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EQUATIONAL CATEGORIES

by Jiri ROSICKÝ

This paper contains some structure theory of equational categories (in the sense of Linton [3]), of Beck categories (see Manes [5]) and of completions that they provide. It continues the previous author's investigations [10 and 11].

We shall work in a given universe \mathcal{U} in Zermelo-Fraenkel set theory with the axiom of choice. The universe \mathcal{U}_f of hereditarily finite sets is permitted. Elements of \mathcal{U} will be called sets, subsets of \mathcal{U} classes and sets (in the sense of ZF) will be called metaclasses. It would be possible to work more generally in a suitable set theory with the above three levels of sets. There are the corresponding levels of categories: small categories, categories and metacategories.

The category of all sets will be denoted by $\mathcal{S}(\mathcal{U})$ (briefly by \mathcal{S}). Under a *concrete category* we will mean a couple (\mathcal{A}, U) consisting of a category \mathcal{A} and of a faithful functor $U: \mathcal{A} \rightarrow \mathcal{S}$. Sometimes we will denote it briefly by \mathcal{A} . A *concrete functor* $F: (\mathcal{A}, U) \rightarrow (\mathcal{A}', U')$ is a functor $F: \mathcal{A} \rightarrow \mathcal{A}'$ such that $U'.F = U$. A *concrete subcategory* means a subcategory such that the inclusion functor is concrete. More generally, we could work with categories over \mathcal{S} , i.e. without the assumption of faithfulness of U . But the much more important generalization consists in the replacement of \mathcal{S} by an arbitrary category \mathcal{X} . In order to be more concise we shall work over \mathcal{S} and shall not present the results in their full generality.

I. EQUATIONAL CATEGORIES.

A *type* t is defined as a metaclass of operation symbols. Their arities are arbitrary cardinals (i.e. the cardinals belonging to \mathcal{U}). t_n will

denote the metaclass of all n -ary operation symbols from t . An *equational metacategory* $(t, l)\text{-Alg}$ consists of all t -algebras satisfying a given metaclass l of equations. If \mathfrak{R} is a t -algebra and $f \in t$ then $f^{\mathfrak{R}}$ denotes the interpretation of f in \mathfrak{R} . The reasons why t and l are meta-classes will appear gradually. Now we only indicate that if t is a class then $t\text{-Alg}$ need not be a category. Some smallness conditions on equational categories are discussed in Reiterman [7].

It is well-known that *monadic categories* coincide with equational categories (\mathfrak{U}, U) such that U has a left adjoint (see Linton [3]) and that Beck's theorem characterizes them as concrete categories such that the underlying functor into \mathfrak{S} has a left adjoint and creates coequalizers of U -absolute pairs. We remark that equational metacategories are quite natural from the point of view of universes. In the case of \mathfrak{U}_φ , $(t, l)\text{-Alg}$ consists of all finite universal algebras from a variety given by a set (in the sense of ZF) of finitary operations. Since a variety often has infinite free algebras over finite sets, $(t, l)\text{-Alg}$ generally is far from being monadic.

Under an ∞ -filtered limit we mean a limit taken over an ordered metaclass such that any of its subsets has a lower bound. The dual concept is an ∞ -filtered colimit. We remark that class-indexed colimits of categories are categories, which is not true for limits (even ∞ -filtered).

1.1. THEOREM. *Any equational metacategory is an ∞ -filtered limit of equational categories. If $\mathfrak{U} \neq \mathfrak{U}_\varphi$ then equational metacategories coincide with ∞ -filtered limits of monadic categories (limits in the sense of concrete categories and concrete functors).*

PROOF. Let $\mathfrak{E} = (t, l)\text{-Alg}$ be an equational metacategory. Assign to each set $s \subset t$ the category $\mathfrak{E}_s = (s, l_s)\text{-Alg}$ where l_s consists of all equations from l written by means of operation symbols of s (i.e. (t, l) is the conservative extension of (s, l_s)). We get the concrete functors

$$R_{s', s}: \mathfrak{E}_{s'} \rightarrow \mathfrak{E}_s, \text{ for } s \subset s',$$

of reducts and similarly $R_s: \mathfrak{E} \rightarrow \mathfrak{E}_s$. It is easy to see that R_s form the limit cone for $R_{s', s}$. Moreover, if $\mathfrak{U} \neq \mathfrak{U}_\varphi$ then \mathfrak{E}_s is monadic.

