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GROUPOIDS, STABILITY COKERNEL SEQUENCES AND DUALITY

by Philip R. HEATH and Klaus Heiner KAMPS

0. INTRODUCTION.

In [HMRS] the authors have defined a stability cokernel sequence in order, among other things, to study the lack of injectivity of the function $[W, Y] \to (W, Y)$ from the based to the free homotopy classes of maps from W to the total space Y of a principal fibration. It is remarked there that duality fails basically because in the free situation a cogroup structure on W will not produce a group structure on (W, X) whereas a group structure on X will.

In this work we use groupoid methods to deduce from the groupoid situation a stability cokernel sequence which generalizes the one in [HMRS] associated with Postnikov factorization. At the same time the groupoid sequence gives rise to a stability cokernel sequence associated with the dual situation of cellular decomposition. Our stability cokernel sequence shows that some essential aspects of duality are preserved and enables one to study the way in which the duality breaks down. This study will appear elsewhere.

In Section 1 we generalize the classification sequence of [HK]. Section 2 develops the stability cokernel sequence for groupoids while Section 3 gives the applications to duality.

As in [HK] we do not work at the maximum level of generality. Our results however go through in a suitable abstract category with homotopy system.

Finally, we would like to thank Professor Peter Hilton for his motivating influence on this work.

1. A MAYER-VIETORIS TYPE SEQUENCE.

In this section we generalize the classification sequence of [HK], 1.3.

Consider the following commutative diagram in a category $\mathcal{C}$.

$$(\#) \qquad \begin{array}{ccc} B & \xrightarrow{\bar{g}} & D \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{g} & C \end{array}$$

1.1. DEFINITION. The square (#) is a *pseudopullback* in the category Gd of groupoids if the canonical morphism $B \to A \sqcap D$ into the pullback induces a bijection $\pi_0 B \xrightarrow{\approx} \pi_0 (A \sqcap D)$ where π_0 denotes the set of components.

Clearly, a pullback in Gd is a pseudopullback. According to [HK], 2.3, we have

1.2. PROPOSITION. *Let (#) be a pullback in the category Top of topological spaces in which* p *or* g *is a fibration and let* π *denote the fundamental groupoid functor. Then* $\pi(\#)$ *is a pseudopullback.*

Consider diagram (#) in Gd and suppose we choose an object $b_0 \in Ob(B)$. Let a_0, d_0, c_0 denote the image of b_0 in A, D and C, respectively.

1.3. THEOREM. *If diagram (#) is a pseudopullback in Gd and if* p *or* g *is a fibration of groupoids and A is connected, then there is a sequence*

$$(MV) \qquad \begin{array}{l} D\{d_0\} \searrow^{p} \\ \qquad\qquad C\{c_0\} \xrightarrow{\Delta} \pi_0 B \xrightarrow{\bar{g}} \pi_0 D \xrightarrow{p} \pi_0 C \\ A\{a_0\} \nearrow_{g} \end{array}$$

of groups and based sets (the base points of $\pi_0 B$, $\pi_0 D$ *and* $\pi_0 C$ *being the components* $\tilde{b_0}$, $\tilde{d_0}$, $\tilde{c_0}$*) with the following exactness properties:*

(i) (MV) is exact in the usual sense at $\pi_0 B$ *and* $\pi_0 D$.

(ii) If $\gamma_1, \gamma_2 \in C\{c_0\}$, *the group of C at* c_0, *then* $\Delta(\gamma_1) = \Delta(\gamma_2)$ *if and only if there are elements* $\alpha \in A\{a_0\}$, $\delta \in D\{d_0\}$ *such that* $\gamma_2 = g(\alpha)\gamma_1 p(\delta)$.

(MV) is a generalization to pseudopullbacks of a special case of R. Brown's Mayer-Vietoris sequence [B2], (2.2). Alternatively, (MV) is an easy generalization of [B1], 4.3, when p is a fibration and an adaptation of the replacement technique of [HK] when g is the fibration.

