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3^{e} COLLOQUE SUR LES CATEGORIES
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EQUATIONAL CATEGORIES

by Jiri ROSICKÝ

This paper contains some structure theory of equational categories (in the sense of Linton [3]), of Beck categories (see Manes [5]) and of completions that they provide. It continues the previous author's investigations [10 and 11].

We shall work in a given universe $\mathfrak{U}$ in Zermelo-Fraenkel set theory with the axiom of choice. The universe $\mathfrak{U}_f$ of hereditarily finite sets is permitted. Elements of $\mathfrak{U}$ will be called sets, subsets of $\mathfrak{U}$ classes and sets (in the sense of ZF) will be called metaclasses. It would be possible to work more generally in a suitable set theory with the above three levels of sets. There are the corresponding levels of categories: small categories, categories and metacategories.

The category of all sets will be denoted by $\mathcal{S}(\mathfrak{U})$ (briefly by $\mathcal{S}$). Under a *concrete category* we will mean a couple $(\mathfrak{A}, U)$ consisting of a category $\mathfrak{A}$ and of a faithful functor $U: \mathfrak{A} \to \mathcal{S}$. Sometimes we will denote it briefly by $\mathfrak{A}$. A *concrete functor* $F: (\mathfrak{A}, U) \to (\mathfrak{A}', U')$ is a functor $F: \mathfrak{A} \to \mathfrak{A}'$ such that $U'. F = U$. A concrete subcategory means a subcategory such that the inclusion functor is concrete. More generally, we could work with categories over $\mathcal{S}$, i.e. without the assumption of faithfulness of U. But the much more important generalization consists in the replacement of $\mathcal{S}$ by an arbitrary category $\mathfrak{X}$. In order to be more concise we shall work over $\mathcal{S}$ and shall not present the results in their full generality.

I. EQUATIONAL CATEGORIES.

A *type* t is defined as a metaclass of operation symbols. Their arities are arbitrary cardinals (i.e. the cardinals belonging to $\mathfrak{U}$). t_n will denote the metaclass of all n-ary operation symbols from t. *An equational metacategory* (t, I)-*Alg* consists of all t-algebras satisfying a given metaclass I of equations. If $\mathfrak{R}$ is a t-algebra and $f \in t$ then $f^{\mathfrak{R}}$ denotes the interpretation of f in $\mathfrak{R}$. The reasons why t and I are metaclasses will appear gradually. Now we only indicate that if t is a class then t-*Alg* need not be a category. Some smallness conditions on equational categories are discussed in Reiterman [7].

It is well-known that *monadic categories* coincide with equational categories $(\mathfrak{A}, U)$ such that U has a left adjoint (see Linton [3]) and that Beck's theorem characterizes them as concrete categories such that the underlying functor into $\mathfrak{S}$ has a left adjoint and creates coequalizers of U-absolute pairs. We remark that equational metacategories are quite natural from the point of view of universes. In the case of $\mathfrak{U}_f$, (t,I)-*Alg* consists of all finite universal algebras from a variety given by a set (in the sense of ZF) of finitary operations. Since a variety often has infinite free algebras over finite sets, (t, I)-*Alg* generally is far from being monadic.

Under an ∞-filtered limit we mean a limit taken over an ordered metaclass such that any of its subsets has a lower bound. The dual concept is an ∞-filtered colimit. We remark that class-indexed colimits of categories are categories, which is not true for limits (even ∞-filtered).

1.1. THEOREM. *Any equational metacategory is an ∞-filtered limit of equational categories. If $\mathfrak{U} \neq \mathfrak{U}_f$ then equational metacategories coincide with ∞-filtered limits of monadic categories (limits in the sense of concrete categories and concrete functors).*

PROOF. Let $\mathfrak{E} = (t, I)$-*Alg* be an equational metacategory. Assign to each set $s \subset t$ the category $\mathfrak{E}_s = (s, I_s)$-*Alg* where I_s consists of all equations from I written by means of operation symbols of s (i.e. (t, I) is the conservative extension of (s, I_s)). We get the concrete functors

$$R_{s',s} : \mathfrak{E}_{s'} \to \mathfrak{E}_s, \quad \text{for } s \subset s',$$

of reducts and similarly $R_s : \mathfrak{E} \to \mathfrak{E}_s$. It is easy to see that R_s form the limit cone for $R_{s',s}$. Moreover, if $\mathfrak{U} \neq \mathfrak{U}_f$ then $\mathfrak{E}_s$ is monadic.