Conversely, let $F_d: \mathfrak{A} \rightarrow \mathfrak{M}_d$ be an ∞ -filtered limit of $F_{d,d'}: \mathfrak{M}_d \rightarrow \mathfrak{M}_{d'}$. If $\mathfrak{M}_d = (t_d, I_d)\text{-Alg}$ then $F_{d,d'}$ induce morphisms

$$e_{d',d}: (t_{d'}, I_{d'}) \rightarrow (t_d, I_d)$$

of equational theories. Then \mathfrak{A} is isomorphic to $(t, I)\text{-Alg}$ where (t, I) is the colimit of $e_{d',d}$.

Under a *monadic completion* of a concrete category \mathfrak{A} we shall mean a reflection of \mathfrak{A} into monadic categories. It therefore consists of a monadic category $\mathfrak{M}(\mathfrak{A})$ and of a concrete functor $M_{\mathfrak{A}}: \mathfrak{A} \rightarrow \mathfrak{M}(\mathfrak{A})$ (briefly denoted by M) such that for any concrete functor $F: \mathfrak{A} \rightarrow \mathfrak{M}$ monadic there exists a unique concrete functor $\hat{F}: \mathfrak{M}(\mathfrak{A}) \rightarrow \mathfrak{M}$ with $\hat{F} \cdot M = F$. Following Linton [4], $\mathfrak{M}(\mathfrak{A})$ exists iff (\mathfrak{A}, U) is tractable, i.e. iff there exists the codensity monad R_U of U , and R_U then is the monad of $\mathfrak{M}(\mathfrak{A})$. Of course, an *equational completion* of \mathfrak{A} is a reflection $E_{\mathfrak{A}}: \mathfrak{A} \rightarrow \mathfrak{E}(\mathfrak{A})$ of \mathfrak{A} into equational categories.

1.2. COROLLARY. *An equational completion of \mathfrak{A} is a reflection of \mathfrak{A} into equational metacategories. (Proof follows by 1.1.)*

If (\mathfrak{A}, U) is a concrete metacategory then its *canonical type* $t_{\mathfrak{A}}$ consists of all natural transformations $U^n \rightarrow U$ where n runs over cardinals ($U^n(A)$ is defined as $U(A)^n$). Any $A \in \mathfrak{A}$ gives rise to the $t_{\mathfrak{A}}$ -algebra $L(A)$ by the setting

$$\phi^{L(A)} = \phi_A \quad \text{for any } \phi \in t_{\mathfrak{A}}.$$

If $I_{\mathfrak{A}}$ denotes all the equations which hold in $L(\mathfrak{A})$ (i.e. for any $L(A)$, $A \in \mathfrak{A}$) then we get the equational metacategory $\mathfrak{L}(\mathfrak{A}) = (t_{\mathfrak{A}}, I_{\mathfrak{A}})\text{-Alg}$ (see Linton [4]). Then $L_{\mathfrak{A}}: \mathfrak{A} \rightarrow \mathfrak{L}(\mathfrak{A})$ is a concrete functor. Following [4], if \mathfrak{A} is tractable then $L: \mathfrak{A} \rightarrow \mathfrak{L}(\mathfrak{A})$ is the monadic completion of \mathfrak{A} . If $F: \mathfrak{A} \rightarrow (t, I)\text{-Alg}$ is a concrete functor then any term p of type \hat{t} defines the natural transformation $p \circ F \in t_{\mathfrak{A}}$ by means of

$$(p \circ F)_A = p^{F(A)} \quad \text{for all } A \in \mathfrak{A}.$$

1.3. LEMMA. *Any concrete functor $F: \mathfrak{A} \rightarrow (t, I)\text{-Alg}$ can be factorized*

through $L_{\mathcal{Q}}$.

