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HARALD LINDNER

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ENRICHED CATEGORIES AND ENRICHED MODULES

by Harald LINDNER

Our purpose is to show that most of the results on categories enriched over a symmetric monoidal closed category \underline{V} can be formulated and proved in the merely monoidal case. This permits to apply the theory of enriched categories to further examples, to gain a better understanding of the basic notions of (enriched) category theory, and to present enriched category theory more concisely.

An important tool is the notion of enriched modules (Bénabou: «actions of multiplicative categories»), i. e., categories on which a monoidal category acts. We hope to show that the two notions of enriched categories and enriched modules are equally important. These two kinds of objects are the 0-cells of two well-known 2-categories. We have described in previous papers how these two 2-categories can be embedded into a 2-category \mathfrak{U} by introducing 1-cells (and 2-cells) from \underline{V} -categories to \underline{V} -modules, and vice versa. Our examples prove that such 1-cells and 2-cells occur naturally even in the familiar symmetric monoidal closed case.

The key result (1.9) is a characterization of tensored \underline{V} -categories in terms of isomorphisms between enriched categories and enriched modules. We discuss duality, limits and Kan-extensions in our context. Details on further topics such as functor categories will be considered elsewhere. Proofs are usually omitted.

1. THE 2-CATEGORY \mathfrak{U} OF ENRICHED CATEGORIES AND ENRICHED MODULES.

We recall the definition of the 2-category \mathfrak{U} (cf. [15, 17]). Let $\underline{V} = (\underline{V}_0, \otimes, I, a, \lambda, \rho)$ be a monoidal category, i. e., $\otimes: \underline{V}_0 \times \underline{V}_0 \rightarrow \underline{V}_0$ is a functor (written between its arguments), I is an object of \underline{V}_0 , and

$\alpha_{X,Y,Z}: X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$, $\lambda_X: X \rightarrow I \otimes X$, $\rho_X: X \rightarrow X \otimes I$
are compatible natural transformations.

1.1. DEFINITION. A \underline{V} - (left-) module $\underline{A} = (\underline{A}_0, \otimes^A, \alpha^A, \lambda^A)$ consists: of a category \underline{A}_0 , a functor $\otimes^A: \underline{V}_0 \times \underline{A}_0 \rightarrow \underline{A}_0$, and two natural transformations α^A, λ^A

$$\alpha_{X,Y,A}^A: X \otimes^A (Y \otimes^A A) \rightarrow (X \otimes^V Y) \otimes^A A, \quad \lambda_A^A: A \rightarrow I \otimes^A A$$

such that three evident diagrams commute. \underline{A} is called *normal* if α^A and λ^A are both isomorphic; their inverses are then denoted by β^A and ν^A , respectively.

(Cf. [1], 2.3 («actions of multiplicative categories»); [2], 3, Section 1; [15], 5.1; [16], 2; [17], 5.1.)

$(\underline{V}_0, \otimes^V, \alpha^V, \lambda^V)$ is an example of a normal module which we usually denote by \underline{V} , if there is no danger of confusion. Also, we often drop the indices $\underline{A}, \underline{V}, X, Y, A$ of $\otimes^A, \otimes^V, \alpha_{X,Y,A}^A$, etc..., if the context seems to exclude any danger of confusion. We often write $|\underline{A}|$ instead of $|\underline{A}_0|$ for the class of objects of a \underline{V} -module \underline{A} . If $|\underline{A}|$ is a set, \underline{A} is called *small*. If \underline{A} is a tensored \underline{V} -category, \underline{A} is canonically equipped with the structure of a normal \underline{V} -module (cp. 1.9 below).

1.2. DEFINITION. A *1-cell* $F: \underline{A} \rightarrow \underline{B}$ in \mathfrak{U} consists of a functor

$$F_0: \underline{A}_0 \rightarrow \underline{B}_0 \quad (\text{we often omit the index «0»),$$

together with a natural family of morphisms in \underline{V}_0 or \underline{B}_0 , indexed by pairs of objects $A, B \in |\underline{A}|$ or $X \in \underline{V}, A \in |\underline{A}|$, resp.

