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Cat AS A CLOSED MODEL CATEGORY

by R. W. THOMASON *)

In this paper, I shall show that *Cat*, the category of small categories, admits a closed model structure in the sense of Quillen [2, 3, 15, 16]. A closed model structure on a category permits one to do abstract homotopy theory in that category. Prominent examples of categories admitting such a structure are those of topological spaces [15], simplicial sets [15], simplicial spectra [2], shapes [4], differential graded Q-algebras [16], and simplicial commutative algebras [15]. The closed model structure provides analogues in *Cat* of most of the paraphenalia of homotopy theory: homotopy fibres of maps, loops and suspensions of objects, Toda brackets, and indeed general homotopy limits and colimits [1]. The question of the existence of such a structure has thus been raised by several mathematicians interested in the homotopy theory of categories [8, 9, 10].

Let me recall the existing framework for doing homotopy theory of categories. The best reference for further details is Quillen's paper [14].

There is a nerve functor N from *Cat* to the category of simplicial sets. By taking the geometric realization of the nerve $N\mathrm{C}$ of a category C, one produces the classifying space $B\mathrm{C}$. A morphism f in *Cat* is called a weak equivalence if Nf is a weak equivalence of simplicial sets, or what is the same, if Bf is a homotopy equivalence of the classifying spaces. One may regard the homotopy type of $B\mathrm{C}$ as the homotopy type of C, and ask how the structure of C is reflected in its homotopy type.

The closed model structure on *Cat* is lifted from the one on simplicial sets in the following sense. The nerve functor N has the property that a morphism f in *Cat* is a weak homotopy equivalence iff Nf is a weak homotopy equivalence of simplicial sets. Quillen showed ([11], VI,

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3.3.1) that N induces an equivalence between the two homotopy categories obtained by inverting the weak homotopy equivalences. Other examples of equivalences of homotopy categories suggest that there should be an adjoint pair of functors between *Cat* and *Simplicial Sets* such that the adjunction natural transformations are weak homotopy equivalences. The adjoint pair would then induce inverse equivalences of the two homotopy categories.

The functor N does have a left adjoint, the functor

$$c: \textit{Simplicial Sets} \to \textit{Cat}.$$

For X a simplicial set, cX is the category formed by taking the category whose objects are the 0-simplices of X and whose morphisms are freely generated by the 1-simplices of X, with each 1-simplex x regarded as a morphism from $d_1 x$ to $d_0 x$, and then imposing the relations on morphisms

$$d_1 x = d_0 x . d_2 x \quad \text{for each 2-simplex } x$$

([6], II.4). However, the adjunction map $Id \to Nc$ is not in general a weak homotopy equivalence: applied to the boundary of an n-simplex with $n > 2$ this map is

$$\dot{\Delta}[n] \to Nc\dot{\Delta}[n] \approx \Delta[n] .$$

Recall the adjoint pair of endofunctors on *Simplicial Sets*, Sd^2 and Ex^2 from ([12], Section 3, 7). Here Sd^2 is the subdivision functor Sd iterated, and Ex^2 is its right adjoint. These yield an adjoint pair of functors between *Cat* and *Simplicial Sets*, cSd^2 and Ex^2N. I discovered that the adjunction maps

$$Id \to Ex^2NcSd^2 \quad \text{and} \quad cSd^2Ex^2N \to Id$$

are weak homotopy equivalences. For a proof of this, the reader is referred to the paper of Fritsch and Latch [5].

Now *Simplicial Sets* carries a closed model structure, i.e., it has distinguished classes of maps called cofibrations, weak equivalences, and fibrations satisfying certain axioms. The adjoint pair cSd^2 and Ex^2N is used to lift this structure to *Cat*, in the sense that a morphism f in *Cat* will be a weak equivalence or fibration iff $Ex^2 Nf$ is such in *Simplicial Sets*.

Not every category can be fibrant, as would have been true if the attempt of Golasinski [8] to give *Cat* a closed model structure had succeeded.

