline{C}(A, X\otimes A)\otimes A \xrightarrow{e_{A,X\otimes A}} X\otimes A = 1_{X\otimes A}$$

for all $A,\ B \in |\underline{C}|,\ X \in |\underline{V}|$.

The Definition 1.7 can be «translated» to the case of $\underline{V}$-modules (cp. 1.9 below):

1.8. DEFINITION AND PROPOSITION. A *tensored $\underline{V}$-module* consists of a $\underline{V}$-module $\underline{C}$, such that $\lambda^{\underline{C}}$ is isomorphic, together with an adjunction (8) for every $A \in |\underline{C}|$. Every tensored $\underline{V}$-module is normal.

$$(8) \qquad (-\otimes A) \xrightarrow[i_{-,A}]{e_{A,-}} \!\!\dashv\ \underline{C}(A,-): \underline{C} \to \underline{V}.$$

Although the adjunctions (5) and (8) look equal, we should like to emphasize that they are different because $\underline{C}$ denotes a $\underline{V}$-category in 1.7 and a $\underline{V}$-module in 1.8. In particular, the «structure maps» of the 1-cells in (5) and (8) in the nontrivial cases are:

$$(9) \qquad (-\otimes A)_{X,Y}: X \to \underline{C}(Y\otimes A, (X\otimes Y)\otimes A),$$

$$(10) \qquad \underline{C}(A,-)_{X,B}: X\otimes \underline{C}(A,B) \to \underline{C}(A, X\otimes B).$$

1.9. THEOREM. *There is a canonical bijection between:*

*(i) tensored $\underline{V}$-categories,*

*(ii) tensored $\underline{V}$-modules,*

*(iii) isomorphisms between $\underline{V}$-categories and $\underline{V}$-modules such that the underlying functors are identities.*

We must leave the proof to the reader (cp. [17], 5.11).

On applying the Theorem 1.9 to $\underline{A} = \underline{V}$ if $\underline{V}$ is symmetric monoidal closed we recognize the Definition 1.7 of tensored $\underline{V}$-categories as compatible with the classical case (cf. [8], 4).

1.10. REMARK. We stress the importance of the statement (iii) in 1.9: if $\underline{A}$ and/or $\underline{B}$ are tensored $\underline{V}$-categories, the different notions of 1-cells $\underline{A} \to \underline{B}$ in 1.2 are in a bijective correspondence, set up by composing with the isomorphisms between the $\underline{V}$-category and $\underline{V}$-module structures. In particular, these notions are compatible. In this way we can extend most notions in enriched category theory from monoidal closed categories $\underline{V}$ to merely monoidal categories $\underline{V}$.

In the next sections we take the first steps in this direction. Most results are contained in a slightly different form in previous papers (e.g., [17]). The present setting - the 2-category $\mathcal{O}$ - permits a nice formulation.

A common generalization of the two notions of objects in $\mathcal{O}$ appears to be very tempting. In fact, in [18] such a generalization was given. In this way $\underline{V}$-modules and $\underline{V}$-categories can be treated simultaneously. On the other hand, it appears as if additional work were required in order to reinterpret results in terms of the familiar notions of $\underline{V}$-modules and $\underline{V}$-categories. Also, the translation of a notion from $\underline{V}$-categories to $\underline{V}$-modules and vice versa is often quite straightforward.

With regard to 1.9 we may consider 1-cells from a $\underline{V}$-category $\underline{A}$ to a $\underline{V}$-module $\underline{B}$ (in particular $\underline{B} = \underline{V}$) as genuine generalizations of $\underline{V}$-functors. We shall therefore often call these 1-cells and the corresponding 2-cells, $\underline{V}$-functors and $\underline{V}$-natural transformations, respectively.

## 2. DUALITY.

The dual of a $\underline{V}$-category as well as contravariant $\underline{V}$-functors between $\underline{V}$-categories cannot be defined unless $\underline{V}$ is symmetric. In particular, the definition of extraordinary $\underline{V}$-natural transformations requires a symmetry. However, certain parts of this duality for $\underline{V}$-categories are independent of a symmetry (cf. [19, 17]).