1.4. COROLLARY. *If we define*

$$coker(p,g) := C\{c_0\}/\,gA\{a_0\}\times p\,D\{d_0\},$$

the set of double cosets of $gA\{a_0\}$ *and* $p\,D\{d_0\}$ *in* $C\{c_0\}$, *then we have a factorization of* Δ *as*

$$C\{c_0\} \twoheadrightarrow coker(p,g) \xrightarrow{\;j\;} \pi_0 B$$

which induces a bijection $coker(p,g) \approx \bar{g}^{-1}(\tilde{d_0})$. *Thus* $\pi_0 B$ *can be written as a disjoint union of double cosets.*

The first part of the following proposition is a generalization to pseudopullbacks of a special case of [B2], 3.2. The second part is a straightforward generalization of [B1], 4.3 (b).

1.5. PROPOSITION. *If in the situation of 1.3* $gA\{a_0\}$ *is central in* $C\{c_0\}$ *then there is an operation* . *of* $C\{c_0\}$ *on* $\pi_0 B$ *and* Δ *is the restriction of this operation to* $\tilde{b_0}$.
Furthermore if $\tilde{b_1}, \tilde{b_2} \in \pi_0 B$, *then* $\bar{g}(\tilde{b_1}) = \bar{g}(\tilde{b_2})$ *if and only if there exists* $\gamma \in C\{c_0\}$ *such that* $\tilde{b_2} = \gamma . \tilde{b_1}$.

We note that in the case A is 1-connected, 1.3 and 1.5 together generalize [HK], 1.3.

2. THE STABILITY COKERNEL SEQUENCE.

2.1. DEFINITION. If diagram (#) in *Gd* is a pseudopullback, p is a fibration and A is 1-connected, then we call

$$B \xrightarrow{\;\bar{g}\;} D \xrightarrow{\;p\;} C$$

a pseudofibration.

Clearly,

$$F \xrightarrow{\;i\;} E \xrightarrow{\;p\;} B$$

is a pseudofibration if p is a fibration of groupoids and i the inclusion of a fibre.

As a special case of 1.2 we have

2.2. PROPOSITION. *If in*

$$F \xrightarrow{\ i\ } E \xrightarrow{\ p\ } B$$

p is a fibration in Top and i the inclusion of a fibre, then

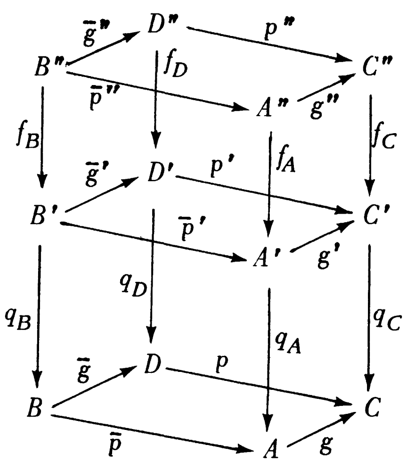
$$\pi F \xrightarrow{\pi i} \pi E \xrightarrow{\pi p} \pi B$$

is a pseudofibration.

Pseudofibrations give rise to an exact sequence of the type [HK], 1.3 (see also 1.3 and 1.5 of this paper with A 1-connected).

Consider the following commutative diagram in Gd.

(##)



We choose an object $b_0'' \in Ob(B'')$ and get induced objects in each groupoid of the diagram. The notation is obvious.

2.3. THEOREM. *If in diagram (##) the middle and top squares are pseudo-pullbacks and the vertical morphisms over B and D are pseudofibrations and either* g', g'' *or* p' *and* p'' *are fibrations and* A', A'' *are connected, then there is an exact sequence (SC):*

$$stab\, b_0'' \rightarrowtail \bar{g}^{-1}\, stab\, d_0'' \xrightarrow{\ \partial\ } coker(p'', g'') \xrightarrow{\ \phi\ } coker(p', g')$$