To see this, define the n^{th} homotopy group of a based object C of *Cat* or *Simplicial Sets* as the group of based maps in the homotopy category from the homotopy type of an n-sphere into C. Using a closed model structure as in ([15], 1.16, Corollary 1), one sees that, for C fibrant, $\pi_n C$ is isomorphic to the set of maps $S^n \to C$ in *Cat* or *Simplicial Sets*, modulo an equivalence relation of homotopy. Here S^n is any cofibrant object of the homotopy type of an n-sphere. (In general, not every map between the homotopy types of S^n and C is induced by a map of given simplicial sets representing these types, unless the source is cofibrant and the target fibrant.) For a product of fibrant objects, this implies that the homotopy group of the product is the product of the homotopy groups. If all categories were fibrant, this would then hold for any family of categories C_i indexed by a set I. As the nerve functor preserves products and induces an equivalence of homotopy categories, this would also imply

$$\pi_n(\Pi N C_i) \approx \Pi\, \pi_n(N C_i).$$

Now let I be any infinite set, and each C_i the category with two objects, 0, 1 and two distinct non-identity morphisms which run from 0 to 1. Then $\pi_1 N C_i$ is $\mathbb{Z}$, and $\Pi \pi_1 N C_i$ is the group of all functions $I \to \mathbb{Z}$. However, calculation of $\pi_1(\Pi N C_i)$ by means of the usual presentation of π_1 of a simplicial set ([6], II.7) shows it is the group of all *bounded* functions $I \to \mathbb{Z}$. As this differs from the previous group, the C_i cannot be fibrant in any closed model structure on *Cat* for which the nerve functor induces an equivalence of homotopy categories.

I would like to thank Dan Kan for suggesting the problem of lifting a closed model structure from *Simplicial Sets* to *Cat*. John Moore, Joe Neisendorfer, and Paul Selick helped me realize why I couldn't prove the homotopy groups of a product of arbitrary simplicial sets were the products

of the homotopy groups. William Dwyer, Rudolf Fritsch and Dana Latch simplified some of my first proofs and explained them to me. Finally, Don Anderson, Dan Quillen and Steve Wilson were generally helpful.

1. A closed model structure on a category $\mathcal{C}$ consists of three distinguished classes of morphisms: the cofibrations, the fibrations, and the weak equivalences. The structure is required to satisfy the axioms CM 1 through CM 5 below. These axioms are given in the form used in [16], II.1; they are equivalent to those in [15].

CM 1 : The category $\mathcal{C}$ has finite limits and colimits.

CM 2: For any composable pair f, g of morphisms in $\mathcal{C}$, if any two of f, g, gf are weak equivalences, so is the third.

CM 3 : If f is a retract of g and g is a weak equivalence, fibration, or cofibration, then f is also such.

Recall a morphism $f: X \to Y$ is a retract of $g: W \to Z$ if there are morphisms

$$i: X \to W, \quad r: W \to X, \quad j: Y \to Z \quad \text{and} \quad s: Z \to W$$

such that $ri = 1_X$, $sj = 1_Y$ (so X is a retract of W and Y is a retract of Z) and also $gi = jf$, $fr = sg$.

CM 4 (Lifting): Given a diagram of solid arrows (1.1) with i a cofibration and p a fibration, if either i or p is also a weak equivalence, then there is a dotted arrow f making the diagram commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "f"] & Y \end{tikzcd} \tag{1.1}$$

CM 5 (Factorization): Any morphism f may be factored both as $f = pi$ and $f = qj$, where p and q are fibrations, i and j are cofibrations, and p and j are weak equivalences.

There is a slightly stronger notion of a proper closed model structure, as discussed in [2, 16]. These structures must satisfy an additional axiom:

PROPRIETY: Consider the diagram (1.2)

$$(1.2)\qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r, "g"] & D \end{tikzcd}$$

If it is a pushout square with i a cofibration and f a weak equivalence, then g is a weak equivalence. If it is a pullback square with p a fibration and g a weak equivalence, then f is a weak equivalence.

