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FACTORIZATION THEOREMS FOR GEOMETRIC MORPHISMS, I

by P. T. JOHNSTONE

0. INTRODUCTION.

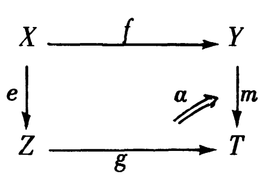
This paper is the first of two in which we aim to investigate a number of factorization structures (= bicategory structures in the sense of Mac Lane [14] or Isbell [10]) which can be imposed on the category $\mathfrak{Top}$ of toposes and geometric morphisms. Since factorization theorems play an important role in the study of the category of topological spaces and continuous maps (see, for example, [3, 5 and 7]) and of the category of small categories and functors [15], both of which are embeddable in $\mathfrak{Top}$, it seems worthwhile to investigate whether these factorization theorems have topos-theoretic generalizations. On the other hand, since $\mathfrak{Top}$ is in fact a bicategory (in the sense of Bénabou [2]!), it is necessary to interpret the term «factorization structure» in an «up-to-isomorphism» sense which is weaker than the usual one. We therefore begin by giving a precise definition.

A *factorization structure* on a bicategory $\mathcal{K}$ consists of a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$, both closed under composition and containing all equivalences of $\mathcal{K}$, and satisfying the following conditions:

(a) For every morphism $f: X \to Y$ of $\mathcal{K}$, there exists a factorization

$X \xrightarrow{e} Q \xrightarrow{m} Y$ with $me = f$, $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

(b) The elements of $\mathcal{E}$ are *orthogonal* to those of $\mathcal{M}$, i.e. given a square



commuting up to a 2-isomorphism α, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists $h: Z \to Y$ and 2-isomorphisms

$$\beta: he \to f \text{ and } \gamma: g \to mh \text{ such that } \alpha = (m*\beta)(\gamma*e).$$

Moreover, h is unique up to a 2-isomorphism which is itself uniquely determined by β and γ.

Given such a structure, we can derive the following familiar properties of $\mathcal{E}$ and $\mathcal{M}$ by straightforward generalizations of the usual 1-categorical methods:

0.1. LEMMA. *(i) $\mathcal{E} \cap \mathcal{M}$ consists precisely of the equivalences of $\mathcal{K}$.*

(ii) Given a composable pair

$$X \xrightarrow{f} Y \xrightarrow{g} Z \quad \text{with} \quad gf \in \mathcal{E} \text{ and } f \in \mathcal{E},$$

we have $g \in \mathcal{E}$. (The dual result holds for $\mathcal{M}$.)

(iii) Any morphism orthogonal to the whole of $\mathcal{M}$ is in $\mathcal{E}$. Hence $\mathcal{E}$ is determined by $\mathcal{M}$, and vice versa.

(iv) The factorization in part (a) of the definition is unique up to equivalence, the equivalence being itself unique up to unique 2-isomorphism.

We may also deduce the (pseudo-)functoriality of the factorization in (a), and the fact that it provides, for each object Y, a left adjoint for the inclusion $\mathcal{M}/Y \to \mathcal{K}/Y$, at least at the 1-categorical level; to check that these functors are defined on 2-arrows, we need a stronger version of condition (b), referring to squares which commute up to a non-invertible 2-arrow.

The most familiar example of a factorization system in $\mathcal{T}op$ is that given by surjections and inclusions [11, 4.14]. It is well-known that this corresponds to the surjection-inclusion factorization in the category of topological spaces, and to the factorization of functors into those which are surjective on objects and those which are full and faithful. However, it has a number of undesirable properties: it is not stable under pullback (a pullback of a surjection need not be surjective), and the class of inclusions is in some sense too restrictive (it is far from containing even all the split monomorphisms in $\mathcal{T}op$, whereas surjections need not be epi in any reasonable 2-categorical sense). Thus we are provided with additional reasons for seeking other factorization theorems in $\mathcal{T}op$.

1. THE LOCALIC FACTORIZATION.

In the present paper, we shall be concerned with a factorization which in one sense goes to the opposite extreme from the surjection-inclusion one, in that it makes the class $\mathcal{E}$ as small as one might reasonably expect.