To a monoidal category $\underline{V} = (\underline{V}_0, \otimes, I, a, \lambda, \rho)$ we may assign an opmonoidal (cp. (2); the brackets are shifted the other direction) category $\underline{V}^t = (\underline{V}_0, \otimes^t, I, a^t, \lambda^t, \rho^t)$, *the transpose of* $\underline{V}$ by:

(1)
$$\begin{tikzcd} \underline{V}_0 \times \underline{V}_0 \arrow[rr, "Tw"] \arrow[dr, "\otimes^t"'] & & \underline{V}_0 \times \underline{V}_0 \arrow[dl, "\otimes"] \\ & \underline{V}_0 & \end{tikzcd}$$

(2) $a^t_{X,Y,Z} := a_{Z,Y,X}$ ; (3) $\lambda^t := \rho$ ; (4) $\rho^t := \lambda$

($Tw$ denotes twisting of the arguments, i.e., $Tw(X,Y) = (Y,X)$ etc.).

Clearly $\underline{V}^{tt} = \underline{V}$. Symmetries $\gamma$ for $\underline{V}$ are in bijection with monoidal functors $\Gamma = (1_{\underline{V}_0}, \gamma, 1_I): \underline{V}^t \to \underline{V}$ which are quasi-involutive, i.e., $\Gamma(\Gamma^t) = 1$ (but $\Gamma\Gamma$ is not defined). By inverting $a^t$ we obtain an (honest) monoidal category $\underline{V}^s = (\underline{V}_0, \otimes^t, (a^t)^{-1}, \lambda^t, \rho^t)$ (cf. e.g. [17], 1.3). To a $\underline{V}$-category $\underline{A}$ we assign a $\underline{V}^t$-category $\underline{A}^t$ by

$$(5) \qquad \underline{A}^t(A,B) := \underline{A}(B,A), \ \iota^t_A := \iota_A, \ \mu^t_{A,B,C} := \mu_{C,B,A}.$$

This construction extends to 1-cells and 2-cells. It is a 2-functor, contravariant with respect to 2-cells (cf. [17], 2.9-2.11). The extension to the 2-category $\mathfrak{V}$ is straightforward. The general idea is to reinterpret the diagrams in terms of $\underline{V}^t$. This turns a $\underline{V}$-left module $\underline{A}$ into a $\underline{V}$-right module $\underline{A}^t = (\underline{A}_0, \otimes^t, a^t, \lambda^t)$:

$$(6) \qquad \otimes^t = \otimes \circ Tw, \ a^t_{A,X,Y} := a_{Y,X,A}, \ \lambda^t_A := \lambda_A$$

and correspondingly for 1-cells and 2-cells (cp. [2], 3 Section 3).

2.1. DEFINITION. Let $\underline{A}$ be a $\underline{V}$-category and let $\underline{B}$ be a right (!) $\underline{V}$-module. A *contravariant $\underline{V}$-functor from $\underline{A}$ to $\underline{B}$* is a $\underline{V}^t$-functor from the $\underline{V}^t$-category $\underline{A}^t$ to the $\underline{V}^t$-left module $\underline{B}^t$.

A contravariant $\underline{V}$-functor $F: \underline{A} \to \underline{B}$ consists therefore of a contravariant functor $F_0 : \underline{A}_0 \to \underline{B}_0$, together with a natural family of maps

$$F_{A,B}: FA \otimes \underline{A}(B,A) \to FB$$

such that two evident diagrams commute (cp. [17], 5+6; [19]). The contravariant hom functors

$$\underline{A}(C,-): \underline{A} \to \underline{V} \quad (\underline{A}(C,-)_{A,B} := \mu^A_{B,A,C})$$

are an example (here $\underline{V}$ denotes the $\underline{V}$-right module $(\underline{V}_0, \otimes, a, \rho)$). $\underline{V}$-bifunctors (distributors) may be defined in this situation (cp. [3], 6 Section 2; [17], 7.4 (d)). An important example is the Hom-bifunctor $\underline{A}(-,-)$ for a $\underline{V}$-category $\underline{A}$. There is an evident way of defining extraordinary $\underline{V}$-natural transformations from $X \in |\underline{C}|$ to a distributor with values in a $\underline{V}$-bimodule (cp. [1], 2.3) in the case $\underline{C} = \underline{V}$, $X = I$, such that $\iota^A$ ($\underline{A}$ a $\underline{V}$-category) is extraordinary $\underline{V}$-natural. In the general case a symmetry for $\underline{V}$ is required. The extraordinary $\underline{V}$-naturality of $\mu^A_{A,B,C}$ with respect to

$\underline{B}$ can be defined for a merely monoidal category $\underline{V}$ (cp. 3.6 below).

## 3. LIMITS.

We consider the notion of ($\underline{V}$-) limits in the 2-category $\mho$. This general notion combines and generalizes the two essentially equivalent (in the spirit of 1.9) notions of $\underline{V}$-limits as considered in [4], [17] 6.3, [19].