The standard example of such a structure is the category *Simplicial Sets*, monomorphisms being the cofibrations, and Kan fibrations being the fibrations. The weak equivalences are the maps whose geometric realizations are homotopy equivalences. For details one may consult [2], [3] VIII or [15] II, Section 3.

A morphism which is both a cofibration and a weak equivalence is called a trivial cofibration. A morphism which is both a fibration and a weak equivalence is called a trivial fibration. An object X is cofibrant if the unique morphism from the initial object, $\emptyset \to X$, is a cofibration. An object X is fibrant if the unique morphism to the terminal object, $X \to *$, is a fibration.

2. From the introduction, recall the functor $Ex^2 N: Cat \to Simplicial\ Sets$.

DEFINITION 2.1. A morphism f in *Cat* is a *weak equivalence* iff $Ex^2 Nf$ is a weak equivalence in *Simplicial Sets*.

DEFINITION 2.2. A morphism f in *Cat* is a *fibration* iff $Ex^2 N f$ is a fibration in *Simplicial Sets*, i.e., $Ex^2 N f$ is a Kan fibration.

DEFINITION 2.3. A morphism i in *Cat* is a *cofibration* iff it has the lifting property with respect to trivial fibrations required by the axiom CM4. That is, $i: A \to B$ is a cofibration iff, whenever one has the diagram of solid arrows (1.1) with p a fibration and a weak equivalence in *Cat*, then there exists a dotted arrow f making the diagram commute.

For any small category C , π_1 C is the fundamental groupoid of C , i.e., the category obtained by formally inverting all morphisms in C . It is equal to the fundamental groupoid of the nerve, $\pi_1 N$ C , as defined by [6], II, Section 7, and is equivalent to the path groupoid of B C .

PROPOSITION 2.4. *The following are equivalent for a morphism* f: C → D *in Cat :*

1° f is a weak equivalence, i. e., $Ex^2 N f$ is a weak equivalence.

2° $N f$ is a weak equivalence.

3° f induces an equivalence of groupoids π_1 C → π_1 D, and for any local coefficient system F on D, *f induces isomorphisms :*

$$H_*(\mathrm{C}; F.f) \approx H_*(\mathrm{D}; F).$$

PROOF. Statements 1 and 2 are equivalent because, for any map g in *Simplicial Sets*, g is a weak equivalence iff $Ex^2 g$ is. This holds because there is a natural weak equivalence $Id \to Ex^2$ by [12], Lemma 3.7.

From the remarks above, one sees statement 3 is equivalent to the assertion that Nf induces an equivalence of fundamental groupoids and an isomorphism of homology with any local coefficient system. But it is well-known (e.g., [15], II, 3.19, Proposition 4) that this is equivalent to Nf being a weak equivalence.

Recall that the simplicial n-simplex $\Delta[n]$ contains subsimplicial sets $\Delta[n,k]$ called *k-horns*, for $k = 0, 1, \dots, n$. The non-degenerate simplices of $\Delta[n,k]$ are those of $\Delta[n]$, except that the non-degenerate top n-simplex and the $n-1$-simplex opposite the vertex k are missing ([3], VIII, 3.3 ; [6], IV, 2).

In the Introduction it was observed that $Ex^2 N$ has a left adjoint cSd^2 . Applying this functor to the horns $\Delta[n,k] \hookrightarrow \Delta[n]$, one gets inclusions in *Cat* :

$$cSd^2\Delta[n,k] \hookrightarrow cSd^2\Delta[n] .$$

Call these inclusions the *categorical horns.*

PROPOSITION 2.5. *The following are equivalent for a morphism* p : C → D *in Cat :*

1° p is a fibration, i.e., $Ex^2 Np$ is a Kan fibration.