We recall that a geometric morphism $f: \mathcal{F} \to \mathcal{E}$ is said to be *localic* if «$\mathcal{F}$ is generated relative to $\mathcal{E}$ by subobjects of 1», i.e. if 1 is an object of generators for $\mathcal{F}$ over $\mathcal{E}$ in the sense of [11, 4.43]. Explicitly, this means that for every object Y of $\mathcal{F}$ we can find an object X of $\mathcal{E}$ and a diagram of the form

$$f^*X \longleftarrow\!\!\!< S \longrightarrow\!\!\!\!\to Y$$

in $\mathcal{F}$. (One should think of $S \twoheadrightarrow Y$ as an «X-indexed cover of Y by subobjects of 1».)

The following lemma is an easy special case of [11, 4.44] and we omit the proof:

1.1. LEMMA. *Let*

$$\mathcal{G} \xrightarrow{\ g\ } \mathcal{F} \xrightarrow{\ f\ } \mathcal{E}$$

be a composable pair of geometric morphisms.

(i) If f and g are localic, so is the composite fg.

(ii) If fg is localic, so is g.

For localic morphisms, the relative Giraud Theorem [11, 4.46] can be made more explicit:

1.2. LEMMA. *The following conditions on an $\mathcal{E}$-topos $f: \mathcal{F} \to \mathcal{E}$ are equivalent:*

(i) f is localic.

(ii) There exists an internal poset A in $\mathcal{E}$ such that $\mathcal{F}$ is equivalent to a sheaf subtopos of $\mathcal{E}^{A^{op}}$.

(iii) There exists an internal locale (= complete Heyting algebra) A in $\mathcal{E}$ such that $\mathcal{F}$ is equivalent to the topos $\mathcal{E}[A]$ of $\mathcal{E}$-valued sheaves (for the canonical topology) on A.

PROOF. (iii) ⇒ (ii) is trivial; and (ii) ⇒ (i) by a relativized version of the proof of [11, 5.34]. For (i) ⇒ (iii), we take $A = f_*(\Omega_{\mathcal{F}})$ and work through the proof of the relative Giraud theorem (cf. also [11, 5.37]).

If $f: \mathcal{F} \to \mathcal{E}$ is not localic, the class of objects of $\mathcal{F}$ which can be expressed as quotients of subobjects of «constant objects» f^*X is clearly of interest. We shall (until further notice) write $\mathcal{G}$ for the full subcategory of $\mathcal{F}$ consisting of all such objects: we shall say that f is *hyperconnected* if f^* induces an equivalence between $\mathcal{E}$ and $\mathcal{G}$, i.e. if f^* is full and faithful and its image in $\mathcal{F}$ is closed under subobjects and quotients.

1.3. LEMMA. *Let $f: \mathcal{F} \to \mathcal{E}$ be a geometric morphism. and let $\mathcal{G}$ be the full subcategory of $\mathcal{F}$ defined as above. Then $\mathcal{G}$ is a topos, and the inclusion $\mathcal{G} \to \mathcal{F}$ is the inverse image of a geometric morphism $h: \mathcal{F} \to \mathcal{G}$.*

PROOF. Since f^* preserves finite products (and since a product of epimorphisms in $\mathcal{F}$ is epi), it is clear that the class of objects of $\mathcal{G}$ is closed under finite products. It is also closed under arbitrary subobjects in $\mathcal{F}$ (in particular, under equalizers), since pullbacks of epis are epi. So $\mathcal{G}$ has finite limits, and the inclusion $h^*: \mathcal{G} \to \mathcal{F}$ preserves them.