3.1. DEFINITION. (i) A *$\underline{V}$-natural pair $(P,\pi)$* from $E: \underline{A} \to \underline{V}$ to $F: \underline{A} \to \underline{B}$ consists of an object $P \in |\underline{B}|$, together with a 2-cell $\pi$:

a) $\pi: E \to \underline{B}(P,-) \circ F$ if $\underline{B}$ is a $\underline{V}$-category,

b) $\pi: (-\otimes P) \circ E \to F$ if $\underline{B}$ is a $\underline{V}$-module.

(ii) A *$\underline{V}$-limit (mean cotensorproduct)* of $E$ and $F$ is a $\underline{V}$-natural pair $(P,\pi)$ from $E$ to $F$ which is universal, i.e.,

a) the commutative diagram (1) (for all $A \in |\underline{A}|$) sets up a bijection (2) (for all $X \in |\underline{V}|$) between $\underline{V}$-natural pairs $(O,\omega)$ from $(-\otimes X) \circ E$ to $F$ and morphisms $p: X \to \underline{B}(O,P)$ in $\underline{B}_0$.

$$(1) \qquad \begin{tikzcd} EA \otimes X \arrow[rr, "\omega_A"] \arrow[dr, "\pi_A \otimes p"'] & & \underline{B}(O, FA) \\ & \underline{B}(P, FA) \otimes \underline{B}(O,P) \arrow[ur, "\mu^{\underline{B}}_{O,P,FA}"'] & \end{tikzcd}$$

$$(2) \qquad \frac{(-\otimes X) \circ E \xrightarrow{\ \omega\ } \underline{B}(O,-) \circ F}{X \xrightarrow{\ p\ } \underline{B}(O,P)}$$

If (2) is a bijection merely for $X = I$, then $(P,\pi)$ is called a *limit (weak mean cotensorproduct) of $E$ and $F$.*

b) the commutative diagram (3) (for all $A \in |\underline{A}|$) sets up a bijection (4) between $\underline{V}$-natural pairs $(O,\omega)$ from $E$ to $F$ and morphisms $p: O \to P$ in $\underline{B}_0$.

$$(3) \qquad \begin{tikzcd} EA \otimes O \arrow[rr, "\omega_A"] \arrow[dr, "1 \otimes p"'] & & FA \\ & EA \otimes P \arrow[ur, "\pi_A"'] & \end{tikzcd}$$

$$(4) \qquad \frac{(-\otimes O)\circ E \xrightarrow{\omega} F}{O \xrightarrow{p} P}$$

If $\underline{B}$ is a tensored $\underline{V}$-category, both notions of $\underline{V}$-limits 3.1 a, b are easily seen to be compatible, i.e., the canonical bijection between (conjugate) 2-cells

$$E \to \underline{B}(P,-)\circ F \quad \text{and} \quad (-\otimes P)\circ E \to F$$

preserves $\underline{V}$-limits (for the calculus of conjugate 2-cells, cp. e.g. [7], 1.6; [11]; [17], 4; [20], IV.7).

3.2. THEOREM *(Covariant Yoneda-Lemma). Let* $\underline{A}$ *be a* $\underline{V}$*-category and let* $\underline{B}$ *be either a* $\underline{V}$*-category or a* $\underline{V}$*-module. If* $C \in |\underline{A}|$ *and* $F: \underline{A} \to \underline{B}$ *is a 1-cell, then* $(FC, F_{C,-})$ *is a* $\underline{V}$*-limit of* $\underline{A}(C,-): \underline{A} \to \underline{V}$ *and* $F$.
(Cp. e.g. [4], 3.1; [5], 5.1; [17], 6.4; [19], 2, Theorem 3.)

PROOF. Let $\underline{B}$ be a $\underline{V}$-category.

$$F_{C,-}: \underline{A}(C,-) \to \underline{B}(FC,-)\circ F$$

is a 1-cell according to 1.6 a. If $p: X \to \underline{B}(O, FC)$ is any morphism in $\underline{V}_0$, the composition $\omega_A := \mu^{B}_{O,FC,FA}(F_{C,A}\otimes p)$ yields a 1-cell

$$\omega: (-\otimes X)\circ \underline{A}(C,-) \to \underline{B}(O,-)\circ F$$

(cp. 1.6 a, e). The morphism $p$ is uniquely determined by $\omega$ via

$$p = \omega_C(\iota^A_C \otimes X)(\lambda^V_X).$$

The converse is now obvious. The proof is analogous for a $\underline{V}$-module $\underline{B}$.