2° Given any categorical horn i and any solid arrow diagram (2.1), there is a dotted arrow as shown which makes the diagram commute.

$$\text{(2.1)} \qquad \begin{tikzcd} c\,Sd^2\Delta[n,k] \arrow[r] \arrow[d, hook, "i"'] & C \arrow[d, "p"] \\ c\,Sd^2\Delta[n] \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

PROOF. Under the adjoint functors $c\,Sd^2 \to Ex^2 N$, there is a natural bijective correspondence between diagrams (2.1) in *Cat* and diagrams (2.2)

$$\text{(2.2)} \qquad \begin{tikzcd} \Delta[n,k] \arrow[r] \arrow[d, hook] & Ex^2 N\,C \arrow[d, "Ex^2 Np"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & Ex^2 N\,D \end{tikzcd}$$

involving horns in *Simplicial Sets*. Under this correspondence, statement 2 is equivalent to the condition that $Ex^2 Np$ is a Kan fibration, i.e., to statement 1.

There is an explicit description of $c\,Sd^2\Delta[n,k]$ and $c\,Sd^2\Delta[n]$, which will be needed later. Let K be any simplicial set which is equivalent to an old-fashioned simplicial complex with an ordered set of vertices. Then $Sd\,K$ is the old-fashioned barycentric subdivision of the simplicial complex K. The vertices of SdK are the non-degenerate simplices of K, and there is an edge e of SdK with

$$d_0\, e = v, \quad d_1\, e = w$$

iff v is a face of w considered as simplices of K. From this, one sees that the category $c\,SdK$ is the poset of non-degenerate faces of K, ordered by inclusion. Further, $N\,c\,SdK$ is then SdK, as noted in the third paragraph of [14], Section 1. Applying this to $K = Sd\Delta[n,k]$, $Sd\Delta[n]$, one gets that $c\,Sd^2\Delta[n,k]$ and $c\,Sd^2\Delta[n]$ are the posets of faces of the simplicial complexes $Sd\,\Delta[n,k]$ and $Sd\Delta[n]$. Further,

$$c\,Sd^2\Delta[n,k] \hookrightarrow c\,Sd^2\Delta[n]$$

is the obvious inclusion and is a weak equivalence, as N sends it to the weak equivalence $Sd\Delta[n,k] \to Sd\Delta[n]$.

PROPOSITION 2.6. *The following are equivalent for a morphism* $p: X \to Y$ *in* Cat *:*

1° p is a trivial fibration; i. e., $Ex^2 Np$ *is a trivial fibration in Simplicial Sets.*

2° Given any solid arrow diagram (2.3) involving the canonical morphism $c\,Sd^2\dot{\Delta}[n] \to c\,Sd^2\Delta[n]$, *there is a dotted arrow making the diagram commute.*

$$
(2.3) \qquad
\begin{tikzcd}
c\,Sd^2\dot{\Delta}[n] \arrow[r] \arrow[d, hook] & X \arrow[d, "p"] \\
c\,Sd^2\Delta[n] \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
$$

PROOF. The equivalence follows from the bijective correspondence between diagrams (2.3) and diagrams (2.4) in *Simplicial Sets*.

$$
(2.4) \qquad
\begin{tikzcd}
\dot{\Delta}[n] \arrow[r] \arrow[d, hook] & Ex^2 N X \arrow[d, "Ex^2 Np"] \\
\Delta[n] \arrow[r] \arrow[ur, dashed] & Ex^2 N Y
\end{tikzcd}
$$

3. I shall now proceed to verify the proposed closed model structure on *Cat* satisfies the axioms.

Axiom CM1 is well-known. Axioms CM2 and CM3 for weak equivalences and fibrations hold in *Cat* because they do in *Simplicial Sets*, and because a morphism f in *Cat* is a weak equivalence or fibration iff $Ex^2 Nf$ is such in *Simplicial Sets*.

It is just a little harder to verify CM3 for cofibrations.

The half of Axiom CM4 dealing with the case where p is a weak equivalence is true by definition of cofibration. The half of the Axiom Propriety dealing with fibre squares holds as it does in *Simplicial Sets*, because $Ex^2 N$ preserves fibre squares (it is a right adjoint), and because f is a weak equivalence iff $Ex^2 Nf$ is.