If Y is in $\mathcal{G}$, we have a canonical way of choosing the objects X and S which prove it, as follows: form the pullback

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ f^*f_*\tilde{Y} & \xrightarrow{\epsilon} & \tilde{Y} \end{array}$$

where $\tilde{Y}$ is the partial-map representer for Y [11, 1.25] and ϵ is the counit of $(f^* \dashv f_*)$. Now if we are given X and S as before, there is a unique $\alpha: f^*X \to \tilde{Y}$ such that

$$\begin{array}{ccc} S & \twoheadrightarrow & Y \\ \downarrow & & \downarrow \\ f^*X & \xrightarrow{\alpha} & \tilde{Y} \end{array}$$

is a pullback: and then α factors as

$$f^*X \xrightarrow{f^*\bar{\alpha}} f^*f_*\tilde{Y} \xrightarrow{\epsilon} \tilde{Y}.$$

So we get a factorization of $S \twoheadrightarrow Y$ through $P \to Y$, and in particular the latter must be epi; thus we have the canonical choice

$$X = f_*\tilde{Y}, \quad S = P.$$

For an arbitrary object Y of $\mathfrak{F}$, let us form the pullback P as above and then consider the image factorization

$$P \twoheadrightarrow h_*Y \rightarrowtail Y$$

of $P \to Y$. h_* is clearly a functor $\mathfrak{F} \to \mathfrak{F}$, since all the constructions involved in its definition are functorial; and in fact h_*Y is in $\mathfrak{G}$, since P is a subobject of $f^*f_*\tilde{Y}$. Moreover, the monomorphism $h_*Y \rightarrowtail Y$ defines a natural transformation from h_* to the identity, which is an isomorphism precisely when Y is in $\mathfrak{G}$. It now follows at once that $h_*: \mathfrak{F} \to \mathfrak{G}$ is right adjoint to the inclusion h^*.

The adjunction $(h^* \dashv h_*)$ is coreflective and therefore comonadic, and we saw earlier that h^* is left exact. So by [11, 2.32] $\mathfrak{G}$ is a topos and h is a geometric morphism.

1.4. THEOREM. *The pair (hyperconnected morphisms, localic morphisms) is a factorization structure on* $\mathfrak{Top}$.

PROOF. (a) Given a morphism $f: \mathfrak{F} \to \mathfrak{E}$, define $\mathfrak{G}$ and $h: \mathfrak{F} \to \mathfrak{G}$ as in Lemma 1.3. It is clear that h is hyperconnected, and since the image of f^* is contained in $\mathfrak{G}$, we can regard f^* as a composite

$$\mathfrak{E} \xrightarrow{g^*} \mathfrak{G} \xrightarrow{h^*} \mathfrak{F}.$$

Also, if Y is any object of $\mathfrak{G}$ and X any object of $\mathfrak{E}$, we have natural bijections

$$\begin{aligned} hom(g^*X, Y) \approx hom(h^*g^*X, h^*Y) &= hom(f^*X, h^*Y) \\ &\approx hom(X, f_*h^*Y), \end{aligned}$$

so that the composite f_*h^* is a right adjoint g_* for g^*. It is clear that g^* is left exact, so we have a factorization $f \approx gh$ in $\mathfrak{Top}$. The fact that

g is localic is immediate from the definition of $\mathcal{G}$.

(b) Suppose given a commutative square (up to isomorphism)

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{k} & \mathcal{F} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{G} & \xrightarrow{g} & \mathcal{E} \end{array}$$

where f is localic and h is hyperconnected. For any object Y of $\mathcal{F}$, we have a diagram

$$f^*X \leftarrowtail S \twoheadrightarrow Y$$

in $\mathcal{F}$, and hence a diagram

$$h^*g^*X \approx k^*f^*X \leftarrowtail k^*S \twoheadrightarrow k^*Y$$

in $\mathcal{H}$. Since the image of h^* is closed under subobjects and quotients, it follows that it contains k^*Y, and hence we have a factorization $l^*: \mathcal{F} \to \mathcal{G}$ (unique up to unique isomorphism) of k^* through h^*. As before, we find that the composite $l_* = k_*h^*$ is right adjoint to k^*; l^* is left exact since h^* creates finite limits; and since h^* is full and faithful the isomorphism

$$h^*g^* \approx k^*f^* \approx h^*l^*f^*$$

uniquely induces an isomorphism $g^* \approx l^*f^*$.

Although localic morphisms have been fairly widely studied, the class of hyperconnected morphisms has received less attention. The next result provides various equivalent characterizations of this class.