Conversely, let $F_d : \mathfrak{A} \to \mathfrak{M}_d$ be an ∞-filtered limit of $F_{d,d'} : \mathfrak{M}_d \to \mathfrak{M}_{d'}$. If $\mathfrak{M}_d = (t_d, l_d)$-$Alg$ then $F_{d,d'}$ induce morphisms

$$e_{d',d} : (t_{d'}, l_{d'}) \to (t_d, l_d)$$

of equational theories. Then $\mathfrak{A}$ is isomorphic to (t,l)-Alg where (t,l) is the colimit of $e_{d',d}$.

Under a *monadic completion* of a concrete category $\mathfrak{A}$ we shall mean a reflection of $\mathfrak{A}$ into monadic categories. It therefore consists of a monadic category $\mathfrak{M}(\mathfrak{A})$ and of a concrete functor $M_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{M}(\mathfrak{A})$ (briefly denoted by M) such that for any concrete functor $F : \mathfrak{A} \to \mathfrak{M}$ monadic there exists a unique concrete functor $\hat{F} : \mathfrak{M}(\mathfrak{A}) \to \mathfrak{M}$ with $\hat{F}.M = F$. Following Linton [4], $\mathfrak{M}(\mathfrak{A})$ exists iff $(\mathfrak{A}, U)$ is tractable, i.e. iff there exists the codensity monad R_U of U, and R_U then is the monad of $\mathfrak{M}(\mathfrak{A})$. Of course, an *equational completion of* $\mathfrak{A}$ is a reflection $E_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{E}(\mathfrak{A})$ of $\mathfrak{A}$ into equational categories.

1.2. COROLLARY. *An equational completion of $\mathfrak{A}$ is a reflection of $\mathfrak{A}$ into equational metacategories.* (Proof follows by 1.1.)

If $(\mathfrak{A}, U)$ is a concrete metacategory then its *canonical type* $t_{\mathfrak{A}}$ consists of all natural transformations $U^n \to U$ where n runs over cardinals ($U^n(A)$ is defined as $U(A)^n$). Any $A \in \mathfrak{A}$ gives rise to the $t_{\mathfrak{A}}$-algebra $L(A)$ by the setting

$$\phi^{L(A)} = \phi_A \quad \text{for any} \quad \phi \in t_{\mathfrak{A}} .$$

If $l_{\mathfrak{A}}$ denotes all the equations which hold in $L(\mathfrak{A})$ (i.e. for any $L(A)$, $A \in \mathfrak{A}$) then we get the equational metacategory $\mathfrak{L}(\mathfrak{A}) = (t_{\mathfrak{A}}, l_{\mathfrak{A}})$-$Alg$ (see Linton [4]). Then $L_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{L}(\mathfrak{A})$ is a concrete functor. Following [4], if $\mathfrak{A}$ is tractable then $L : \mathfrak{A} \to \mathfrak{L}(\mathfrak{A})$ is the monadic completion of $\mathfrak{A}$. If $F : \mathfrak{A} \to (t,l)$-$Alg$ is a concrete functor then any term p of type t defines the natural transformation $pF \in t_{\mathfrak{A}}$ by means of

$$(pF)_A = p^{F(A)} \quad \text{for all} \quad A \in \mathfrak{A} .$$

1.3. LEMMA. *Any concrete functor $F : \mathfrak{A} \to (t,l)$-$Alg$ can be factorized* *through* $L_{\mathfrak{A}}$.

PROOF. The desired concrete functor

$$\hat{F} : \mathfrak{L}(\mathfrak{A}) \to (t, l)\text{-}Alg \quad \text{with} \quad \hat{F}.L = F$$

is given by means of

$$f^{\hat{F}(\mathfrak{R})} = (fF)^{\mathfrak{R}} \quad \text{for all } \mathfrak{R} \in \mathfrak{L}(\mathfrak{A}) \text{ and all } f \in t.$$