- a) $F_{A,B}: \underline{A}(A, B) \rightarrow \underline{B}(FA, FB)$ if $\underline{A}, \underline{B}$ are \underline{V} -categories,
- b) $F_{A,B}: \underline{A}(A, B) \otimes FA \rightarrow FB$ if \underline{A} is a \underline{V} -category, \underline{B} is a \underline{V} -module,
- c) $F_{X,A}: X \rightarrow \underline{B}(FA, F(X \otimes A))$ if \underline{A} is a \underline{V} -module, \underline{B} is a \underline{V} -category,
- d) $F_{X,A}: X \otimes FA \rightarrow F(X \otimes A)$ if $\underline{A}, \underline{B}$ are \underline{V} -modules,

such that two evident corresponding diagrams commute, e. g. in case c:

$$\begin{array}{ccc} \text{c) (i)} & X \otimes Y \xrightarrow{F_{X,Y \otimes A} \otimes F_{Y,A}} \underline{B}(F(Y \otimes A), F(X \otimes (Y \otimes A))) \otimes \underline{B}(FA, F(Y \otimes A)) & \\ & \downarrow F_{X \otimes Y, A} & \downarrow \mu^B \\ & \underline{B}(FA, F((X \otimes Y) \otimes A)) \xleftarrow{\underline{B}(I, F\alpha^A)} \underline{B}(FA, F(X \otimes (Y \otimes A))) & \end{array}$$

(ii)

$$\begin{array}{ccc}
 I & \xrightarrow{F_{I,A}} & \underline{B}(FA, F(I \otimes A)) \\
 & \searrow \iota_{FA}^{\underline{B}} & \nearrow \underline{B}(1, F\lambda_A^A) \\
 & & \underline{B}(FA, FA)
 \end{array}$$

(cf. e.g., [12], 1; [15], 5.2; [17], 5).

1.3. EXAMPLES. (i) Let C be an object of a \underline{V} -category \underline{A} . The hom functor $\underline{A}_o(C, -): \underline{A}_o \rightarrow \underline{V}_o$, together with the family

$$\underline{A}(C, -)_{A,B} := \mu_{C,A,B}^{\underline{A}}: \underline{A}(A, B) \otimes \underline{A}(C, A) \rightarrow \underline{A}(C, B),$$

is a 1-cell in the sense of 1.2 (b). (Cf. [19]; [17], 5.7.)

(ii) Let C be an object of a \underline{V} -module \underline{B} . The functor $(-\otimes C): \underline{V}_o \rightarrow \underline{B}_o$ together with the family

$$(-\otimes C)_{X,Y} := \alpha_{X,Y,C}^{\underline{B}}: X \otimes (Y \otimes C) \rightarrow (X \otimes Y) \otimes C$$

is a 1-cell from \underline{V} to \underline{B} in the sense of 1.2 (d).

1.4. DEFINITION. The composition of 1-cells $F: \underline{A} \rightarrow \underline{B}$ and $G: \underline{B} \rightarrow \underline{C}$ in \mathcal{U} is defined by composing the underlying functors F_o and G_o and by, e.g.,

$$\underline{A}(A, B) \otimes GFA \xrightarrow{G_{\underline{A}(A,B),FA}} G(\underline{A}(A, B) \otimes FA) \xrightarrow{G(F_{A,B})} GFB$$

if \underline{A} is a \underline{V} -category and $\underline{B}, \underline{C}$ are \underline{V} -modules.

1.5. DEFINITION. A 2-cell $\theta: F \rightarrow H: \underline{A} \rightarrow \underline{B}$ in \mathcal{U} is a natural transformation $\theta: F_o \rightarrow H_o$ such that an evident diagram commutes, e.g. in case c:

c)

$$\begin{array}{ccc}
 X & \xrightarrow{F_{X,A}} & \underline{B}(FA, F(X \otimes A)) \\
 \downarrow H_{X,A} & & \downarrow \underline{B}(1, \theta_{X \otimes A}) \\
 \underline{B}(HA, H(X \otimes A)) & \xrightarrow{\underline{B}(\theta_A, 1)} & \underline{B}(FA, H(X \otimes A))
 \end{array}$$

The composition of 2-cells is evident. We leave to the reader the straightforward proof that these definitions yield a 2-category \mathcal{U} (cf. [15], 5).