1.5. PROPOSITION. *The following conditions on a geometric morphism* $f: \mathcal{F} \to \mathcal{E}$ *are equivalent:*

(i) f *is hyperconnected.*

(ii) f^* *is full and faithful, and its image is closed under subobjects in* $\mathcal{F}$.

(iii) f^* *is full and faithful, and its image is closed under quotients in* $\mathcal{F}$.

(iv) The unit and counit of the adjunction $(f^* \dashv f_*)$ *are both mono.*

(v) f_* *preserves* Ω, *i.e. the comparison map* $f_*(\Omega_{\mathcal{F}}) \to \Omega_{\mathcal{E}}$ *is an*

isomorphism. (Equivalently, $f_*(true_{\mathcal{F}}): 1 \rightarrowtail f_*(\Omega_{\mathcal{F}})$ *is a subobject classifier in* $\mathcal{E}$*.)*

PROOF. (i) ⇒ (ii) and (i) ⇒ (iii) are trivial.

(iii) ⇒ (iv) : if f^* is full and faithful, then the unit of $(f^* \dashv f_*)$ is iso, since we can identify $\mathcal{E}$ with a coreflective subcategory of $\mathcal{F}$. Now consider the counit map $\epsilon_Y : f^* f_* Y \to Y$, and form its image factorization

$$f^* f_* Y \xrightarrow{\ q\ } I \rightarrowtail Y.$$

By assumption, I is in the image of f^*, so from the universal property of f^*, we deduce that q must be split mono and therefore iso. So ϵ_Y is a monomorphism.

(iv) ⇒ (v): First we deduce from (iv) that the unit η of $(f^* \dashv f_*)$ is actually iso. From the triangular identity

$$\begin{array}{ccc} f^* & \xrightarrow{\ f^*\eta\ } & f^* f_* f^* \\ & \searrow^{1_{f^*}} & \downarrow{\scriptstyle \epsilon f^*} \\ & & f^* \end{array}$$

and the fact that ϵ is mono, we deduce that $f^*\eta$ is iso ; but since η is mono, we know that f^* is faithful and so reflects isomorphisms. So η is iso ; hence f^* is full as well as faithful, and so we may identify $\mathcal{E}$ with a coreflective subcategory of $\mathcal{F}$. Since f^* preserves 1, it is easily seen that the square

$$\begin{array}{ccc} f_*(1_{\mathcal{F}}) & \longrightarrow & 1_{\mathcal{F}} \\ {\scriptstyle f_*(true_{\mathcal{F}})}\downarrow & & \downarrow{\scriptstyle true_{\mathcal{F}}} \\ f_*(\Omega_{\mathcal{F}}) & \xrightarrow{\ \epsilon_\Omega\ } & \Omega_{\mathcal{F}} \end{array}$$

commutes, and since ϵ_Ω is mono it must be a pullback. Now if X is any object of $\mathcal{E}$, each subobject $X' \rightarrowtail X$ in $\mathcal{E}$ is mono in $\mathcal{F}$ and hence has a classifying map $\phi : X \to \Omega$, which must factor uniquely through ϵ_Ω. Now in the following diagram, the outer and right-hand square are pullbacks, and the left-hand square commutes (since X' is in $\mathcal{E}$); so an easy diagram-chase shows that the left-hand square is a pullback. On the other hand

$$\begin{array}{ccccc} X' & \longrightarrow & 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle f_*(true)} & & \downarrow{\scriptstyle true} \\ X & \xrightarrow[\bar{\phi}]{} & f_*\Omega & \xrightarrow[\epsilon\Omega]{} & \Omega \end{array}$$

the requirement that this square be a pullback determines $\bar{\phi}$ uniquely; so $f_*(true_{\mathcal{F}})$ is a subobject classifier for $\mathcal{E}$.