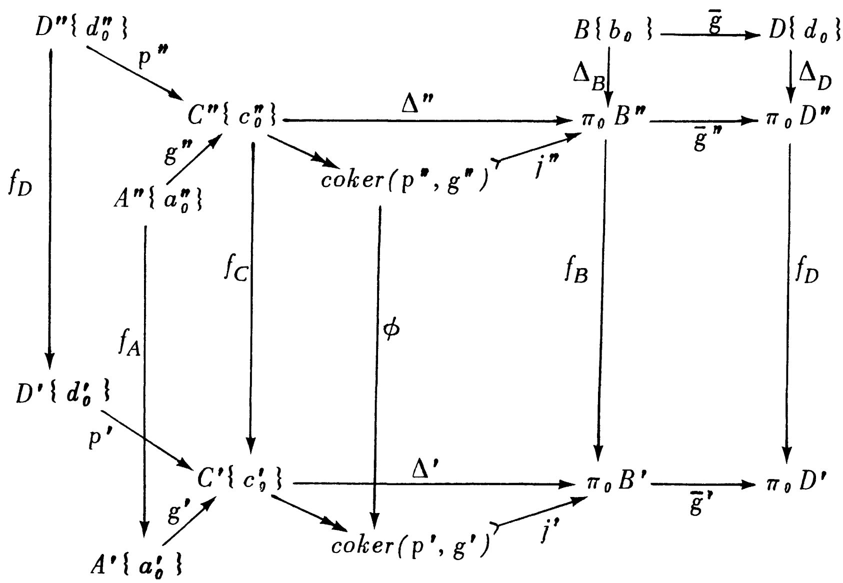
where

$$stab\, b_0'' \subset B\{b_0\}, \quad stab\, d_0'' \subset D\{d_0\}$$

denote the stability (isotropy) groups of $\tilde{b_0''} \in \pi_0 B$ *and* $\tilde{d_0''} \in \pi_0 D''$ *un-*

der the operations of $B\{b_0\}$ *and* $D\{d_0\}$ *on* $\pi_0 B''$ *and* $\pi_0 D''$, *respectively.*

PROOF. We note that (MV) is natural with respect to morphisms of diagram (#), accordingly we have a commutative array with an induced map ϕ where the «rows» are exact in the sense of 1.3 and the last two columns are exact in the sense of 1.3 and 1.5 and in which Δ'' and Δ' factorize as shown by 1.4.



Definition of ∂. Let $\beta \in \bar{g}^{-1}\, stab\ d''_0$, i. e.

$$\beta \in B\{b_0\} \quad \text{and} \quad \bar{g}(\beta) \in stab\ d''_0 .$$

We have

$$\bar{g}''\Delta_B(\beta) = \Delta_D\bar{g}(\beta) = \bar{g}(\beta).\, d''^{\sim}_0 = d''^{\sim}_0 .$$

Thus $\Delta_B(\beta)$ lies in the kernel of $\bar{g}''$. By exactness there exists a unique

$$y'' \in coker(p'', g'') \quad \text{such that} \quad j''(y'') = \Delta_B(\beta) = \beta.\, b''^{\sim}_0 ,$$

so we define $\partial(\beta) := y''$. ●

It is easy to see $stab\ b''_0 \subset \bar{g}^{-1}\, stab\ d''_0$ and to prove exactness at $\bar{g}^{-1}\, stab\ d''_0$.

We prove exactness at $coker(p'', g'')$. Let $\beta \in \bar{g}^{-1}\, stab\ d''_0$. Then

$$j'\phi\partial(\beta) = f_B j''\partial(\beta) = f_B \Delta_B(\beta) = b_0'^{\sim} = j'(1)$$

by exactness, hence $\phi\partial(\beta) = 1$, since j' is injective. Conversely, let

$$y'' \epsilon \; coker(p'', g'') \quad \text{such that} \quad \phi(y'') = 1.$$

Then

$$f_B j''(y'') = j'\phi(y'') = j'(1) = b_0'^{\sim} = f_B(b_0''^{\sim}).$$

Thus by exactness there exists

$$\beta \epsilon B\{b_0\} \quad \text{such that} \quad j''(y'') = \beta . b_0''^{\sim}.$$

We have

$$\bar{g}(\beta).d_0''^{\sim} = \Delta_D \bar{g}(\beta) = \bar{g}''\Delta_B(\beta) = \bar{g}'' j''(y'') = d_0''^{\sim},$$

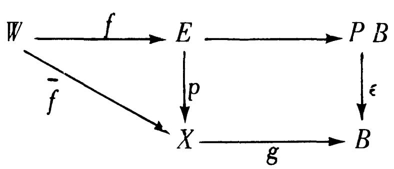
thus $\beta \epsilon \bar{g}^{-1} stab\ d_0''$. Clearly, $\partial(\beta) = y''$. □

3. EXAMPLES.

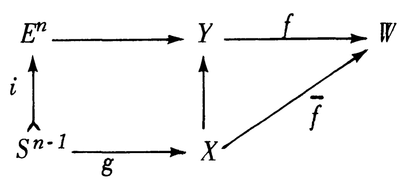
For convenience we work throughout this section in the category of compactly generated Hausdorff spaces.