The weak Yoneda-Lemma is a consequence of 3.2 for $\underline{B} = \underline{V}$: there is a bijection between morphisms $I \to FC$ in $\underline{V}_0$ and 2-cells $\underline{A}(C,-) \to F$. 3.2 also implies the usual Yoneda-Lemma (cf. [4], 3.1) in which $\underline{V}$ is assumed to be symmetric monoidal closed and $\underline{B}$ is a $\underline{V}$-category.

3.3. DEFINITION. A 1-cell $G: \underline{B} \to \underline{C}$ *preserves a* $(\underline{V}\text{-})$ *limit* $(P, \pi)$ of $E: \underline{A} \to \underline{V}$ and $F: \underline{A} \to \underline{B}$ iff

a) if $\underline{B}$, $\underline{C}$ are $\underline{V}$-categories:
$(GP, (G_{P,-}\circ F)\pi)$ is a $(\underline{V}\text{-})$ limit of $E$ and $GF$.

b) if $\underline{B}$ is a $\underline{V}$-category, $\underline{C}$ is a $\underline{V}$-module:

$(GP, (G_{P,-} \circ F)(\pi \otimes GP))$ is a ($\underline{V}$-) limit of $E$ and $GF$.

c) if $\underline{B}$ is a $\underline{V}$-module, $\underline{C}$ is a $\underline{V}$-category:

$(GP, (\underline{C}(GP,-) \circ G \circ \pi)(G_{-,P} \circ E))$ is a ($\underline{V}$-) limit of $E$ and $GF$.

d) if $\underline{B}$, $\underline{C}$ are $\underline{V}$-modules:

$(GP, (G \circ \pi)(G_{-,P} \circ E))$ is a ($\underline{V}$-) limit of $E$ and $GF$.

3.4. PROPOSITION. *Let $\underline{B}$ be a $\underline{V}$-category.*

*(i) For every $C \in |\underline{B}|$ the 1-cell $\underline{B}(C,-): \underline{B} \to \underline{V}$ preserves $\underline{V}$-limits («hom-functors» preserve $\underline{V}$-limits).*

*(ii) Let $E: \underline{A} \to \underline{V}$ and $F: \underline{A} \to \underline{B}$ be 1-cells, $P \in |\underline{B}|$, and let*

$$\pi = \{ \pi_A : EA \to \underline{B}(P, FA) \mid A \in |\underline{A}| \}.$$

*If*

$$(\underline{B}(C,P), \{ (\mu^{\underline{B}}_{C,P,FA}) \cdot (\pi_A \otimes \underline{B}(C,P))) \mid A \in |\underline{A}| \})$$

*is a $\underline{V}$-limit of $E$ and $\underline{B}(C,-) \circ F$ for every $C \in |\underline{B}|$, then $(P, \pi)$ is a $\underline{V}$-limit of $E$ and $F$ («hom-functors» collectively detect $\underline{V}$-limits).*

PROOF. (i) is an immediate consequence of the Definition 3.1. In fact, if only the notion 3.1 (ii) b were known, we would use the assertions in 3.4 as a gauge for the choice of the definition of $\underline{V}$-limits in $\underline{V}$-categories.

(ii) According to our last remark we have only to prove that $\pi$ is a 2-cell in $\mathfrak{V}$. This follows easily on choosing $C := P$.

We can also consider the dual notion of colimits if $\underline{V}$ is merely monoidal.

3.5. DEFINITION. Let $\underline{A}$ be a $\underline{V}$-category, let $F: \underline{A} \to \underline{B}$ be a 1-cell and let $E: \underline{A} \to \underline{V}$ be a contravariant $\underline{V}$-functor. A *$\underline{V}$-natural pair $(P, \pi)$ for $E$ and $F$* consists of an object $P \in |\underline{B}|$, together with a natural family $\pi = \{\pi_A \mid A \in |\underline{A}| \}$:

a) $\pi_A : EA \to \underline{B}(FA, P)$ if $\underline{B}$ is a $\underline{V}$-category;

b) $\pi_A : EA \otimes FA \to P$ if $\underline{B}$ is a $\underline{V}$-module,

such that an evident diagram commutes. A couniversal $\underline{V}$-natural pair is called a *tensorproduct of $E$ with $F$* (over $\underline{A}$).