(ii) ⇒ (v): As above, we identify $\mathcal{E}$ with a coreflective subcategory of $\mathcal{F}$, and form the pullback

$$\begin{array}{ccc} P & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle true_{\mathcal{F}}} \\ f^*\Omega_{\mathcal{F}} & \xrightarrow{\epsilon\Omega} & \Omega_{\mathcal{F}} \end{array} .$$

P is a subobject of $f_*\Omega_{\mathcal{F}}$, so by hypothesis it lies in $\mathcal{E}$. As before, we show that any subobject $X' \rightarrowtail X$ in $\mathcal{E}$ is uniquely obtainable as a pullback of $P \rightarrowtail f_*\Omega$, i.e. the latter is a subobject classifier in $\mathcal{E}$. It now follows that P is a terminal object in $\mathcal{E}$ (and hence also in $\mathcal{F}$); i.e. $P \to 1$ is iso and $P \rightarrowtail f_*\Omega$ must therefore be isomorphic to $f_*(true_{\mathcal{F}})$.

(v) ⇒ (i): Given a morphism $f: \mathcal{F} \to \mathcal{E}$ satisfying (v), form its factorization

$$\mathcal{F} \xrightarrow{h} \mathcal{G} \xrightarrow{g} \mathcal{E}$$

as in 1.4. Then h is hyperconnected and so satisfies (v) by the above argument; so an easy diagram-chase shows that g also satisfies (v). But g is localic, and so by 1.2 $\mathcal{G}$ is equivalent as an $\mathcal{E}$-topos to

$$\mathcal{E}[g_*\Omega_{\mathcal{G}}] \simeq \mathcal{E}[\Omega_{\mathcal{E}}] \simeq \mathcal{E}.$$

That is, g is an equivalence, and so f is hyperconnected.

1.6. COROLLARY. *A hyperconnected geometric morphism is open in the sense of* [*13*].

PROOF. Let $f: \mathcal{F} \to \mathcal{E}$ be hyperconnected. By 1.5 (v) the comparison map $f^*\Omega_{\mathcal{E}} \to \Omega_{\mathcal{F}}$ is, up to isomorphism, just the counit map

$$f^*f_*\Omega_{\mathcal{F}} \longrightarrow \Omega_{\mathcal{F}};$$

and by 1.5 (iv) the latter is mono. Now consider the comparison map

$$f^*(Y^X) \longrightarrow f^*Y^{f^*X}$$

where X and Y are objects of $\mathfrak{E}$. Regarding $\mathfrak{E}$ as a coreflective subcategory of $\mathfrak{F}$, it is easy to see that exponentials in $\mathfrak{E}$ may be computed by forming exponentials in $\mathfrak{F}$ and then coreflecting. So Y^X is isomorphic to $f_*(f^*Y^{f^*X})$, and the above comparison map may again be identified with the counit of $(f^* \dashv f_*)$. Hence it is a monomorphism, and f is open by [13, 1.2].

2. STABILITY UNDER PULLBACKS.

2.1. PROPOSITION. *A pullback of a localic morphism in* $\mathfrak{Top}$ *is localic.*

PROOF. Consider the pullback of

$$\mathfrak{E}[A] \longrightarrow \mathfrak{E} \quad \text{along} \quad f: \mathfrak{F} \longrightarrow \mathfrak{E}.$$

Since $\mathfrak{E}[A]$ is a sheaf subtopos of the presheaf topos $\mathfrak{E}^{A^{op}}$, it follows from [11, 4.47] that the pullback is a sheaf subtopos of $\mathfrak{F}^{f^*A^{op}}$. So by 1.2 the pullback is localic over $\mathfrak{F}$.

Following [12], we shall write $f^\# A$ for the internal locale in $\mathfrak{F}$ corresponding to the pullback of $\mathfrak{E}[A]$ along f. Using the equivalence between internal locales in $\mathfrak{E}$ and localic $\mathfrak{E}$-toposes, we may easily deduce