A concrete category $\mathfrak{A}$ will be called *canonically equational* if $L_{\mathfrak{A}}$ is an isomorphism. An exemple of an equational category $\mathfrak{A}$ which is not canonically equational and such that $\mathfrak{L}(\mathfrak{A})$ is a category (see [6]) shows that, in spite of 1.3, $L_{\mathfrak{A}}$ need not be an equational completion of $\mathfrak{A}$ (and not only owing to the size of $\mathfrak{L}(\mathfrak{A})$). The reason is that $\mathfrak{L}(\mathfrak{A})$ may contain too many operations. Nevertheless, 1.3 makes a monad from $\mathfrak{L}$, which implies that whenever $\mathfrak{L}(\mathfrak{A})$ is not canonically equational then the same holds for $\mathfrak{L}(\mathfrak{L}(\mathfrak{A}))$ (see [11]).

2. BECK CATEGORIES.

We recall that a *Beck category* is a concrete category $(\mathfrak{A}, U)$ such that U creates limits and coequalizers of U-absolute pairs. A *Beck completion* of a concrete category $(\mathfrak{A}, U)$ is a reflection $B_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{B}(\mathfrak{A})$ of $\mathfrak{A}$ into Beck categories.

2.1. THEOREM. *Let $(\mathfrak{A}, U)$ be a small concrete category. Then $M_{\mathfrak{A}}$ is the Beck completion of $\mathfrak{A}$ (it even is the reflection of $\mathfrak{A}$ into Beck metacategories).*

PROOF. Consider a concrete functor $F : \mathfrak{A} \to \mathfrak{B}$ into a Beck metacategory $(\mathfrak{B}, V)$. Denote the forgetful functor of $\mathfrak{M}(\mathfrak{A})$ by W and let S or T be the right Kan extension of F or M resp. along U. We recall that R denotes the codensity monad of U. Let

$$\sigma : S.U \to F, \quad \tau : T.U \to M \quad \text{and} \quad \rho : R.U \to U$$

be the natural transformations exhibiting the corresponding Kan extensions. Clearly

$$\mathbb{W}.T = V.S = R \quad \text{and} \quad \mathbb{W}\tau = V\sigma = \rho.$$

There are unique natural transformations $\bar{\sigma}: S.R \to S$ and $\bar{\tau}: T.R \to T$ such that

$$\sigma.S\rho = \sigma.\bar{\sigma}U \quad \text{and} \quad \tau.T\rho = \tau.\bar{\tau}U.$$

Evidently, $\mathbb{W}\bar{\tau} = V\bar{\sigma} = \mu$ is the multiplication of the monad R.

Any R-algebra (X,h) gives rise to the absolute coequalizer

$$R^2X \underset{Rh}{\overset{\mu_X}{\rightrightarrows}} RX \xrightarrow{h} X$$

The parallel pair on the left is the image by $\mathbb{W}$ or V of

$$TRX \underset{Th}{\overset{\bar{\tau}_X}{\rightrightarrows}} TX \quad \text{or} \quad SRX \underset{Sh}{\overset{\bar{\sigma}_X}{\rightrightarrows}} SX$$

resp. The R-algebra (X,h) is the creating object in $\mathfrak{M}(\mathfrak{A})$ and we shall denote the creating object in $\mathfrak{B}$ by $\hat{F}(X,h)$. It is easy to see that $\hat{F}: \mathfrak{M}(\mathfrak{A}) \to \mathfrak{B}$ is a concrete functor. Since $M(A) = (UA, \rho_A)$, $\hat{F}.M = F$ holds. The unicity of $\hat{F}$ is proved in [10] (see 3.7).

2.2. THEOREM. *Any concrete category $\mathfrak{A}$ has a Beck completion which even is a reflection of $\mathfrak{A}$ into Beck metacategories.*

PROOF. $\mathfrak{A}$ is an ∞-filtered colimit of the diagram consisting of all small full subcategories of $\mathfrak{A}$ together with the inclusions. $\mathfrak{M}$ takes this diagram to the diagram D of monadic categories. Theorem 2.1 together with the fact that an ∞-filtered colimit of Beck categories is a Beck category implies that the colimit of D is the Beck completion of $\mathfrak{A}$.