PROOF. The desired concrete functor

$$\hat{F}: \mathcal{L}(\mathcal{A}) \rightarrow (t, l)\text{-Alg} \quad \text{with} \quad \hat{F}.L = F$$

is given by means of

$$f^{\hat{F}}(\mathfrak{R}) = (fF)\mathfrak{R} \quad \text{for all } \mathfrak{R} \in \mathcal{L}(\mathcal{A}) \quad \text{and all } f \in t.$$

A concrete category \mathcal{A} will be called *canonically equational* if $L_{\mathcal{A}}$ is an isomorphism. An example of an equational category \mathcal{A} which is not canonically equational and such that $\mathcal{L}(\mathcal{A})$ is a category (see [6]) shows that, in spite of 1.3, $L_{\mathcal{A}}$ need not be an equational completion of \mathcal{A} (and not only owing to the size of $\mathcal{L}(\mathcal{A})$). The reason is that $\mathcal{L}(\mathcal{A})$ may contain too many operations. Nevertheless, 1.3 makes a monad from \mathcal{L} , which implies that whenever $\mathcal{L}(\mathcal{A})$ is not canonically equational then the same holds for $\mathcal{L}(\mathcal{L}(\mathcal{A}))$ (see [11]).

2. BECK CATEGORIES.

We recall that a *Beck category* is a concrete category (\mathcal{A}, U) such that U creates limits and coequalizers of U -absolute pairs. A *Beck completion* of a concrete category (\mathcal{A}, U) is a reflection $B_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{A})$ of \mathcal{A} into Beck categories.

2.1. THEOREM. *Let (\mathcal{A}, U) be a small concrete category. Then $M_{\mathcal{A}}$ is the Beck completion of \mathcal{A} (it even is the reflection of \mathcal{A} into Beck metacategories).*

PROOF. Consider a concrete functor $F: \mathcal{A} \rightarrow \mathcal{B}$ into a Beck metacategory (\mathcal{B}, V) . Denote the forgetful functor of $\mathfrak{M}(\mathcal{A})$ by W and let S or T be the right Kan extension of F or M resp. along U . We recall that R denotes the codensity monad of U . Let

$$\sigma: S.U \rightarrow F, \quad \tau: T.U \rightarrow M \quad \text{and} \quad \rho: R.U \rightarrow U$$

be the natural transformations exhibiting the corresponding Kan extensions. Clearly

$$W.T = V.S = R \quad \text{and} \quad W\tau = V\sigma = \rho.$$

There are unique natural transformations $\bar{\sigma}: S.R \rightarrow S$ and $\bar{\tau}: T.R \rightarrow T$ such that

$$\sigma.S\rho = \sigma.\bar{\sigma}U \quad \text{and} \quad \tau.T\rho = \tau.\bar{\tau}U.$$

Evidently, $W\bar{\tau} = V\bar{\sigma} = \mu$ is the multiplication of the monad R .

Any R -algebra (X, h) gives rise to the absolute coequalizer

$$R^2 X \begin{array}{c} \xrightarrow{\mu_X} \\ \xrightarrow{Rb} \end{array} RX \xrightarrow{h} X$$

The parallel pair on the left is the image by W or V of

$$TRX \begin{array}{c} \xrightarrow{\bar{\tau}_X} \\ \xrightarrow{Tb} \end{array} TX \quad \text{or} \quad SRX \begin{array}{c} \xrightarrow{\bar{\sigma}_X} \\ \xrightarrow{sb} \end{array} SX$$

resp. The R -algebra (X, h) is the creating object in $\mathfrak{M}(\mathfrak{A})$ and we shall denote the creating object in \mathfrak{B} by $\hat{F}(X, h)$. It is easy to see that $\hat{F}: \mathfrak{M}(\mathfrak{A}) \rightarrow \mathfrak{B}$ is a concrete functor. Since $M(A) = (UA, \rho_A)$, $\hat{F}.M = F$ holds. The unicity of \hat{F} is proved in [10] (see 3.7).