2.2. COROLLARY. *$f^\#$ is a functor from the category $loc(\mathfrak{E})$ of internal locales in $\mathfrak{E}$ to $loc(\mathfrak{F})$, and is right adjoint to*

$$f_*: loc(\mathfrak{F}) \longrightarrow loc(\mathfrak{E}).$$

PROOF. The functoriality of $f^\#$ follows from the functoriality of pullback along f. Let B be a locale in $\mathfrak{F}$; then locale morphisms $B \to f^\# A$ correspond to geometric morphisms $\mathfrak{F}[B] \to \mathfrak{F}[f^\# A]$ over $\mathfrak{F}$, and hence to morphisms $\mathfrak{F}[B] \to \mathfrak{E}[A]$ over $\mathfrak{E}$. But the orthogonality condition of 1.4 tells us that any such map factors (essentially uniquely) through the hyperconnected part of the composite

$$\mathfrak{F}[B] \xrightarrow{\ g\ } \mathfrak{F} \xrightarrow{\ f\ } \mathfrak{E};$$

and by 1.5 (v) the image of this composite is equivalent to

$$\mathcal{E}[f_* g_*(\Omega_{\mathcal{F}[B]})] \approx \mathcal{E}[f_* B].$$

So, locale morphisms $B \to f^\# A$ in $\mathcal{F}$ correspond to locale morphisms $f_* B \to A$ in $\mathcal{E}$.

To show that the factorization of 1.4 is stable under pullback, it suffices after 2.1 to show that pullbacks of hyperconnected morphisms are hyperconnected. In fact this follows easily (at least for pullbacks along bounded morphisms) from the results on open surjections in [13].

2.3. PROPOSITION. *A pullback of a hyperconnected morphism along a bounded morphism is hyperconnected.*

PROOF. Let

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{k} & \mathcal{G} \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ \mathcal{F} & \xrightarrow{f} & \mathcal{E} \end{array}$$

be a pullback square with f hyperconnected and g bounded. By Corollary 1.6, f is an open surjection, so by [13, Theorem 4.7] k is an open surjection. In particular, this tells us that the unit of $(k^* \dashv k_*)$ is mono, so by 1.5 (iv) it remains to prove that the counit is also mono. To do this, we consider the particular cases:

(a) when g is an inclusion, and

(b) when $\mathcal{G}$ has the form $\mathcal{E}^{C^{op}}$ for some internal category C in $\mathcal{E}$.

Since every bounded morphism is a composite of morphisms of these two types, this will suffice.

(a) Suppose g is an inclusion. Then the counit of $(g^* \dashv g_*)$ is iso, and so the counit of $(k^* \dashv k_*)$ is isomorphic to the counit of the composite adjunction $(k^* g^* \dashv g_* k_*)$. But the latter can be rewritten as

$$(h^* f^* \dashv f_* h_*);$$

the counit of $(f^* \dashv f_*)$ is mono by assumption, and that of $(h^* \dashv h_*)$ is iso, since h is an inclusion. Since h^* preserves monomorphisms the result follows.

(b) Suppose $\mathcal{G} \approx \mathcal{E}^{C^{op}}$; then $\mathcal{H} \approx \mathcal{F}^{f^*C^{op}}$, and we can identify k^*, k_* respectively with the functors:

«apply f^* to discrete fibrations over C»

and

«apply f_* to discrete fibrations over f^*C, then pullback along the unit $C \to f_*f^*C$».

But since f is hyperconnected, the above unit map is an isomorphism, and so the fact that the counit of $(k^* \dashv k_*)$ is mono follows at once from the corresponding fact for f.

2.4. COROLLARY. *The hyperconnected-localic factorization of an arbitrary geometric morphism is stable under pullback along bounded morphisms.*

PROOF. Combine 2.1 and 2.3 with the uniqueness of the factorization (Lemma 0.1 (iv)).

As a by-product of 2.3, we have yet another characterization of hyperconnected morphisms:

2.5. PROPOSITION. *A geometric morphism $f: \mathcal{F} \to \mathcal{E}$ is hyperconnected iff the adjunction $(f_* \dashv f^\#)$ of Corollary 2.2 is a reflection (i.e. $f^\#$ is full and faithful).*