2.3. COROLLARY. *Beck categories coincide with ∞-filtered colimits of monadic categories.*

2.3 is proved in [10]; the indicated colimits are class-indexed and of concrete categories. $B_{\mathfrak{A}}$ is full iff there is a full concrete functor from $\mathfrak{A}$ into a Beck category. If $B_{\mathfrak{A}}$ is full then U reflects limits and coequalizers of U-absolute pairs. The converse is true provided U creates limits (see [10], 4.3).

3. BIRKHOFF SUBCATEGORIES.

A *Birkhoff subcategory* is defined as a full subcategory of an equational metacategory closed under products, subalgebras and homomorphic images (see Manes [5]). Birkhoff subcategories behave well in weakly compact universes. A universe $\mathfrak{U}$ is weakly compact if for any tree which is a class and all its levels are sets there exists a path through it. $\mathfrak{U}_f$ is weakly compact following König's Lemma. Concerning weakly compact cardinals, see e.g. [2].

The use of weak compactness in our situation lies in the following lemma (if $\mathfrak{B} \subset \mathfrak{D} \subset \mathfrak{A}$ then an extension of ϕ to $\mathfrak{D}$ means a natural transformation $\psi : W^n \to W$, where W is the restriction of U on $\mathfrak{D}$ such that $\psi_B = \phi_B$ for all $B \in \mathfrak{B}$).

3.1. LEMMA. *Let $\mathfrak{U}$ be weakly compact, $(\mathfrak{B}, V)$ be a full concrete subcategory of a concrete category $(\mathfrak{A}, U)$ and $\phi : V^n \to V$ be a natural transformation which can be extended to $\mathfrak{B} \cup \mathcal{C}$ for any set $\mathcal{C}$ of objects of $\mathfrak{A}$. Then ϕ can be extended to $\mathfrak{A}$.*

PROOF. The class of objects of $\mathfrak{A}$ which do not belong to $\mathfrak{B}$ is a union of an ascending chain $\mathcal{C}_0 \subset \dots \mathcal{C}_\alpha \subset \dots$ of its subsets indexed by all ordinals. Consider the tree T which consists of all extensions of ϕ to $\mathcal{C}_\alpha$ with the ordering given by the restrictions. Since any $\mathcal{C}_\alpha$ is a set, levels of T are sets too. Then a path through T provides the extension of ϕ to $\mathfrak{A}$.

3.2. THEOREM. *Let $\mathfrak{U}$ be weakly compact and $\mathfrak{A}$ be a concrete category. Then the concrete functor $K : \mathfrak{B}(\mathfrak{A}) \to \mathfrak{L}(\mathfrak{A})$ given by 2.2 is the full embedding making $\mathfrak{B}(\mathfrak{A})$ a Birkhoff subcategory of $\mathfrak{L}(\mathfrak{A})$.*

PROOF. The fullness of K is proved in 6.9 of [10] in the case of $\mathfrak{U}_f$. The proof for a weakly compact universe is the same because 3.1 plays the role of 6.8 of [10]. Since the underlying functor W of $\mathfrak{B}(\mathfrak{A})$ creates limits any morphism $f : A \to A'$ of $\mathfrak{B}(\mathfrak{A})$ with

$$W(A) = W(A') \quad \text{and} \quad W(f) = id_{WA}$$

must be the identity on $A = A'$. Hence K is a full embedding. The fact that $\mathfrak{B}(\mathfrak{A})$ is a Birkhoff subcategory of $\mathfrak{L}(\mathfrak{A})$ is proved in 6.10 of [10] for $\mathfrak{U}_{\mathfrak{f}}$ and the weakly compact case is the same, again.

3.3. COROLLARY. *If $\mathfrak{U}$ is weakly compact then Beck categories coincide with Birkhoff subcategories of equational metacategories.*

3.4. COROLLARY. *Let $\mathfrak{U}$ be weakly compact and $(\mathfrak{A}, U)$ be a concrete category such that U creates limits. Then the following conditions are equivalent:*

(i) There exists a full concrete embedding of $\mathfrak{A}$ into an equational metacategory.

(ii) U reflects coequalizers of U-absolute pairs.