This leaves only the half of CM4 dealing with the case where i is a weak equivalence, the factorization axiom CM5, and half of Propriety to be verified.

4. In order to verify these remaining axioms, it is necessary to consider a new class of morphisms in *Cat*, the Dwyer maps. These maps will turn out to have good properties with respect to taking pushouts in *Cat*.

Recall that a subcategory A of a category B is called a sieve (crible) if for every morphism $b: B \to B'$ in B where B' is an object of A, then also B is an object of A and b is a morphism in A. In particular, A must be a full subcategory of B. Dually, a subcategory A of B is a cosieve if for every $b: B \to B'$ in B with B in A, then B' and b are also in A.

DEFINITION 4.1. A morphism $i: A \to B$ in *Cat* is a *Dwyer map* if i embeds A as a sieve in B, and there is a cosieve W in B containing A. Further, it is required that $i: A \to W$ be a reflection, i.e., it has a right adjoint $r: W \to A$ such that $ri = 1_A$ and the adjunction map $1 \to ri$ is the identity. (Note two related inclusions are named «i».)

Note if $i: A \to W$ has a right adjoint r, it may be chosen to satisfy the extra conditions, as is easily seen using the fact that A is a full subcategory of W.

The definition is best clarified by giving the examples of Dwyer maps that will be most useful:

PROPOSITION 4.2. *Let $L \subset K$ be an inclusion of simplicial sets which arise from ordered simplicial complexes. Then the induced $c\,Sd^2 L \to c\,Sd^2 K$ is a Dwyer map.*

PROOF. As noted after the proof of Proposition 2.5, $cSd^2 K$ is the poset of faces of the simplicial complex SdK, ordered by inclusion; there is a similar description of $c\,Sd^2 L$. The cosieve W in $cSd^2 K$ is the subposet of all simplices of SdK that meet SdL. Note that as $SdL \subset SdK$ is subdivided, any simplex σ of SdK that meets SdL does so in a face; i.e., $\sigma \cap SdL$ is a, not necessarily proper, face of σ. Then the reflection : $r: W \to c\,Sd^2 L$ sends each object σ of W to the non-empty face $\sigma \cap SdL$. Clearly, $ri = 1$ and $ir \to 1$ at σ is the inclusion of the face $\sigma \cap SdL$ in

σ. The verifications that W is a cosieve in $c\,Sd^2K$ and $c\,Sd^2L$ is a sieve in $c\,Sd^2K$ are easy.

The idea is roughly that for $i: A \to B$ to be a Dwyer map, A must be a «deformation retract» of its «star neighborhood» W. Consider now a pushout diagram in *Cat*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & C \arrow[d, "j"] \\ B \arrow[r, "g"] & B \coprod_A C \end{tikzcd} \tag{4.1}$$

One can apply the nerve functor to this and obtain a diagram in *Simplicial Sets*. Compare this to the pushout of $N\mathrm{B}$ and $N\mathrm{C}$ under NA. There will be a canonical map

$$N\mathrm{B} \coprod_{N\mathrm{A}} N\mathrm{C} \longrightarrow N(\mathrm{B} \coprod_{\mathrm{A}} \mathrm{C}) \tag{4.2}$$

induced by Nj and Ng. In general, this map is neither an isomorphism nor a weak equivalence. The reader may find examples in the work of Fritsch and Latch [5], Section 3. The utility of Dwyer maps arise from the proposition:

PROPOSITION 4.3. *In a pushout diagram (4.1) in Cat, if i is a Dwyer map, the canonical map (4.2) is a weak equivalence. Furthermore, j is a Dwyer map.*

COROLLARY 4.4. *In a pushout diagram (4.1) in Cat with i Dwyer if f is a weak equivalence then g is, and if i is a weak equivalence then j is.*

PROOF. The corresponding property holds for pushouts in *Simplicial Sets*. Note as i is an inclusion of categories, Ni is an inclusion of simplicial sets. As i is Dwyer, Proposition 4.3 gives that the nerve of the pushout in *Cat* is weak homotopy equivalent to the pushout of nerves. The result now follows using Proposition 2.4 and the property for pushouts in *Simplicial Sets* corresponding to the corollary.