PROOF. If the adjunction is a reflection, then the counit map

$$f_*f^\#(\Omega_{\mathcal{E}}) \longrightarrow \Omega_{\mathcal{E}}$$

is an isomorphism in $loc(\mathcal{E})$. But from the definition of $f^\#$, it is clear that $f^\#(\Omega_{\mathcal{E}}) \approx \Omega_{\mathcal{F}}$, and so this tells us that f_* preserves Ω, i.e. f is hyperconnected. Conversely, if f is hyperconnected, then so is its pullback $\mathcal{F}[f^\#A] \to \mathcal{E}[A]$ for any locale A in $\mathcal{E}$; so the hyperconnected-localic factorization of the composite

$$\mathcal{F}[f^\#A] \longrightarrow \mathcal{F} \xrightarrow{\ f\ } \mathcal{E}$$

is equivalent to

$$\mathcal{F}[f^\#A] \longrightarrow \mathcal{E}[A] \longrightarrow \mathcal{E},$$

and hence the counit $f_*f^\#A \to A$ is an isomorphism in $loc(\mathcal{E})$.

3. EXAMPLES AND APPLICATIONS.

Since any morphism between localic $\mathcal{S}$-toposes is localic by 1.1, the hyperconnected-localic factorization clearly cannot give us any useful information about morphisms of spatial toposes. However, for morphisms of presheaf toposes, we do obtain a familiar factorization. In what follows, we shall assume for notational convenience that $\mathcal{S}$ is the topos of constant sets, but since our arguments are constructive they will in fact work over any base topos.

3.1. PROPOSITION. *Let $f: D \to C$ be a functor between small categories.*

(i) If f is faithful, then the induced geometric morphism $\mathcal{S}^{D^{op}} \to \mathcal{S}^{C^{op}}$ is localic.

(ii) If f is full and essentially surjective on objects, then the induced geometric morphism $\mathcal{S}^{D^{op}} \to \mathcal{S}^{C^{op}}$ is hyperconnected.

PROOF. (i) Suppose f is faithful. Now every functor $D^{op} \to \mathcal{S}$ may be expressed as a quotient of a coproduct of representable functors; but faithfulness of f implies that the canonical natural transformation

$$hom_D(\cdot, d) \longrightarrow hom_C(f(\cdot), f(d)) = f^*(hom_C(\cdot, f(d)))$$

is a monomorphism. Since a coproduct of monos is mono, the result follows.

(ii) If f is full and essentially surjective, then C is equivalent to the quotient category D/Q, where Q is the congruence on D induced by f (i.e. the set of parallel pairs (α, β) such that $f\alpha = f\beta$). Hence we can identify $\mathcal{S}^{C^{op}}$ with the full subcategory of $\mathcal{S}^{D^{op}}$ consisting of presheaves which respect the congruence Q; since this subcategory is clearly closed under subobjects and quotients in $\mathcal{S}^{D^{op}}$, the result follows.

3.2. COROLLARY. *For a functor $f: D \to C$, the hyperconnected-localic factorization of the induced morphism $\mathcal{S}^{D^{op}} \to \mathcal{S}^{C^{op}}$ corresponds to the factorization*

$$D \longrightarrow D/Q \longrightarrow C$$

of f, where Q is the congruence on D induced by f. (In particular if f

is a monoid homomorphism, this is just the usual image factorization in the category of monoids.)

It is also of interest to characterise those morphisms for which the hyperconnected-localic factorization coincides with the more usual surjection-inclusion one. In this direction we have the following result.

3.3. PROPOSITION. *The following conditions on a geometric morphism* $f: \mathcal{F} \to \mathcal{E}$ *are equivalent:*

(i) The hyperconnected-localic and surjection-inclusion factorizations of f coincide.

(ii) f can be factored as the composite of a hyperconnected morphism and an inclusion.

(iii) The counit of $(f^* \dashv f_*)$ *is mono.*

PROOF. (i) ⇔ (ii) is trivial. For (ii) ⇒ (iii), we note that both hyperconnected morphisms and inclusions satisfy (iii), and that this property is stable under composition. Conversely, if f satisfies (iii), form its surjection-inclusion factorization

$$\mathcal{F} \xrightarrow{\ h\ } \mathcal{G} \xrightarrow{\ g\ } \mathcal{E}.$$

Then the comonad on $\mathcal{F}$ induced by $(h^* \dashv h_*)$ is the same as that induced by $(f^* \dashv f_*)$, so its counit is mono; but h is also surjective, so by 1.5 (iv) it is hyperconnected.

