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ROSWITHA HARTING

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LOCALLY INJECTIVE G-SHEAVES OF ABELIAN GROUPS

by Roswitha HARTING

The problem of the existence of enough injective abelian group objects in an elementary topos with a natural number object leads to the construction of the internal (parametrized) coproduct of abelian group objects [4]. From certain properties of this parametrized coproduct we earlier derived some further consequences [5], among them the surprising result that «all internal notions of injectivity for abelian group objects are equivalent».

In the following summary we shall apply this result to $Shv(X)^{G^{op}}$, the topos of set-valued sheaves on a topological space X with a left action of a group-valued sheaf G .

We require the following results and definitions [4, 5] (where \mathfrak{E} denotes an elementary topos with natural number object and $Ab(\mathfrak{E})$ the category of abelian group objects in \mathfrak{E}).

(0.1) THEOREM AND DEFINITION. *For any object X in \mathfrak{E} the functor $X^*: Ab(\mathfrak{E}) \rightarrow Ab(\mathfrak{E}/X)$ has a left adjoint $\Theta_X: Ab(\mathfrak{E}/X) \rightarrow Ab(\mathfrak{E})$ which respects monomorphisms and is faithful.*

For $A(x) \in Ob Ab(\mathfrak{E}/X)$ the abelian group object $\Theta_X A(x)$ in \mathfrak{E} is called parametrized coproduct of $A(x)$. (We use «parametrized» to emphasize that the indexing object is in general not just a set but for example a set with an action of a group on it.)

A consequence of this theorem is the following proposition [5]:

(0.2) PROPOSITION. *If $Ab(\mathfrak{E})$ has enough injectives, then so does $Ab(\mathfrak{E}^A)$ for any internal category A in \mathfrak{E} .*

In the following the internal Hom-functor $Ab(\mathfrak{E})^{op} \times Ab(\mathfrak{E}) \rightarrow Ab(\mathfrak{E})$

is denoted by $\text{Hom}(-, -)$. For A, B abelian group objects in \mathfrak{E} , $\text{Hom}(A, B)$ is the abelian group object in \mathfrak{E} that internalizes the abelian group of group-morphisms from A to B .

(0.3) DEFINITION. An abelian group object B in \mathfrak{E} is called

(i) internally injective if for every monomorphism $A \rightarrow C$ in $\text{Ab}(\mathfrak{E})$, $\text{Hom}(C, B) \rightarrow \text{Hom}(A, B)$ is an epimorphism in \mathfrak{E} .

(ii) locally injective if for every diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{m} & C \\ f \downarrow & & \\ B & & \end{array}$$

in $\text{Ab}(\mathfrak{E})$ there exists a cover $U \rightarrow 1$ of \mathfrak{E} and a morphism $g: U^*C \rightarrow U^*B$ in $\text{Ab}(\mathfrak{E}/U)$ such that

$$\begin{array}{ccc} U^*A & \xrightarrow{U^*m} & U^*C \\ U^*f \downarrow & \searrow g & \\ U^*B & & \end{array}$$

commute s.

(0.4) PROPOSITION. The following conditions on an abelian group object B in \mathfrak{E} are equivalent (for (iii) we suppose that $\text{Ab}(\mathfrak{E})$ has enough injectives):

(i) B is locally injective.

(ii) For every diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{m} & C \\ f \downarrow & & \\ B & & \end{array}$$

in $\text{Ab}(\mathfrak{E})$ there exist a cover $U \rightarrow 1$ of \mathfrak{E} and a morphism

$$g: U \rightarrow \text{Hom}(C, B) \text{ in } \mathfrak{E}$$

such that the diagram

$$\begin{array}{ccc}
 \text{Hom}(C, B) & \xrightarrow{\text{Hom}(m, B)} & \text{Hom}(A, B) \\
 \uparrow g & & \uparrow f \\
 U & \longrightarrow & I
 \end{array}$$

commutes in \mathfrak{E} .

(iii) There exists a cover $U \twoheadrightarrow I$ of \mathfrak{E} such that $\eta_B: B \twoheadrightarrow B^U$ is an injective effacement [2]. Here $\eta: \text{id} \text{Ab}(\mathfrak{E}) \rightarrow \pi_U \cdot U^*$ denotes the unit of the adjunction $U^* \dashv \pi_U$, and the abelian group object structure of B^U is induced by that of B .

(A monomorphism $m: A \twoheadrightarrow C$ in $\text{Ab}(\mathfrak{E})$ is an injective effacement iff for every monomorphism $f: A \twoheadrightarrow B$ in $\text{Ab}(\mathfrak{E})$ there exists a morphism $g: B \rightarrow C$ in $\text{Ab}(\mathfrak{E})$ such that $g \cdot f = m$.)