Let $\mathbf{1}$ be the category with two objects 0, 1, and one non-identity arrow $0 \to 1$.

LEMMA 4.5. *A subcategory* A *of* B *is a sieve iff there is a functor* χ *:* B $\to$ 1 *such that* $\chi^{-1}(0) =$ A. *If such a functor exists, it is unique. Call it the characteristic functor of* A. *Dually,* A $\subset$ B *is a cosieve iff there is a functor* χ : B $\to$ 1 *such that* $\chi^{-1}(1) =$ A.

PROOF. Define χ on objects by $\chi(B) = 0$ or 1 as the object B of B is in A or not. If A is a sieve, there is an obvious unique way to extend this to morphisms to get a functor χ : B $\to$ 1 with $\chi^{-1}(0) =$ A. Conversely for any χ : B $\to$ 1, $\chi^{-1}(0)$ is a sieve in B, essentially because $\{0\}$ is a sieve in 1.

I will now give the proof of Proposition 4.3. Let A $\to$ W $\to$ B be a factorization of i: A $\to$ B which satisfies the conditions of Definition 4.1. Let V $\subset$ B be the full subcategory of B whose objects are the objects of B not in A. This V is a cosieve in B as A is a sieve. One sees easily that W $\cap$ V is a cosieve in W and in V, and that B is the pushout of W and V under W $\cap$ V.

I claim NB is the pushout of NW and NV under N(W $\cap$ V). For a p-simplex of NB is a string of p composable maps in B:

$$B_0 \to B_1 \to \dots \to B_p .$$

As V and W are both cosieves in B, the entire string lives in V or W if B_0 does. As V $\cup$ W contains all objects of B, this shows the disjoint union of NV and NW maps onto NB. The strings common to NV and NW in NB are precisely those in NV $\cap$ NW $= N$(V $\cap$ W). This shows NB is the pushout as claimed. Consider now the pushout squares in diagram

$$(4.3)\qquad \begin{array}{ccc} A & \xrightarrow{\quad f \quad} & C \\ \downarrow & & \downarrow \\ W & \xrightarrow{\qquad} & W \coprod_A C = W' \\ \downarrow & & \downarrow \\ B & \xrightarrow[\quad g \quad]{} & B \coprod_A C = D . \end{array}$$

Let W' be the pushout of W and C under A, and V' be the full subcategory of D whose objects are those not contained in C. Here D is the pushout of B and C under A . I claim $j: \mathrm{C} \to \mathrm{D}$ is a Dwyer map, and that $\mathrm{C} \to \mathrm{W}' \to \mathrm{D}$ is a factorization satisfying the conditions of Definition 4.1. First C is a sieve in D ; for the characteristic functor $\chi: \mathrm{B} \to 1$ of A in B , together with the functor $\mathrm{C} \to 1$ sending everything to 0 , induce a functor $\chi: \mathrm{D} \to 1$ with $\chi^{-1}(0) = \mathrm{C}$. Then Lemma 4.5 shows C is a sieve. A similar argument shows W' is a cosieve in D . Finally, the retraction $r: \mathrm{W} \to A$ and natural transformation $ir \to 1$ induce a retraction

$$r' = r \coprod_A \mathrm{C} : \mathrm{W}' = \mathrm{W} \coprod_A \mathrm{C} \to A \coprod_A \mathrm{C} = \mathrm{C}$$

and a natural transformation $jr' \to 1$. The proper identities are easily verified. Note that V' is a cosieve in D as C is a sieve.