3.4. COROLLARY. *Let* $f: D \to C$ *be a functor between small categories. Then f is full iff the counit of the adjunction* $(f^* \dashv \varprojlim_f)$ *is a monomorphism.*

PROOF. Clearly, f is full iff the second half of the factorization of 3.2 is full as well as faithful, i.e. iff the geometric morphism it induces is an inclusion. So this result is immediate from 3.3. (In fact it is not hard to prove directly.)

In [8 and 9], P. Freyd has investigated *exponential varieties* in Grothendieck toposes, i.e. full subcategories which are closed under the formation of arbitrary limits, colimits and power objects. If $\mathcal{E}$ is such a subcategory of $\mathcal{F}$, then the inclusion $\mathcal{E} \to \mathcal{F}$ has both left and right adjoints, and so we have a geometric morphism $f: \mathcal{F} \to \mathcal{E}$ which is connected and atomic (i.e. its inverse image is full, faithful and logical). Now any such morphism must in fact be hyperconnected, since the bijection

$$hom(X, \Omega_{\mathcal{E}}) \approx hom(f^*X, f^*\Omega_{\mathcal{E}}) \approx hom(f^*X, \Omega_{\mathcal{F}})$$

implies that every subobject of f^*X in $\mathcal{F}$ is in the image of f^*. A. Joyal has recently given a simple proof of Freyd's representation theorem [9] which depends on the fact that connected atomic morphisms are stable under (bounded) pullback; this follows easily from 2.3 and the fact (proved in [1]) that atomic morphisms are stable under pullback.

Freyd's own proof of the representation theorem involves studying locales in the topos $\mathcal{C}(G)$ of continuous G-sets, where G is a topological group. He proves that every such locale is obtained from a locale in the topos $\mathcal{S}^G$ of all G-sets by applying the coreflection functor

$$\mathcal{S}^G \longrightarrow \mathcal{C}(G).$$

Since this coreflection is the direct image of a hyperconnected morphism $\mathcal{S}^G \to \mathcal{C}(G)$, this fact is easily seen to be a special case of our Proposition 2.5.

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REFERENCES.

1. M. BARR & R. DIACONESCU, Atomic toposes, *J. Pure Appl. Algebra* 17 (1980) 1-24.
2. J. BENABOU, Introduction to bicategories, *Lecture Notes in Math.* 47, Springer (1967), 1-77.
3. A. BŁASZCZYK, A factorization theorem and its application to extremally disconnected resolutions, *Colloq. Math.* 28 (1973), 33-40.
4. M. J. BROCKWAY, D. Phil. Thesis, Oxford University, 1980.
5. P. J. COLLINS, Concordant mappings and the concordant-dissonant factorization of an arbitrary continuous function, *Proc. A. M. S.* 27 (1971), 587-591.
6. M. COSTE, *La démonstration de Diaconescu du théorème de Barr*, Séminaire Bénabou, Université Paris-Nord 1975 (unpublished).
7. R. DYCKOFF, Factorization theorems and projective spaces in Topology, *Math. Zeit.* 127 (1972), 256-264.
8. P. FREYD, The axiom of choice, *J. Pure Appl. Alg.* (to appear).
9. P. FREYD, *All topoi are localic, or, Why permutation models prevail.* Preprint, University of Pennsylvania, 1979.
10. J. R. ISBELL, Subobjects, adequacy, completeness and categories of algebras, *Rozprawy Mat.* 36 (1964).
11. P. T. JOHNSTONE, *Topos theory,* Academic Press, London, 1977.
12. P. T. JOHNSTONE, Factorization and pullback theorems for localic geometric morphisms, *Univ. Cath. de Louvain, Sém. Math. Pures,* Rapport 79 (1979).
13. P. T. JOHNSTONE, Open maps of toposes, *Man. Math.* 31 (1980), 217-247.
14. S. MAC LANE, Duality for groups, *Bull. A. M. S.* 56 (1950), 485-516.
15. R. STREET & R. F. C. WALTERS, The comprehensive factorization of a functor, *Bull. A. M. S.* 79 (1973), 936-941.

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