Proof follows by 3.3 and Theorem 4.3 of [10].

3.5. LEMMA *(Goralčík, Koubek). Let $\mathfrak{U}$ be weakly compact and $(\mathfrak{B}, V)$ be a Birkhoff subcategory of an equational category $(\mathfrak{E}, U)$. Then any natural transformation $\phi: V^n \to V$ can be extended to $\mathfrak{E}$.*

PROOF. Let $\mathfrak{E} = (t, I)\text{-}Alg$. For any set $\mathcal{C}$ of algebras from $\mathfrak{B}$ there exists a term $p(\mathcal{C})$ of type t such that

$$p(\mathcal{C})^{\mathfrak{R}} = \phi_{\mathfrak{R}} \quad \text{for all } \mathfrak{R} \in \mathcal{C}$$

(see the proof of the Proposition from [11]).

Let $\mathfrak{Z}$ be a set of algebras from $\mathfrak{E}$ not belonging to $\mathfrak{B}$. Define the equivalence relation on the metaclass of all n-ary terms of type t as follows:

$$p \sim q \quad \text{iff} \quad p^{\mathfrak{R}} = q^{\mathfrak{R}} \quad \text{for all } \mathfrak{R} \in \mathfrak{Z}.$$

Since $\sim$ has only a set of equivalence classes, there is an equivalence class T such that $\mathfrak{B}$ is the union of all $\mathcal{C}$ such that $p(\mathcal{C}) \in T$. The setting $\psi_{\mathfrak{R}} = p^{\mathfrak{R}}$ for $p \in T$ gives the extension of ϕ to $\mathfrak{B} \cup \mathfrak{Z}$.

Thus the result follows from 3.1.

In an arbitrary $\mathfrak{U}$, 3.5 holds for any $\mathfrak{E}$ such that the number of its objects has the tree property. Goralčík and Koubek (see [1]) have proved

3.5 in $\mathfrak{U}_{\mathfrak{f}}$ for any equational metacategory $\mathfrak{S}$ by means of the following argument. Under a regular extension of ϕ we shall mean an extension of ϕ which is induced by a term (in the sense of the beginning of the proof of 3.5) on any set of algebras. The proof of 3.5 shows that ϕ can be regularly extended to each $\mathfrak{R} \in \mathfrak{S}$. The assertion now follows from the fact that there is a maximal regular extension of ϕ. Indeed, any filtered set of regular extensions has an upper bound. However, the last statement holds only for $\mathfrak{U}_{\mathfrak{f}}$ because otherwise ∞-filtered does not mean filtered.

The next theorem is the extension of Birkhoff variety Theorem to the equational case (for $\mathfrak{S} = (t, l)$-Alg being monadic implies that $t_{\mathfrak{S}} = t$). This theorem was proved by Reiterman (see [8]) in $\mathfrak{U}_{\mathfrak{f}}$. Goralčík and Koubek (see [1]) have generalized the theorem to any equational metacategory (in $\mathfrak{U}_{\mathfrak{f}}$ again; see the above remark) and they have also simplified Reiterman's original proof. Their proof starts from 3.5 and works for a weakly compact universe, too. For the reader's convenience, we write down how 3.6 follows from 3.5. We remark that this derivation is analogous to the proof of 6.10 of [10] (6.10 uses the types $t_{\mathcal{C}_a}$, 3.6 $t_{\mathfrak{M}(\mathcal{C}_a)}$; these types generally are distinct). We mention that 3.6 does not infer that $\mathfrak{B}$ is equational.

3.6 .THEOREM *(Reiterman). Let $\mathfrak{U}$ be weakly compact and $\mathfrak{B}$ be a Birkhoff subcategory of an equational category $\mathfrak{S}$. Then $\mathfrak{B}$ is determined in $\mathfrak{S}$ by equations of the type $t_{\mathfrak{S}}$ (i. e. if $\phi_{\mathfrak{R}} = \psi_{\mathfrak{R}}$ for any $\phi, \psi \in t_{\mathfrak{S}}$ having the same restriction on $\mathfrak{B}$ then $\mathfrak{R} \in \mathfrak{B}$).*