3.6. THEOREM *(Contravariant Yoneda-Lemma). Let* $\underline{A}$ *be a* $\underline{V}$*-category and let* $\underline{B}$ *be either a* $\underline{V}$*-category or a* $\underline{V}$*-module. If* $C \in |\underline{A}|$ *and* $F: \underline{A} \to \underline{B}$ *is a 1-cell, then* $(FC, F_{-,C})$ *is a* $\underline{V}$*-colimit of the contravariant* $\underline{V}$*-functor* $\underline{A}(-, C): \underline{A} \to \underline{V}$ *and* $F$.

(Cp. e.g. [17], 6.10; [19].) The proof is dual to the proof of 3.2.

The proof of the following proposition is straightforward.

3.7. PROPOSITION. *Adjoint 1-cells preserve* $\underline{V}$*-limits.*

## 4. KAN EXTENSIONS.

The definition of Kan extensions can be formulated in any 2-category: $(K, \kappa)$ is called a *Kan extension* of a 1-cell $J: \underline{A} \to \underline{D}$ along a 1-cell $F: \underline{A} \to \underline{B}$ iff $K: \underline{B} \to \underline{D}$ is a 1-cell and $\kappa: KF \to J$ is a 2-cell (cf. (1)), such that the assignment (2) is a bijection (3) for every 1-cell $L: \underline{B} \to \underline{D}$.

(1)
$$\begin{tikzcd} \underline{A} \arrow[rr, "F"] \arrow[dr, "J"'] & \arrow[d, phantom, "\overset{\kappa}{\leftarrow}"] & \underline{B} \arrow[dl, "K"] \\ & \underline{D} & \end{tikzcd}$$

(2) $$(\chi: L \to K) \mapsto \kappa(\chi \circ F) \ ; \qquad (3) \quad \frac{\chi: L \to K}{\omega: LF \to J}.$$

A 1-cell $R: \underline{D} \to \underline{E}$ *respects a Kan extension* $(K, \kappa)$ of $J$ along $F$ iff $(RK, R\kappa)$ is a Kan extension of $RJ$ along $F$. If, in particular, $\underline{D}$ is a $\underline{V}$-category, the hom-functors of $\underline{D}$ need not respect Kan extensions. The Kan extensions respected by all hom-functors are called *pointwise Kan extensions* (if we assume $(RK, R\kappa)$ to be a Kan extension for every hom-functor $R$, then $(K, \kappa)$ can be shown to be a Kan extension).

4.1. DEFINITION. Let $\underline{V}$ be a symmetric monoidal category, and let

$$J: \underline{A} \to \underline{D}, \quad F: \underline{A} \to \underline{B}, \quad K: \underline{B} \to \underline{D}$$

be 1-cells and let $\kappa: KF \to J$ be a 2-cell (cp. (1)). $(K, \kappa)$ is called a $\underline{V}$*-Kan extension of* $J$ *along* $F$ iff:

a) (if $\underline{D}$ is a $\underline{V}$-category) the commutative diagram (4) (for all $A \in |\underline{A}|$)

sets up a bijection (5) (for all $X \in |\underline{V}|$ and 1-cells $L: \underline{B} \to \underline{D}$) between extraordinary $\underline{V}$-natural transformations $\chi$ and $\omega$.

(4)
$$\begin{tikzcd} X \arrow[rr, "\omega A"] \arrow[dr, "\chi FA"'] & & \underline{D}(LFA, JA) \\ & \underline{D}(LFA, KFA) \arrow[ur, "\underline{D}(1, \kappa_A)"'] & \end{tikzcd}$$

(5)
$$\frac{\chi : X \to Hom_{\underline{D}} \circ (L^0 \otimes K)}{\omega : X \to Hom_{\underline{D}} \circ ((LF)^0 \otimes J)}$$

b) (if $\underline{D}$ is a $\underline{V}$-module) the commutative diagram (6) (for all $A \in |\underline{A}|$) sets up a bijection (7) between 1-cells $\chi$ and $\omega$.

(6)
$$\begin{tikzcd} X \otimes LFA \arrow[rr, "\omega A"] \arrow[dr, "\chi FA"'] & & JA \\ & KFA \arrow[ur, "\kappa_A"'] & \end{tikzcd}$$

(7)
$$\frac{\chi : (X \otimes -) \circ L \to K}{\omega : (X \otimes -) \circ L \circ F \to J}$$

A 1-cell $R: \underline{D} \to \underline{E}$ is said to *respect a $\underline{V}$-Kan extension* $(K, \kappa)$ iff $(RK, R\kappa)$ is a $\underline{V}$-Kan extension of $RJ$ along $F$.