1.6. EXAMPLES OF 2-CELLS IN \mathcal{U} . Let $F: \underline{A} \rightarrow \underline{B}$ be a 1-cell in \mathcal{U} and let $A \in |\underline{A}|$. We consider the four cases a-d in 1.2:

a) $F_{A,-}: \underline{A}(A, -) \rightarrow \underline{B}(FA, -) \circ F$ (cf. (1)),

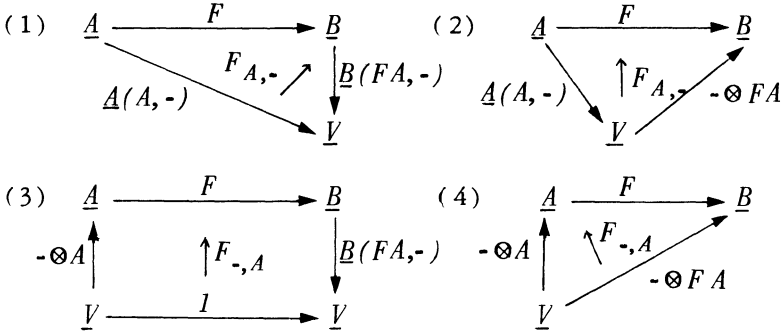
b) $F_{A,-}: (- \otimes FA) \circ \underline{A}(A, -) \rightarrow F$ (cf. (2)),

c) $F_{-,A}: I_{\underline{V}} \rightarrow \underline{B}(FA, -) \circ F \circ (- \otimes A)$ (cf. (3)),

d) $F_{-,A}: - \otimes FA \rightarrow F \circ (- \otimes A)$ (cf. (4)),

e) $\mu_{A,B,-}^A: (- \otimes \underline{A}(A, B)) \circ \underline{A}(B, -) \rightarrow \underline{A}(A, -)$ is a 2-cell. This is a specialization of b (cp. 1.3 (i)).

f) $\alpha_{-,Y,A}: (- \otimes (Y \otimes A)) \rightarrow (- \otimes A) \circ (- \otimes Y)$ is a 2-cell. This is a specialisation of d (cp. 1.3 (ii)).



In this setup we are able to extend the usual definition of tensored \underline{V} -categories (cf. [8], 4), in which \underline{V} had to be symmetric monoidal closed, to the case of a merely monoidal category (cp. [10], 9).

1.7. DEFINITION. A *tensored \underline{V} -category* consists of a \underline{V} -category \underline{C} together with an adjunction (5) in \mathcal{C} for every $A \in |\underline{C}|$ (cf. 1.3 (i)):

$$(5) \quad (- \otimes A) \xrightleftharpoons[i_{-,A}]{e_{A,-}} \underline{C}(A, -): \underline{C} \rightarrow \underline{V}.$$

Although a tensored \underline{V} -category consists of a \underline{V} -category \underline{C} together with additional data, rather than a specific property of \underline{C} , it is customary to denote a tensored \underline{V} -category by the same symbol as the «underlying» \underline{V} -category \underline{C} . This is of course justified to some extent, since (co-)adjoints are determined uniquely up to isomorphism. The reader is invited to draw the commutative diagrams, provided by 1.7, for later reference.

As an example we list the adjunction equations:

$$(6) \quad \underline{C}(A, B) \xrightarrow{i_{\underline{C}(A, B), A}} \underline{C}(A, C(A, B) \otimes A) \xrightarrow{\underline{C}(A, e_{A, B})} \underline{C}(A, B) = I_{\underline{C}(A, B)},$$

$$(7) \quad X \otimes A \xrightarrow{i_{X,A} \otimes A} \underline{C}(A, X \otimes A) \otimes A \xrightarrow{e_{A, X \otimes A}} X \otimes A = I_{X \otimes A}$$

for all $A, B \in |\underline{C}|$, $X \in |\underline{V}|$.

The Definition 1.7 can be «translated» to the case of \underline{V} -modules (cp. 1.9 below):

1.8. DEFINITION AND PROPOSITION. A *tensoried \underline{V} -module* consists of a \underline{V} -module \underline{C} , such that $\lambda^{\underline{C}}$ is isomorphic, together with an adjunction (8) for every $A \in |\underline{C}|$. Every tensoried \underline{V} -module is normal.

$$(8) \quad (- \otimes A) \xrightleftharpoons[i_{-,A}]{e_{A,-}} \underline{C}(A, -): \underline{C} \rightarrow \underline{V}.$$

Although the adjunctions (5) and (8) look equal, we should like to emphasize that they are different because \underline{C} denotes a \underline{V} -category in 1.7 and a \underline{V} -module in 1.8. In particular, the «structure maps» of the 1-cells in (5) and (8) in the nontrivial cases are:

$$(9) \quad (- \otimes A)_{X,Y}: X \rightarrow \underline{C}(Y \otimes A, (X \otimes Y) \otimes A),$$

$$(10) \quad \underline{C}(A, -)_{X,B}: X \otimes \underline{C}(A, B) \rightarrow \underline{C}(A, X \otimes B).$$

1.9. THEOREM. *There is a canonical bijection between:*

(i) *tensoried \underline{V} -categories,*

(ii) *tensoried \underline{V} -modules,*

(iii) *isomorphisms between \underline{V} -categories and \underline{V} -modules such that the underlying functors are identities.*

We must leave the proof to the reader (cp. [17], 5.11).