2.2. THEOREM. *Any concrete category \mathfrak{A} has a Beck completion which even is a reflection of \mathfrak{A} into Beck metacategories.*

PROOF. \mathfrak{A} is an ∞ -filtered colimit of the diagram consisting of all small full subcategories of \mathfrak{A} together with the inclusions. \mathfrak{M} takes this diagram to the diagram D of monadic categories. Theorem 2.1 together with the fact that an ∞ -filtered colimit of Beck categories is a Beck category implies that the colimit of D is the Beck completion of \mathfrak{A} .

2.3. COROLLARY. *Beck categories coincide with ∞ -filtered colimits of monadic categories.*

2.3 is proved in [10]; the indicated colimits are class-indexed and of concrete categories. $B_{\mathfrak{A}}$ is full iff there is a full concrete functor from \mathfrak{A} into a Beck category. If $B_{\mathfrak{A}}$ is full then U reflects limits and coequalizers of U -absolute pairs. The converse is true provided U creates limits (see [10], 4.3).

3. BIRKHOFF SUBCATEGORIES.

A *Birkhoff subcategory* is defined as a full subcategory of an equational metacategory closed under products, subalgebras and homomorphic images (see Manes [5]). Birkhoff subcategories behave well in weakly compact universes. A universe \mathcal{U} is weakly compact if for any tree which is a class and all its levels are sets there exists a path through it. \mathcal{U}_ℓ is weakly compact following König's Lemma. Concerning weakly compact cardinals, see e. g. [2].

The use of weak compactness in our situation lies in the following lemma (if $\mathcal{B} \subset \mathcal{D} \subset \mathcal{A}$ then an extension of ϕ to \mathcal{D} means a natural transformation $\psi: \mathcal{W}^{\mathcal{D}} \rightarrow \mathcal{W}$, where \mathcal{W} is the restriction of U on \mathcal{D} such that $\psi_B = \phi_B$ for all $B \in \mathcal{B}$).

3.1. LEMMA. *Let \mathcal{U} be weakly compact, (\mathcal{B}, V) be a full concrete subcategory of a concrete category (\mathcal{A}, U) and $\phi: V^n \rightarrow V$ be a natural transformation which can be extended to $\mathcal{B} \cup \mathcal{C}$ for any set \mathcal{C} of objects of \mathcal{A} . Then ϕ can be extended to \mathcal{A} .*

PROOF. The class of objects of \mathcal{A} which do not belong to \mathcal{B} is a union of an ascending chain $\mathcal{C}_0 \subset \dots \subset \mathcal{C}_\alpha \subset \dots$ of its subsets indexed by all ordinals. Consider the tree T which consists of all extensions of ϕ to \mathcal{C}_α with the ordering given by the restrictions. Since any \mathcal{C}_α is a set, levels of T are sets too. Then a path through T provides the extension of ϕ to \mathcal{A} .

3.2. THEOREM. *Let \mathcal{U} be weakly compact and \mathcal{A} be a concrete category. Then the concrete functor $K: \mathcal{B}(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{A})$ given by 2.2 is the full embedding making $\mathcal{B}(\mathcal{A})$ a Birkhoff subcategory of $\mathcal{L}(\mathcal{A})$.*

PROOF. The fullness of K is proved in 6.9 of [10] in the case of \mathcal{U}_ℓ . The proof for a weakly compact universe is the same because 3.1 plays the role of 6.8 of [10]. Since the underlying functor \mathcal{W} of $\mathcal{B}(\mathcal{A})$ creates limits any morphism $f: A \rightarrow A'$ of $\mathcal{B}(\mathcal{A})$ with

$$\mathcal{W}(A) = \mathcal{W}(A') \quad \text{and} \quad \mathcal{W}(f) = id_{\mathcal{W}A}$$

must be the identity on $A = A'$. Hence K is a full embedding. The fact that $\mathfrak{B}(\mathfrak{A})$ is a Birkhoff subcategory of $\mathfrak{L}(\mathfrak{A})$ is proved in 6.10 of [10] for \mathfrak{U}_φ and the weakly compact case is the same, again.