(0.5) LEMMA. An abelian group object B in \mathfrak{E} is internally injective iff for every monomorphism $m: A \twoheadrightarrow C$ in $\text{Ab}(\mathfrak{E})$ and every generalized element $f: V \rightarrow \text{Hom}(A, B)$ in \mathfrak{E} there exist an object U , an epimorphism $h: U \twoheadrightarrow V$ and a morphism $g: U \rightarrow \text{Hom}(C, B)$ in \mathfrak{E} such that

$$\begin{array}{ccc}
 \text{Hom}(C, B) & \xrightarrow{\text{Hom}(m, B)} & \text{Hom}(A, B) \\
 \uparrow g & & \uparrow f \\
 U & \xrightarrow{h} & V
 \end{array}$$

commutes.

(0.6) THEOREM. An abelian group object B in \mathfrak{E} is locally injective iff B is internally injective.

In the following we shall study the meaning of this result in the topos $\mathfrak{E} = \text{Shv}(X)^{G^{op}}$, where X denotes a topological space (resp. a locale [7, 9]) and G a group-valued sheaf on X . Then $\text{Ab}(\text{Shv}(X)^{G^{op}})$ is the category of abelian group-valued sheaves on X equipped with a left action of G compatible with the abelian group structure. So $\text{Ab}(\text{Shv}(X)^{G^{op}})$ is the category of G -modules on X , and will be denoted from now on by $G\text{-Mod}(X)$.

(1.1) SOME REMARKS.

(i) The following diagram of forgetful functors commutes:

$$\begin{array}{ccc}
 G\text{-Mod}(X) & \xrightarrow{V''} & \text{Shv}(X)^{G^{\text{op}}} \\
 \downarrow V & & \downarrow \bar{V} \\
 \text{Ab}(\text{Shv}(X)) & \xrightarrow{V'} & \text{Shv}(X)
 \end{array}$$

All these forgetful functors create epimorphisms and monomorphisms, they all have a left adjoint, and they respect injectives [10].

(ii) In $\text{Ab}(\text{Shv}(X))$ the notions of injectivity and internal injectivity coincide [5]. For A, B in $\text{Ab}(\text{Shv}(X))$ the internal Hom is obtained as follows: for U open in X , $\text{Hom}(A, B)(U)$ is defined to be

$$\text{Hom}_{\text{Ab}(\text{Shv}(U))}(A/U, B/U).$$

(iii) $G\text{-Mod}(X)$ has enough injectives. (This follows immediately from (0.2).)

(iv) In $G\text{-Mod}(X)$ the internal Hom is obtained as follows: For A, B G -modules on X , $\text{Hom}(A, B)$ is defined to be

$$\text{Hom}(VA, VB) \text{ in } \text{Ab}(\text{Shv}(X))$$

equipped with the following action of G : for U open in X ,

$$\begin{array}{ccc}
 GU \times \text{Hom}_{\text{AbShv}(U)}(VA/U, VB/U) & \rightarrow & \text{Hom}_{\text{AbShv}(U)}(VA/U, VB/U) \\
 (s, h) & \longmapsto & s \circ h
 \end{array}$$

is defined by:

$$(s \circ h)_{\mathbb{W}}(x) := (s/\mathbb{W}) \cdot h_{\mathbb{W}}((s^{-1}/\mathbb{W}) \cdot x),$$

where $\mathbb{W} \subset U$, \mathbb{W} open in X and $x \in A\mathbb{W}$.

(1.2) PROPOSITION. Let B be a G -module on X .

(i) B is internally injective iff VB is injective in $\text{Ab}(\text{Shv}(X))$.

(ii) If B is internally injective, then B^G is injective.

PROOF. (i): Suppose B to be internally injective. To show that B is injective in $\text{Ab}(\text{Shv}(X))$, it is sufficient to show that B is internally injective in $\text{Ab}(\text{Shv}(X))$ (cf. (1.1) (ii)). So let $m: A \twoheadrightarrow C$ be a mono-

morphism in $Ab(Shv(X))$. Let G operate trivially on A and C ; then $m: A \rightarrow C$ becomes a monomorphism in $G\text{-Mod}(X)$. Since B is internally injective it follows that $Hom(C, B) \rightarrow Hom(A, B)$ is an epimorphism in $Shv(X)^{G^{op}}$, and hence an epimorphism in $Shv(X)$ (cf. (1.1) (i) and (iv)). So B is injective in $Ab(Shv(X))$.

The other implication is equally easy.