On applying the Theorem 1.9 to $\underline{A} = \underline{V}$ if \underline{V} is symmetric monoidal closed we recognize the Definition 1.7 of tensoried \underline{V} -categories as compatible with the classical case (cf. [8], 4).

1.10. REMARK. We stress the importance of the statement (iii) in 1.9: if \underline{A} and/or \underline{B} are tensoried \underline{V} -categories, the different notions of 1-cells $\underline{A} \rightarrow \underline{B}$ in 1.2 are in a bijective correspondence, set up by composing with the isomorphisms between the \underline{V} -category and \underline{V} -module structures. In particular, these notions are compatible. In this way we can extend most no-

tions in enriched category theory from monoidal closed categories \underline{V} to merely monoidal categories \underline{V} .

In the next sections we take the first steps in this direction. Most results are contained in a slightly different form in previous papers (e. g., [17]). The present setting - the 2-category \mathfrak{U} - permits a nice formulation.

A common generalization of the two notions of objects in \mathfrak{U} appears to be very tempting. In fact, in [18] such a generalization was given. In this way \underline{V} -modules and \underline{V} -categories can be treated simultaneously. On the other hand, it appears as if additional work were required in order to re-interpret results in terms of the familiar notions of \underline{V} -modules and \underline{V} -categories. Also, the translation of a notion from \underline{V} -categories to \underline{V} -modules and vice versa is often quite straightforward.

With regard to 1.9 we may consider 1-cells from a \underline{V} -category \underline{A} to a \underline{V} -module \underline{B} (in particular $\underline{B} = \underline{V}$) as genuine generalizations of \underline{V} -functors. We shall therefore often call these 1-cells and the corresponding 2-cells, \underline{V} -functors and \underline{V} -natural transformations, respectively.

2. DUALITY.

The dual of a \underline{V} -category as well as contravariant \underline{V} -functors between \underline{V} -categories cannot be defined unless \underline{V} is symmetric. In particular, the definition of extraordinary \underline{V} -natural transformations requires a symmetry. However, certain parts of this duality for \underline{V} -categories are independent of a symmetry (cf. [19, 17]).

To a monoidal category $\underline{V} = (\underline{V}_0, \otimes, I, \alpha, \lambda, \rho)$ we may assign an opmonoidal (cp. (2); the brackets are shifted the other direction) category $\underline{V}^t = (\underline{V}_0, \otimes^t, I, \alpha^t, \lambda^t, \rho^t)$, the transpose of \underline{V} by:

$$(1) \quad \begin{array}{ccc} \underline{V}_0 \times \underline{V}_0 & \xrightarrow{Tw} & \underline{V}_0 \times \underline{V}_0 \\ & \searrow \otimes^t & \swarrow \otimes \\ & \underline{V}_0 & \end{array}$$

(2) $\alpha^t_{X,Y,Z} := \alpha_{Z,Y,X}$; (3) $\lambda^t := \rho$; (4) $\rho^t := \lambda$
 (Tw denotes twisting of the arguments, i. e., $Tw(X, Y) = (Y, X)$ etc.).

Clearly $\underline{V}^{tt} = \underline{V}$. Symmetries γ for \underline{V} are in bijection with monoidal functors $\Gamma = (I_{\underline{V}_0}, \gamma, I_I): \underline{V}^t \rightarrow \underline{V}$ which are quasi-involutive, i. e., $\Gamma(\Gamma^t) = I$ (but $\Gamma\Gamma$ is not defined). By inverting α^t we obtain an (honest) monoidal category $\underline{V}^s = (\underline{V}_0, \otimes^t, (\alpha^t)^{-1}, \lambda^t, \rho^t)$ (cf. e. g. [17], 1.3). To a \underline{V} -category \underline{A} we assign a \underline{V}^t -category \underline{A}^t by

$$(5) \quad \underline{A}^t(A, B) := \underline{A}(B, A), \quad \iota_A^t := \iota_A, \quad \mu_{A,B,C}^t := \mu_{C,B,A}.$$