3.3. COROLLARY. *If \mathfrak{U} is weakly compact then Beck categories coincide with Birkhoff subcategories of equational metacategories.*

3.4. COROLLARY. *Let \mathfrak{U} be weakly compact and (\mathfrak{A}, U) be a concrete category such that U creates limits. Then the following conditions are equivalent:*

(i) *There exists a full concrete embedding of \mathfrak{A} into an equational metacategory.*

(ii) *U reflects coequalizers of U -absolute pairs.*

Proof follows by 3.3 and Theorem 4.3 of [10].

3.5. LEMMA (Goralčík, Koubek). *Let \mathfrak{U} be weakly compact and (\mathfrak{B}, V) be a Birkhoff subcategory of an equational category (\mathfrak{E}, U) . Then any natural transformation $\phi: V^n \rightarrow V$ can be extended to \mathfrak{E} .*

PROOF. Let $\mathfrak{E} = (t, l)\text{-Alg}$. For any set \mathcal{C} of algebras from \mathfrak{B} there exists a term $p(\mathcal{C})$ of type t such that

$$p(\mathcal{C})^{\mathfrak{R}} = \phi^{\mathfrak{R}} \quad \text{for all } \mathfrak{R} \in \mathcal{C}$$

(see the proof of the Proposition from [11]).

Let \mathcal{Z} be a set of algebras from \mathfrak{E} not belonging to \mathfrak{B} . Define the equivalence relation on the meta-class of all n -ary terms of type t as follows:

$$p \sim q \quad \text{iff } p^{\mathfrak{R}} = q^{\mathfrak{R}} \quad \text{for all } \mathfrak{R} \in \mathcal{Z}.$$

Since \sim has only a set of equivalence classes, there is an equivalence class T such that \mathfrak{B} is the union of all \mathcal{C} such that $p(\mathcal{C}) \in T$. The setting $\psi^{\mathfrak{R}} = p^{\mathfrak{R}}$ for $p \in T$ gives the extension of ϕ to $\mathfrak{B} \cup \mathcal{Z}$.

Thus the result follows from 3.1.

In an arbitrary \mathfrak{U} , 3.5 holds for any \mathfrak{E} such that the number of its objects has the tree property. Goralčík and Koubek (see [1]) have proved

3.5 in \mathcal{U}_φ for any equational metacategory \mathcal{E} by means of the following argument. Under a regular extension of ϕ we shall mean an extension of ϕ which is induced by a term (in the sense of the beginning of the proof of 3.5) on any set of algebras. The proof of 3.5 shows that ϕ can be regularly extended to each $\mathfrak{R} \in \mathcal{E}$. The assertion now follows from the fact that there is a maximal regular extension of ϕ . Indeed, any filtered set of regular extensions has an upper bound. However, the last statement holds only for \mathcal{U}_φ because otherwise ∞ -filtered does not mean filtered.

The next theorem is the extension of Birkhoff variety Theorem to the equational case (for $\mathcal{E} = (t, l)\text{-Alg}$ being monadic implies that $t_{\mathcal{E}} = t$). This theorem was proved by Reiterman (see [8]) in \mathcal{U}_φ . Goralčík and Koubek (see [1]) have generalized the theorem to any equational metacategory (in \mathcal{U}_φ again; see the above remark) and they have also simplified Reiterman's original proof. Their proof starts from 3.5 and works for a weakly compact universe, too. For the reader's convenience, we write down how 3.6 follows from 3.5. We remark that this derivation is analogous to the proof of 6.10 of [10] (6.10 uses the types $t_{\mathcal{C}_a}$, 3.6 $t_{\mathfrak{M}(\mathcal{C}_a)}$; these types generally are distinct). We mention that 3.6 does not infer that \mathcal{B} is equational.