4.2. THEOREM. *Let $\underline{V}$ be symmetric monoidal and let $\underline{D}$ be a $\underline{V}$-category.*

*(i) Every $\underline{V}$-Kan extension $(K, \kappa)$ of $J: \underline{A} \to \underline{D}$ along $F: \underline{A} \to \underline{B}$ is a Kan extension.*

*(ii) Every pointwise Kan extension $(K, \kappa)$ of $J: \underline{A} \to \underline{D}$ along $F: \underline{A} \to \underline{B}$ is a $\underline{V}$-Kan extension.*

We remark that every Kan extension is a $\underline{V}$-Kan extension in the case $\underline{V} = Ens$, the category of sets. This is certainly the reason why $\underline{V}$-Kan extensions apparently have not yet been considered in the literature. The usual connection between pointwise Kan extensions and $\underline{V}$-limits remains valid if $\underline{V}$ is merely monoidal:

4.3. THEOREM. *$(K, \kappa)$ is a pointwise Kan extension of $J: \underline{A} \to \underline{D}$ along $F: \underline{A} \to \underline{B}$ iff $(KB, \pi_B)$, determined by*

$$\pi_B := D(KB, \kappa) \circ K_{B,-} \circ F$$

*is a* $\underline{V}$*-limit of* $\underline{B}(B,-) \circ F$ *and* $J$ *for every* $B \in |\underline{B}|$.

4.4. REMARK. Several other notions may be defined for merely monoidal categories $\underline{V}$ by means of Kan extension. E. g., a 1-cell $F: \underline{A} \to \underline{B}$ is called *codense* iff $(1_{\underline{B}}, 1_F)$ is a Kan extension of $F$ along $F$. Also, final and initial 1-cells (in the non-topological sense) may be defined (cf. [15], 4 (10)-(12)).

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**REFERENCES.**

(We abbreviate *Lecture Notes in Mathematics, Springer*, by *LN*.)

1. BENABOU, J., Introduction to bicategories, *LN* 47 (1967), 1-77.
2. BENABOU, J., Les catégories multiplicatives, *Rapport Inst. Math. Pure et App. Univ. Cath. Louvain* 27 (1972).
3. BENABOU, J., Les distributeurs, *Ibidem* 33 (1973).
4. BORCEUX, F. & KELLY, G.M., A notion of limit for enriched categories, *Bull. Austral. Math. Soc.* 12 (1975), 49-72.
5. DAY, B.J. & KELLY, G.M., Enriched functor categories, *LN* 106 (1969), 178-191.
6. EILENBERG, S. & KELLY, G.M., A generalization of the functorial calculus, *J. Algebra* 3 (1966), 366-375.
7. GRAY, J.W., Formal category theory: Adjunctions for 2-categories, *LN* 391 (1974).
8. KELLY, G.M., Adjunction for enriched categories, *LN* 106 (1969), 166-177.
9. KELLY, G.M., Doctrinal adjunction, *LN* 420 (1974), 257-280.
10. KELLY, G.M., Saunders MacLane and category theory, In *Saunders Mac Lane Selected papers* (edited by I. Kaplansky), Springer (1979), 527-543.
11. KELLY, G.M. & STREET, R., Review of the elements of 2-categories, *LN* 420 (1974), 75-103.
12. KOCK, A., Strong functors and monoidal monads, *Arch. of Math.* 23 (1972), 113-120.
13. LAWVERE, F.W., Ordinal sums and equational doctrines, *LN* 80 (1969), 141-155.
14. LINDNER, H., Adjunctions in monoidal categories, *Manuscripta Math.* 26 (1978) 123-139.
15. LINDNER, H., Center and trace: a) *Seminarberichte Fernuniv. Hagen* 7 (1980), 149-181; b) *Arch. of Math.*, 35 (1980), 476-496.
16. LINDNER, H., Monads generated by monoids, *Manuscripta Math.* 15 (1975), 139-152.
17. LINDNER, H., *Monoidale und geschlossene Kategorien*, Habilitationsschrift, Univ. Düsseldorf, 1976.
18. LINTON, F.E.J., Sur les choix de variance prédestinés, Exposé oral, Colloque Amiens 1975 (non publié).
19. LINTON, F.E.J., The multilinear Yoneda-Lemmas, *LN* 195 (1971), 209-229.
20. MACLANE S., *Categories for the working mathematician*, Springer, 1971.