This construction extends to 1-cells and 2-cells. It is a 2-functor, contravariant with respect to 2-cells (cf. [17], 2.9-2.11). The extension to the 2-category \mathfrak{C} is straightforward. The general idea is to reinterpret the diagrams in terms of \underline{V}^t . This turns a \underline{V} -left module \underline{A} into a \underline{V} -right module $\underline{A}^t = (\underline{A}_0, \otimes^t, \alpha^t, \lambda^t)$:

$$(6) \quad \otimes^t = \otimes \circ Tw, \quad \alpha_{A,X,Y}^t := \alpha_{Y,X,A}, \quad \lambda_A^t := \lambda_A$$

and correspondingly for 1-cells and 2-cells (cp. [2], 3 Section 3).

2.1. DEFINITION. Let \underline{A} be a \underline{V} -category and let \underline{B} be a right (!) \underline{V} -module. A *contravariant \underline{V} -functor from \underline{A} to \underline{B}* is a \underline{V}^t -functor from the \underline{V}^t -category \underline{A}^t to the \underline{V}^t -left module \underline{B}^t .

A contravariant \underline{V} -functor $F: \underline{A} \rightarrow \underline{B}$ consists therefore of a contravariant functor $F_0: \underline{A}_0 \rightarrow \underline{B}_0$, together with a natural family of maps

$$F_{A,B}: F A \otimes \underline{A}(B, A) \rightarrow F B$$

such that two evident diagrams commute (cp. [17], 5+6; [19]). The contravariant hom functors

$$\underline{A}(C, -): \underline{A} \rightarrow \underline{V} \quad (\underline{A}(C, -)_{A,B} := \mu_{B,A,C}^A)$$

are an example (here \underline{V} denotes the \underline{V} -right module $(\underline{V}_0, \otimes, \alpha, \rho)$). \underline{V} -bifunctors (distributors) may be defined in this situation (cp. [3], 6 Section 2; [17], 7.4 (d)). An important example is the Hom-bifunctor $\underline{A}(-, -)$ for a \underline{V} -category \underline{A} . There is an evident way of defining extraordinary \underline{V} -natural transformations from $X \in |\underline{C}|$ to a distributor with values in a \underline{V} -bimodule (cp. [1], 2.3) in the case $\underline{C} = \underline{V}$, $X = I$, such that ι^A (\underline{A} a \underline{V} -category) is extraordinary \underline{V} -natural. In the general case a symmetry for \underline{V} is required. The extraordinary \underline{V} -naturality of $\mu_{A,B,C}^A$ with respect to

\underline{B} can be defined for a merely monoidal category \underline{V} (cp. 3.6 below).

3. LIMITS.

We consider the notion of (\underline{V} -) limits in the 2-category \mathcal{U} . This general notion combines and generalizes the two essentially equivalent (in the spirit of 1.9) notions of \underline{V} -limits as considered in [4], [17] 6.3, [19].

3.1. DEFINITION. (i) A \underline{V} -natural pair (P, π) from $E: \underline{A} \rightarrow \underline{V}$ to $F: \underline{A} \rightarrow \underline{B}$ consists of an object $P \in |\underline{B}|$, together with a 2-cell π :

- a) $\pi: E \rightarrow \underline{B}(P, -) \circ F$ if \underline{B} is a \underline{V} -category,
- b) $\pi: (- \otimes P) \circ E \rightarrow F$ if \underline{B} is a \underline{V} -module.

(ii) A \underline{V} -limit (mean cotensorproduct) of E and F is a \underline{V} -natural pair (P, π) from E to F which is universal, i. e.,

- a) the commutative diagram (1) (for all $A \in |\underline{A}|$) sets up a bijection
- (2) (for all $X \in |\underline{V}|$) between \underline{V} -natural pairs (O, ω) from $(- \otimes X) \circ E$ to F and morphisms $p: X \rightarrow \underline{B}(O, P)$ in \underline{B}_0 .

$$(1) \quad \begin{array}{ccc} EA \otimes X & \xrightarrow{\omega_A} & \underline{B}(O, FA) \\ & \searrow \pi_A \otimes p & \nearrow \mu_{O, P, FA}^{\underline{B}} \\ & & \underline{B}(P, FA) \otimes \underline{B}(O, P) \end{array}$$

$$(2) \quad \frac{(- \otimes X) \circ E \xrightarrow{\omega} \underline{B}(O, -) \circ F}{X \xrightarrow{p} \underline{B}(O, P)}$$

If (2) is a bijection merely for $X = I$, then (P, π) is called a *limit (weak mean cotensorproduct) of E and F* .