3.6. THEOREM (Reiterman). *Let \mathcal{U} be weakly compact and \mathcal{B} be a Birkhoff subcategory of an equational category \mathcal{E} . Then \mathcal{B} is determined in \mathcal{E} by equations of the type $t_{\mathcal{E}}$ (i. e. if $\phi_{\mathfrak{R}} = \psi_{\mathfrak{R}}$ for any $\phi, \psi \in t_{\mathcal{E}}$ having the same restriction on \mathcal{B} then $\mathfrak{R} \in \mathcal{B}$).*

PROOF. Let $\mathcal{E} = (t, l)\text{-Alg}$ and U be the forgetful functor of \mathcal{E} . Following [10] 6.5, \mathcal{B} is a union of an ascending chain $\mathfrak{M}_0 \subset \dots \mathfrak{M}_\alpha \subset \dots$ of monadic categories indexed by all the ordinals. Let \mathfrak{R} be an algebra from \mathcal{E} not belonging to \mathcal{B} and n be the cardinality of $U(\mathfrak{R})$. For any a there are n -ary terms p_a, q_a of type t such that the equation $p_a^- = q_a^-$ holds on \mathfrak{M}_a and not on \mathfrak{R} . Since there is only a set of mappings $U(\mathfrak{R})^n \rightarrow U(\mathfrak{R})$ we may assume that

$$(p_a)^{\mathfrak{R}} = (p_\beta)^{\mathfrak{R}} \quad \text{and} \quad (q_a)^{\mathfrak{R}} = (q_\beta)^{\mathfrak{R}} \quad \text{for any } a, \beta.$$

Since terms induce natural transformations, following 3.1 and 3.5 there are $\phi, \psi \in t_{\mathfrak{E}}$ having the same restriction on \mathfrak{B} and such that $\phi_{\mathfrak{A}} \neq \psi_{\mathfrak{A}}$.

4. EQUATIONAL COMPLETION.

4.1. LEMMA. *Let \mathfrak{A} be a concrete category and $\phi, \psi \in t_{\mathfrak{E}(\mathfrak{A})}$ such that $\phi E_{\mathfrak{A}} = \psi E_{\mathfrak{A}}$. Then $\phi = \psi$.*

PROOF. Let $\mathfrak{E}(\mathfrak{A}) = (t, l)\text{-Alg}$ and consider the type $t' = t \cup \{\phi, \psi\}$. Then l is a theory of type t' , and let $l' = l \cup \{\phi = \psi\}$. Then

$$\mathfrak{E} = (t', l)\text{-Alg} \quad \text{and} \quad \mathfrak{E}' = (t', l')\text{-Alg}$$

are equational categories. Denote by $T: \mathfrak{E}' \rightarrow \mathfrak{E}$ the inclusion and by $R: \mathfrak{L}(\mathfrak{E}(\mathfrak{A})) \rightarrow \mathfrak{E}$ the reduct. Since $\phi E_{\mathfrak{A}} = \psi E_{\mathfrak{A}}$, there is a concrete functor $F: \mathfrak{A} \rightarrow \mathfrak{E}'$ such that

$$R \cdot L_{\mathfrak{E}(\mathfrak{A})} \cdot E_{\mathfrak{A}} = T \cdot F.$$

Hence there is a concrete functor

$$\hat{F}: \mathfrak{E}(\mathfrak{A}) \rightarrow \mathfrak{E}' \quad \text{with} \quad T \cdot \hat{F} = R \cdot L_{\mathfrak{E}(\mathfrak{A})}.$$

But it follows that $\phi = \psi$.

Now, we can state the main theorem.

4.2. THEOREM. *Let \mathfrak{U} be weakly compact and \mathfrak{A} a concrete category. Then an equational completion of \mathfrak{A} exists iff $\mathfrak{B}(\mathfrak{A})$ is equational and $B_{\mathfrak{A}}$ is in this case the equational completion of \mathfrak{A} .*