Consider the two commutative diagrams of well pointed spaces

(3.1.a)

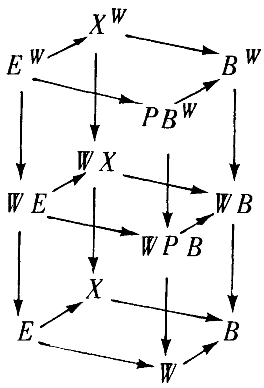


(3.1.b)

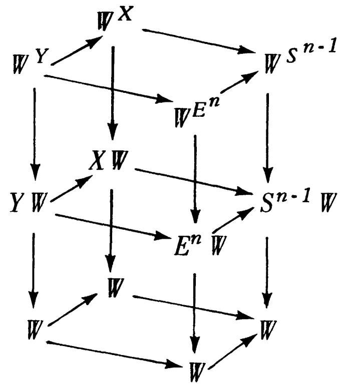


where the first square is a pullback, PB the paths on B eminating from the base point and ϵ the path fibration evaluating each path at its end-point, and where the second square is a pushout, S^{n-1} the $(n-1)$-sphere, E^n the n-ball, $n \geq 1$.

The above diagrams give rise to the following diagrams (3.2.a) and (3.2.b) where the vertical maps are evaluation maps and hence fibrations and where W^Y and YW for example denote the based, respectively, the free maps from Y to W.



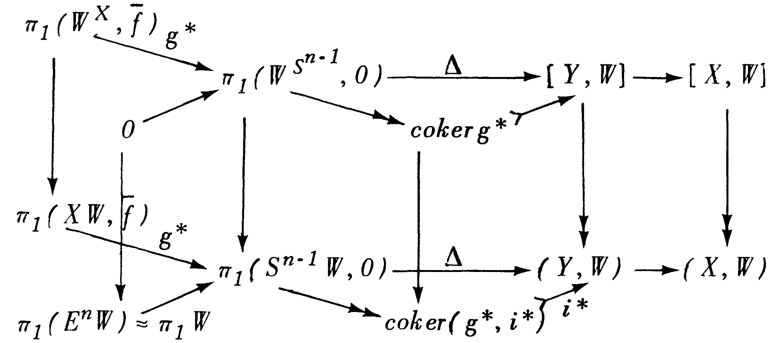
(3.2.a)



(3.2.b)

3.3.a. EXAMPLE. If in the situation (3.1.a) we apply the fundamental groupoid functor to (3.2.a) the resulting diagram satisfies 2.3. We deduce the existence of the sequence of [HMRS], Theorem 4.3.

3.3.b. EXAMPLE. In the situation of (3.1.b) if W is path connected, then taking π of (3.2.b) there are sequences



where 0 denotes the constant map into the base point, and

$$stab\, f \rightarrowtail stab\, \bar{f} \longrightarrow coker\, g^* \longrightarrow coker(g^*, i^*),$$

where $stab\, f$, $stab\, \bar{f} \subset \pi_1 W$ represent the stability subgroups of f and $\bar{f}$ under the respective actions of $\pi_1 W$ on $[Y, W]$ and $[X, W]$.

Thus we see that when the situation of (3.1.a) is seen in the wider context presented here, there are some essential aspects of duality that are preserved. Our formulation also allows by 1.4 a comparison of the based homotopy classes $[Y, W]$ with the free ones, dualizing the situation of (3.1.a).

We note that (Y, W) is classified as the disjoint union of double cosets of the group $\pi_1(S^{n-1}W, 0)$.

There is of course in general no operation of $\pi_1(S^{n-1}W, 0)$ on (Y, W), but we are able by our groupoid methods to deduce that for W path connected $\pi_1(S^{n-1}W, 0)$ is isomorphic to the semidirect product of $\pi_1 W$ and $\pi_n W$. Details will appear elsewhere.

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