- b) the commutative diagram (3) (for all $A \in |\underline{A}|$) sets up a bijection
- (4) between \underline{V} -natural pairs (O, ω) from E to F and morphisms $p: O \rightarrow P$ in \underline{B}_0 .

$$(3) \quad \begin{array}{ccc} EA \otimes O & \xrightarrow{\omega_A} & FA \\ & \searrow I \otimes p & \nearrow \pi_A \\ & & EA \otimes P \end{array}$$

$$(4) \quad \frac{(- \otimes O) \circ E \xrightarrow{\omega} F}{O \xrightarrow{P} P}$$

If \underline{B} is a tensored \underline{V} -category, both notions of \underline{V} -limits 3.1 a, b are easily seen to be compatible, i. e., the canonical bijection between (conjugate) 2-cells

$$E \rightarrow \underline{B}(P, -) \circ F \quad \text{and} \quad (- \otimes P) \circ E \rightarrow F$$

preserves \underline{V} -limits (for the calculus of conjugate 2-cells, cp. e. g. [7], 1.6; [11]; [17], 4; [20], IV.7).

3.2. THEOREM (*Covariant Yoneda-Lemma*). Let \underline{A} be a \underline{V} -category and let \underline{B} be either a \underline{V} -category or a \underline{V} -module. If $C \in |\underline{A}|$ and $F: \underline{A} \rightarrow \underline{B}$ is a 1-cell, then $(FC, F_{C, -})$ is a \underline{V} -limit of $\underline{A}(C, -): \underline{A} \rightarrow \underline{V}$ and F .

(Cp. e. g. [4], 3.1; [5], 5.1; [17], 6.4; [19], 2, Theorem 3.)

PROOF. Let \underline{B} be a \underline{V} -category.

$$F_{C, -}: \underline{A}(C, -) \rightarrow \underline{B}(FC, -) \circ F$$

is a 1-cell according to 1.6 a. If $p: X \rightarrow \underline{B}(O, FC)$ is any morphism in \underline{V}_0 , the composition $\omega_A := \mu_{O, FC, FA}^B(F_{C, A} \otimes p)$ yields a 1-cell

$$\omega: (- \otimes X) \circ \underline{A}(C, -) \rightarrow \underline{B}(O, -) \circ F$$

(cp. 1.6 a, e). The morphism p is uniquely determined by ω via

$$p = \omega_C(\iota_C^A \otimes X)(\lambda_X^V).$$

The converse is now obvious. The proof is analogous for a \underline{V} -module \underline{B} .

The weak Yoneda-Lemma is a consequence of 3.2 for $\underline{B} = \underline{V}$: there is a bijection between morphisms $I \rightarrow FC$ in \underline{V}_0 and 2-cells $\underline{A}(C, -) \rightarrow F$. 3.2 also implies the usual Yoneda-Lemma (cf. [4], 3.1) in which \underline{V} is assumed to be symmetric monoidal closed and \underline{B} is a \underline{V} -category.

3.3. DEFINITION. A 1-cell $G: \underline{B} \rightarrow \underline{C}$ preserves a (\underline{V} -) limit (P, π) of $E: \underline{A} \rightarrow \underline{V}$ and $F: \underline{A} \rightarrow \underline{B}$ iff

a) if $\underline{B}, \underline{C}$ are \underline{V} -categories:

($GP, (G_P \circ F)\pi$) is a (\underline{V} -) limit of E and GF .

b) if \underline{B} is a \underline{V} -category, \underline{C} is a \underline{V} -module :

$(GP, (G_{P,-} \circ F)(\pi \otimes GP))$ is a $(\underline{V}-)$ limit of E and GF .

c) if \underline{B} is a \underline{V} -module, \underline{C} is a \underline{V} -category :

$(GP, (\underline{C}(GP, -) \circ G \circ \pi)(G_{-,P} \circ E))$ is a $(\underline{V}-)$ limit of E and GF .

d) if $\underline{B}, \underline{C}$ are \underline{V} -modules :

$(GP, (G \circ \pi)(G_{-,P} \circ E))$ is a $(\underline{V}-)$ limit of E and GF .