PROOF. If $\mathfrak{B}(\mathfrak{A})$ is equational then $B_{\mathfrak{A}}$ evidently is the equational completion of \mathfrak{A} . Assume that $\mathfrak{E}(\mathfrak{A}) = (t, l)\text{-Alg}$ exists. There appear concrete functors

$$\hat{E}: \mathfrak{B}(\mathfrak{A}) \rightarrow \mathfrak{E}(\mathfrak{A}) \quad \text{and} \quad \hat{L}: \mathfrak{E}(\mathfrak{A}) \rightarrow \mathfrak{L}(\mathfrak{A})$$

with

$$\hat{E} \cdot B = E \quad \text{and} \quad \hat{L} \cdot E = L$$

(concerning \hat{L} see 1.2). Since $\hat{L} \cdot \hat{E}$ is the functor K from 3.2, $\mathfrak{B}(\mathfrak{A})$ is a Birkhoff subcategory of $\mathfrak{L}(\mathfrak{A})$ and hence of $\mathfrak{E}(\mathfrak{A})$, too. Following 3.6 and 4.1, $\mathfrak{E}(\mathfrak{A}) = \mathfrak{B}(\mathfrak{A})$ holds.

Hence a Beck category which is not equational can not have an equational completion. Concerning such a Beck category see [10], 6.1.

A concrete category \mathcal{A} will be called *stable* if

$$\phi_{\mathfrak{R}} = (\phi L)^{\mathfrak{R}} \text{ for all } \phi \in t\mathcal{Q}(\mathcal{A}) \text{ and all } \mathfrak{R} \in \mathcal{L}(\mathcal{A}).$$

4.3. THEOREM. *Consider the following statements for a given concrete category \mathcal{A} :*

- (i) $L_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A})$ is the Beck completion of \mathcal{A} .
- (ii) $L_{\mathcal{A}}: \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A})$ is the equational completion of \mathcal{A} .
- (iii) \mathcal{A} is stable and $\mathcal{L}(\mathcal{A})$ is a category.

Then (i) \Rightarrow (ii) \Rightarrow (iii). If \mathcal{U} is weakly compact, then all three statements are equivalent.

PROOF. Clearly (i) \Rightarrow (ii). Assume (ii) and let $\phi \in t\mathcal{Q}(\mathcal{A})$. Let $\psi \in t\mathcal{Q}(\mathcal{A})$ be given by the setting

$$\psi_{\mathfrak{R}} = (\phi L)^{\mathfrak{R}} \text{ for all } \mathfrak{R} \in \mathcal{L}(\mathcal{A}).$$

Since $\phi L = \psi L$, $\phi = \psi$ holds by 4.1 and thus \mathcal{A} is stable.

Let \mathcal{U} be weakly compact and (iii) hold. Following 3.6, $\mathcal{B}(\mathcal{A})$ is determined in $\mathcal{L}(\mathcal{A})$ by equations of type $t\mathcal{Q}(\mathcal{A})$ and hence by equations of type $t\mathcal{A}$ because \mathcal{A} is stable. However, if an equation of type $t\mathcal{A}$ holds in $\mathcal{B}(\mathcal{A})$ then it also holds in \mathcal{A} and thus in $\mathcal{L}(\mathcal{A})$, too. Therefore we have $\mathcal{B}(\mathcal{A}) = \mathcal{L}(\mathcal{A})$ and (i) is proved.

If $\mathcal{L}(\mathcal{A})$ is monadic then \mathcal{A} clearly is stable and thus 4.3 implies that 2.1 holds for any tractable \mathcal{A} in a weakly compact \mathcal{U} . Concerning an example of a stable concrete category \mathcal{A} such that $\mathcal{L}(\mathcal{A})$ is not monadic, see [11]. If \mathcal{A} is stable then it is easy to see that $\mathcal{L}(\mathcal{A})$ is canonically equational. I do not know whether the converse is true or not. Notice nevertheless that the consequence of 3.6 is that if \mathcal{U} is weakly compact and $\mathcal{L}(\mathcal{A})$ a canonically equational category, then $B_{\mathcal{A}}$ is the equational completion of \mathcal{A} .

4.4. PROPOSITION. *Let $(t, l)\text{-Alg}$ be an equational completion of \mathcal{A} . Then*

$(t_{\mathcal{Q}}, l_{\mathcal{Q}})$ is a conservative extension of (t, l) .

PROOF. The result is given by the fact that for all terms p, q of type t , $p E_{\mathcal{Q}} = q E_{\mathcal{Q}}$ implies, by 4.1, that the equation $p = q$ follows from l .

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