PROOF. Let $\mathfrak{S} = (t, l)$-Alg and U be the forgetful functor of $\mathfrak{S}$. Following [10] 6.5, $\mathfrak{B}$ is a union of an ascending chain $\mathfrak{M}_0 \subset \dots \mathfrak{M}_a \subset \dots$ of monadic categories indexed by all the ordinals. Let $\mathfrak{R}$ be an algebra from $\mathfrak{S}$ not belonging to $\mathfrak{B}$ and n be the cardinality of $U(\mathfrak{R})$. For any a there are n-ary terms p_a, q_a of type t such that the equation $p_a^- = q_a^-$ holds on $\mathfrak{M}_a$ and not on $\mathfrak{R}$. Since there is only a set of mappings $U(\mathfrak{R})^n \to U(\mathfrak{R})$ we may assume that

$$(p_a)^{\mathfrak{R}} = (p_\beta)^{\mathfrak{R}} \quad \text{and} \quad (q_a)^{\mathfrak{R}} = (q_\beta)^{\mathfrak{R}} \quad \text{for any } a, \beta .$$

Since terms induce natural transformations, following 3.1 and 3.5 there are $\phi, \psi \in t_{\mathfrak{E}}$ having the same restriction on $\mathfrak{B}$ and such that $\phi_{\mathfrak{R}} \neq \psi_{\mathfrak{R}}$.

4. EQUATIONAL COMPLETION.

4.1. LEMMA. *Let $\mathfrak{A}$ be a concrete category and $\phi, \psi \in t_{\mathfrak{E}(\mathfrak{A})}$ such that $\phi E_{\mathfrak{A}} = \psi E_{\mathfrak{A}}$. Then $\phi = \psi$.*

PROOF. Let $\mathfrak{E}(\mathfrak{A}) = (t, I)\text{-}Alg$ and consider the type $t' = t \cup \{\phi, \psi\}$. Then I is a theory of type t', and let $I' = I \cup \{\phi = \psi\}$. Then

$$\mathfrak{E} = (t', I)\text{-}Alg \quad \text{and} \quad \mathfrak{E}' = (t', I')\text{-}Alg$$

are equational categories. Denote by $T: \mathfrak{E}' \to \mathfrak{E}$ the inclusion and by $R: \mathfrak{L}(\mathfrak{E}(\mathfrak{A})) \to \mathfrak{E}$ the reduct. Since $\phi E_{\mathfrak{A}} = \psi E_{\mathfrak{A}}$, there is a concrete functor $F: \mathfrak{A} \to \mathfrak{E}'$ such that

$$R . L_{\mathfrak{E}(\mathfrak{A})} . E_{\mathfrak{A}} = T . F.$$

Hence there is a concrete functor

$$\hat{F}: \mathfrak{E}(\mathfrak{A}) \to \mathfrak{E}' \quad \text{with} \quad T . \hat{F} = R . L_{\mathfrak{E}(\mathfrak{A})}.$$

But it follows that $\phi = \psi$.

Now, we can state the main theorem.

4.2. THEOREM. *Let $\mathfrak{U}$ be weakly compact and $\mathfrak{A}$ a concrete category. Then an equational completion of $\mathfrak{A}$ exists iff $\mathfrak{B}(\mathfrak{A})$ is equational and $B_{\mathfrak{A}}$ is in this case the equational completion of $\mathfrak{A}$.*

PROOF. If $\mathfrak{B}(\mathfrak{A})$ is equational then $B_{\mathfrak{A}}$ evidently is the equational completion of $\mathfrak{A}$. Assume that $\mathfrak{E}(\mathfrak{A}) = (t, I)\text{-}Alg$ exists. There appear concrete functors

$$\hat{E}: \mathfrak{B}(\mathfrak{A}) \to \mathfrak{E}(\mathfrak{A}) \quad \text{and} \quad \hat{L}: \mathfrak{E}(\mathfrak{A}) \to \mathfrak{L}(\mathfrak{A})$$

with

$$\hat{E} . B = E \quad \text{and} \quad \hat{L} . E = L$$

(concerning $\hat{L}$ see 1.2). Since $\hat{L} . \hat{E}$ is the functor K from 3.2, $\mathfrak{B}(\mathfrak{A})$ is a Birkhoff subcategory of $\mathfrak{L}(\mathfrak{A})$ and hence of $\mathfrak{E}(\mathfrak{A})$, too. Following 3.6 and 4.1, $\mathfrak{E}(\mathfrak{A}) = \mathfrak{B}(\mathfrak{A})$ holds.