3.4. PROPOSITION. Let \underline{B} be a \underline{V} -category.

(i) For every $C \in |\underline{B}|$ the 1-cell $\underline{B}(C, -) : \underline{B} \rightarrow \underline{V}$ preserves \underline{V} -limits («hom-functors» preserve \underline{V} -limits).

(ii) Let $E : \underline{A} \rightarrow \underline{V}$ and $F : \underline{A} \rightarrow \underline{B}$ be 1-cells, $P \in |\underline{B}|$, and let

$$\pi = \{ \pi_A : EA \rightarrow \underline{B}(P, FA) \mid A \in |\underline{A}| \}.$$

If

$$(\underline{B}(C, P), \{ (\mu_{C,P,FA}^{\underline{B}}) \cdot (\pi_A \otimes \underline{B}(C, P)) \mid A \in |\underline{A}| \})$$

is a \underline{V} -limit of E and $\underline{B}(C, -) \circ F$ for every $C \in |\underline{B}|$, then (P, π) is a \underline{V} -limit of E and F («hom-functors» collectively detect \underline{V} -limits).

PROOF. (i) is an immediate consequence of the Definition 3.1. In fact, if only the notion 3.1 (ii) b were known, we would use the assertions in 3.4 as a gauge for the choice of the definition of \underline{V} -limits in \underline{V} -categories.

(ii) According to our last remark we have only to prove that π is a 2-cell in \mathcal{U} . This follows easily on choosing $C := P$.

We can also consider the dual notion of colimits if \underline{V} is merely monoidal.

3.5. DEFINITION. Let \underline{A} be a \underline{V} -category, let $F : \underline{A} \rightarrow \underline{B}$ be a 1-cell and let $E : \underline{A} \rightarrow \underline{V}$ be a contravariant \underline{V} -functor. A \underline{V} -natural pair (P, π) for E and F consists of an object $P \in |\underline{B}|$, together with a natural family $\pi = \{ \pi_A \mid A \in |\underline{A}| \}$:

a) $\pi_A : EA \rightarrow \underline{B}(FA, P)$ if \underline{B} is a \underline{V} -category;

b) $\pi_A : EA \otimes FA \rightarrow P$ if \underline{B} is a \underline{V} -module,

such that an evident diagram commutes. A couniversal \underline{V} -natural pair is called a *tensorproduct of E with F (over \underline{A})*.

3.6. THEOREM (*Contravariant Yoneda-Lemma*). Let \underline{A} be a \underline{V} -category and let \underline{B} be either a \underline{V} -category or a \underline{V} -module. If $C \in |\underline{A}|$ and $F: \underline{A} \rightarrow \underline{B}$ is a 1-cell, then $(FC, F_{-,C})$ is a \underline{V} -colimit of the contravariant \underline{V} -functor $\underline{A}(-, C): \underline{A} \rightarrow \underline{V}$ and F .

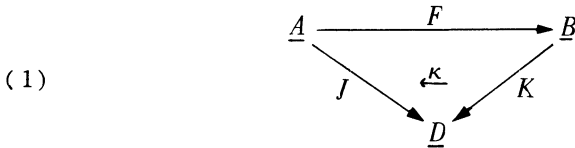
(Cp. e. g. [17], 6.10; [19].) The proof is dual to the proof of 3.2.

The proof of the following proposition is straightforward.

3.7. PROPOSITION. *Adjoint 1-cells preserve \underline{V} -limits.*

4. KAN EXTENSIONS.

The definition of Kan extensions can be formulated in any 2-category: (K, κ) is called a *Kan extension* of a 1-cell $J: \underline{A} \rightarrow \underline{D}$ along a 1-cell $F: \underline{A} \rightarrow \underline{B}$ iff $K: \underline{B} \rightarrow \underline{D}$ is a 1-cell and $\kappa: KF \rightarrow J$ is a 2-cell (cf. (1)), such that the assignment (2) is a bijection (3) for every 1-cell $L: \underline{B} \rightarrow \underline{D}$.