(ii): Let

$$\begin{array}{ccc} A & \xrightarrow{m} & C \\ f \downarrow & & \\ B^G & & \end{array}$$

be a diagram in $G\text{-Mod}(X)$ and suppose B to be internally injective. We have a sequence of natural bijections between the following sets:

$$\begin{array}{l} \hline A \rightarrow B^G \text{ in } G\text{-Mod}(X) \\ \hline G \rightarrow V^*(Hom(A, B)) \text{ in } Shv(X)^{G^{op}} \\ \hline 1 \rightarrow \bar{V}V^*(Hom(A, B)) \text{ in } Shv(X) \\ \hline 1 \rightarrow V'(Hom(VA, VB)) \text{ in } Shv(X) \\ \hline VA \rightarrow VB \text{ in } Ab(Shv(X)). \end{array} \quad \begin{array}{l} \bar{F} \dashv \bar{V}, \bar{F}(1) = G \\ \bar{V}V^* = V'V, (1.1) (iv) \end{array}$$

So $f: A \rightarrow B^G$ determines, and is determined by, a morphism $\bar{f}: VA \rightarrow VB$ in $Ab(Shv(X))$. B is supposed to be internally injective, so, by (i), VB is injective in $Ab(Shv(X))$. Hence there is a morphism h in $Ab(Shv(X))$ such that

$$\begin{array}{ccc} VA & \xrightarrow{Vm} & VC \\ \bar{f} \downarrow & & \nearrow h \\ VB & & \end{array}$$

commutes. As above h determines a morphism $\hat{h}: C \rightarrow B^G$ in $G\text{-Mod}(X)$, and it is easy to verify that $\hat{h}m = f$. So B^G is injective in $G\text{-Mod}(X)$.

(1.3) REMARK. Let ΔZ be the ring-valued sheaf on X associated to the

constant presheaf with value Z . Then $U \mapsto \Delta Z(U)[GU]$ defines an abelian group-valued presheaf with the usual left action of G , where we denote by $\Delta Z(U)[GU]$ the group-ring over GU . The associated sheaf is a G -module on X and is denoted by $Z[G]$. Some obvious calculations show that there is a natural isomorphism

$$\text{Hom}(Z[G], A) \approx A^G.$$

In the following the composite

$$A \xrightarrow{\eta_A} A^G \xrightarrow{\approx} \text{Hom}(Z[G], A)$$

is again denoted by η_A (cf. (0.4) (iii)).

(1.4) PROPOSITION. *Let B be a G -module on X . The following conditions on B are equivalent:*

- (i) *$\text{Hom}(Z[G], B)$ is an injective G -module on X .*
- (ii) *There exists an epimorphism $D \twoheadrightarrow 1$ in $\text{Shv}(X)^{G^{op}}$ such that B^D is an injective G -module on X .*
- (iii) *$\eta_B: B \twoheadrightarrow \text{Hom}(Z[G], B)$ is an injective effacement in $G\text{-Mod}(X)$.*
- (iv) *There exists an epimorphism $D \twoheadrightarrow 1$ in $\text{Shv}(X)^{G^{op}}$ such that $\eta_B: B \twoheadrightarrow B^D$ is an injective effacement in $G\text{-Mod}(X)$.*
- (v) *B is injective in $\text{Ab}(\text{Shv}(X))$.*
- (vi) *B is internally injective in $G\text{-Mod}(X)$.*
- (vii) *There exists an open cover of X , $X = \bigcup_{i \in I} U_i$, such that B/U_i is injective in $\text{Ab}(\text{Shv}(U_i))$ for every $i \in I$.*

PROOF. (iii) $\xrightarrow{(1.3)}$ (iv) $\xrightarrow{(1.1) (iii), (0.4) (iii), (0.6)}$ (vi) $\xrightarrow{(1.2) (ii)}$
 $\xrightarrow{\quad} (i) \xrightarrow{\quad} (iii)$.
 (ii) \Rightarrow (iv) $\xrightarrow{\text{above}}$ (i) $\xrightarrow{(1.3)}$ (ii).
 (vi) $\xleftarrow{(1.2) (i)}$ (v) $\xleftarrow{[6]}$ (vii).

So they are all equivalent.

(1.5) REMARK. If X is reduced to a point, then $\text{Shv}(X)^{G^{op}}$ is the topos

of G -sets, $Ab(Shv(X)^{G^{op}})$ is the category of left $Z[G]$ -modules, and $Ab(Shv(X))$ is the category of abelian groups. For two $Z[G]$ -modules A, B the internal Hom, $Hom(A, B)$, is the abelian group of all Z -linear maps from A to B equipped with the following G -action:

$$(s \circ f)(x) = sf(s^{-1}x) \text{ for } s \in G, f \in Hom_Z(A, B), x \in A.$$

$\eta_B: B \rightarrow Hom(Z[G], B)$ is here defined by

$$\eta_B(b)(s) = b \text{ for every } s \in G.$$

We remark that in the above case the Proposition (1.4) remains true if G is replaced by a monoid M and consequently $Hom(Z[G], B)$ by B^M (cf. [3]).

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Mathematisches Institut
Universitätsstr. 1
D-4000 DÜSSELDORF. R. F. A.