Hence a Beck category which is not equational can not have an equational completion. Concerning such a Beck category see [10], 6.1.

A concrete category $\mathfrak{A}$ will be called *stable* if

$$\phi_{\mathfrak{R}} = (\phi L)^{\mathfrak{R}} \quad \text{for all} \quad \phi \in t_{\mathfrak{L}(\mathfrak{A})} \quad \text{and all } \mathfrak{R} \in \mathfrak{L}(\mathfrak{A}).$$

4.3. THEOREM. *Consider the following statements for a given concrete category* $\mathfrak{A}$:

(i) $L_{\mathfrak{A}}: \mathfrak{A} \to \mathfrak{L}(\mathfrak{A})$ *is the Beck completion of* $\mathfrak{A}$.

(ii) $L_{\mathfrak{A}}: \mathfrak{A} \to \mathfrak{L}(\mathfrak{A})$ *is the equational completion of* $\mathfrak{A}$.

(iii) $\mathfrak{A}$ *is stable and* $\mathfrak{L}(\mathfrak{A})$ *is a category.*

Then (i) ⇒ (ii) ⇒ (iii). If $\mathfrak{U}$ *is weakly compact, then all three statements are equivalent.*

PROOF. Clearly (i) ⇒ (ii). Assume (ii) and let $\phi \in t_{\mathfrak{L}(\mathfrak{A})}$. Let $\psi \in t_{\mathfrak{L}(\mathfrak{A})}$ be given by the setting

$$\psi_{\mathfrak{R}} = (\phi L)^{\mathfrak{R}} \quad \text{for all} \quad \mathfrak{R} \in \mathfrak{L}(\mathfrak{A}).$$

Since $\phi L = \psi L$, $\phi = \psi$ holds by 4.1 and thus $\mathfrak{A}$ is stable.

Let $\mathfrak{U}$ be weakly compact and (iii) hold. Following 3.6, $\mathfrak{B}(\mathfrak{A})$ is determined in $\mathfrak{L}(\mathfrak{A})$ by equations of type $t_{\mathfrak{L}(\mathfrak{A})}$ and hence by equations of type $t_{\mathfrak{A}}$ because $\mathfrak{A}$ is stable. However, if an equation of type $t_{\mathfrak{A}}$ holds in $\mathfrak{B}(\mathfrak{A})$ then it also holds in $\mathfrak{A}$ and thus in $\mathfrak{L}(\mathfrak{A})$, too. Therefore we have $\mathfrak{B}(\mathfrak{A}) = \mathfrak{L}(\mathfrak{A})$ and (i) is proved.

If $\mathfrak{L}(\mathfrak{A})$ is monadic then $\mathfrak{A}$ clearly is stable and thus 4.3 implies that 2.1 holds for any tractable $\mathfrak{A}$ in a weakly compact $\mathfrak{U}$. Concerning an example of a stable concrete category $\mathfrak{A}$ such that $\mathfrak{L}(\mathfrak{A})$ is not monadic, see [11]. If $\mathfrak{A}$ is stable then it is easy to see that $\mathfrak{L}(\mathfrak{A})$ is canonically equational. I do not know whether the converse is true or not. Notice nevertheless that the consequence of 3.6 is that if $\mathfrak{U}$ is weakly compact and $\mathfrak{L}(\mathfrak{A})$ a canonically equational category, then $B_{\mathfrak{A}}$ is the equational completion of $\mathfrak{A}$.

4.4. PROPOSITION. *Let (t, I)-Alg be an equational completion of* $\mathfrak{A}$. *Then*

$(t_{\mathfrak{A}}, I_{\mathfrak{A}})$ is a conservative extension of (t, I).

PROOF. The result is given by the fact that for all terms p, q of type t, $pE_{\mathfrak{A}} = qE_{\mathfrak{A}}$ implies, by 4.1, that the equation $p = q$ follows from I.

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