(2) $(\chi: L \rightarrow K) \mapsto \kappa(\chi \circ F)$; (3) $\frac{\chi: L \rightarrow K}{\omega: LF \rightarrow J}$

A 1-cell $R: \underline{D} \rightarrow \underline{E}$ respects a Kan extension (K, κ) of J along F iff $(RK, R\kappa)$ is a Kan extension of RJ along F . If, in particular, \underline{D} is a \underline{V} -category, the hom-functors of \underline{D} need not respect Kan extensions. The Kan extensions respected by all hom-functors are called *pointwise Kan extensions* (if we assume $(RK, R\kappa)$ to be a Kan extension for every hom-functor R , then (K, κ) can be shown to be a Kan extension).

4.1. DEFINITION. Let \underline{V} be a symmetric monoidal category, and let

$$J: \underline{A} \rightarrow \underline{D}, \quad F: \underline{A} \rightarrow \underline{B}, \quad K: \underline{B} \rightarrow \underline{D}$$

be 1-cells and let $\kappa: KF \rightarrow J$ be a 2-cell (cp. (1)). (K, κ) is called a *\underline{V} -Kan extension of J along F* iff:

- a) (if \underline{D} is a \underline{V} -category) the commutative diagram (4) (for all $A \in |\underline{A}|$)

sets up a bijection (5) (for all $X \in |\underline{V}|$ and 1-cells $L: \underline{B} \rightarrow \underline{D}$) between extraordinary \underline{V} -natural transformations χ and ω .

$$(4) \quad \begin{array}{ccc} X & \xrightarrow{\omega_A} & \underline{D}(LFA, JA) \\ XFA & \searrow & \nearrow \underline{D}(I, \kappa_A) \\ & \underline{D}(LFA, KFA) & \end{array}$$

$$(5) \quad \frac{\chi: X \rightarrow \text{Hom}_{\underline{D}} \circ (L^0 \otimes K)}{\omega: X \rightarrow \text{Hom}_{\underline{D}} \circ ((LF)^0 \otimes J)}$$

b) (if \underline{D} is a \underline{V} -module) the commutative diagram (6) (for all $A \in |\underline{A}|$) sets up a bijection (7) between 1-cells χ and ω .

$$(6) \quad \begin{array}{ccc} X \otimes LFA & \xrightarrow{\omega_A} & JA \\ XFA & \searrow & \nearrow \kappa_A \\ & KFA & \end{array}$$

$$(7) \quad \frac{\chi: (X \otimes -) \circ L \rightarrow K}{\omega: (X \otimes -) \circ L \circ F \rightarrow J}$$

A 1-cell $R: \underline{D} \rightarrow \underline{E}$ is said to respect a \underline{V} -Kan extension (K, κ) iff $(RK, R\kappa)$ is a \underline{V} -Kan extension of RJ along F .

4.2. THEOREM. Let \underline{V} be symmetric monoidal and let \underline{D} be a \underline{V} -category.

(i) Every \underline{V} -Kan extension (K, κ) of $J: \underline{A} \rightarrow \underline{D}$ along $F: \underline{A} \rightarrow \underline{B}$ is a Kan extension.

(ii) Every pointwise Kan extension (K, κ) of $J: \underline{A} \rightarrow \underline{D}$ along $F: \underline{A} \rightarrow \underline{B}$ is a \underline{V} -Kan extension.

We remark that every Kan extension is a \underline{V} -Kan extension in the case $\underline{V} = \text{Ens}$, the category of sets. This is certainly the reason why \underline{V} -Kan extensions apparently have not yet been considered in the literature. The usual connection between pointwise Kan extensions and \underline{V} -limits remains valid if \underline{V} is merely monoidal:

4.3. THEOREM. (K, κ) is a pointwise Kan extension of $J: \underline{A} \rightarrow \underline{D}$ along $F: \underline{A} \rightarrow \underline{B}$ iff (KB, π_B) , determined by

$$\pi_B := D(KB, \kappa) \circ K_{B, \cdot} \circ F$$

is a \underline{V} -limit of $\underline{B}(B, -) \circ F$ and J for every $B \in |\underline{B}|$.

4.4. REMARK. Several other notions may be defined for merely monoidal categories \underline{V} by means of Kan extension. E. g., a 1-cell $F: \underline{A} \rightarrow \underline{B}$ is called *codense* iff $(I_{\underline{B}}, I_F)$ is a Kan extension of F along F . Also, final and initial 1-cells (in the non-topological sense) may be defined (cf. [15], 4 (10)-(12)).

Mathematisches Institut II
 Universität Düsseldorf
 Universitätstr. 1
 D-4000 DÜSSELDORF

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