Now I claim that the functor $g : \mathrm{B} \to \mathrm{D}$ induces isomorphisms

$$\mathrm{V} \approx \mathrm{V}', \quad \mathrm{V} \cap \mathrm{W} \approx \mathrm{V}' \cap \mathrm{W}' .$$

For g certainly induces an isomorphism of the quotients

$$(4.4) \qquad \mathrm{B}/A \approx \mathrm{B} \coprod_A \mathrm{C} / A \coprod_A \mathrm{C} = \mathrm{D}/\mathrm{C} .$$

But B/A is the category V with one new object $*$ added, and with a unique new morphism $* \to V$ for each object V of $\mathrm{V} \cap \mathrm{W}$. This description makes sense, i.e., composition of morphisms is defined, as $\mathrm{V} \cap \mathrm{W}$ is a cosieve in V . The object $*$ is the image of A in the quotient. The category D/C has a similar description with V' replacing V and $\mathrm{V}' \cap \mathrm{W}'$ replacing $\mathrm{V} \cap \mathrm{W}$. Thus the isomorphism $\mathrm{B}/A \approx \mathrm{D}/\mathrm{C}$ is seen to imply the existence of isomorphisms

$$\mathrm{V} \approx \mathrm{V}', \quad \mathrm{V} \cap \mathrm{W} \approx \mathrm{V}' \cap \mathrm{W}'$$

induced by g , as claimed.

Just as $N\mathrm{B}$ is the pushout of $N\mathrm{W}$ and $N\mathrm{V}$ under $N(\mathrm{W} \cap \mathrm{V})$, $N\mathrm{D}$ is the pushout of $N\mathrm{W}'$ and $N\mathrm{V}'$ under $N(\mathrm{W}' \cap \mathrm{V}')$. By the above, this implies $N\mathrm{D}$ is the pushout of $N\mathrm{W}'$ and $N\mathrm{V}$ under $N(\mathrm{W} \cap \mathrm{V})$.

As a final preparation note that the inclusion and retraction

$$i: A \to \mathrm{W}, \quad r: \mathrm{W} \to A$$

induce inverse homotopy equivalences $NA \approx N\mathrm{W}$, as the natural transformations $1 \to ri$, $ir \to 1$, induce simplicial homotopies

$$1 \approx Nr.Ni, \quad Ni.Nr \approx 1$$

([14], Section 1, Proposition 2). Similarly, the inclusion map $N\mathrm{C} \hookrightarrow N\mathbf{W}'$ is a homotopy equivalence.

Putting all these observations together, one gets a string of isomorphisms and weak equivalences :

$$(4.5) \qquad N\mathrm{B} \coprod_{N\mathrm{A}} N\mathrm{C} \approx N\mathrm{B} \coprod_{N\mathbf{W}} N\mathbf{W}' \approx (N\mathbf{V} \coprod_{N\mathbf{V}\cap\mathbf{W}} N\mathbf{W}) \coprod_{N\mathbf{W}} N\mathbf{W}' \approx$$
$$N\mathbf{V} \coprod_{N\mathbf{V}\cap\mathbf{W}} N\mathbf{W}' \approx N\mathbf{V}' \coprod_{N\mathbf{V}'\cap\mathbf{W}'} N\mathbf{W}' \approx N(\mathrm{B} \coprod_{\mathrm{A}} \mathrm{C}) = N\mathrm{D}$$

as required. The first map is induced by the homotopy equivalences

$$N\mathrm{A} \approx N\mathbf{W}, \quad N\mathrm{C} \approx N\mathbf{W}'$$

and is a weak equivalence by the well-known glueing Lemma of Brown (e. g., [1], Lemma 2.5). This completes the proof of Proposition 4.3.

PROPOSITION 4.6. *Let $L \to K$ be a cofibration in Simplicial Sets. Then $c\,Sd^2 L \to c\,Sd^2 K$ is a cofibration in Cat.*

PROOF. There is a natural bijective correspondence between diagrams (4.6) in *Cat* and diagrams (4.7) in *Simplicial Sets*, induced by the adjointness of $c\,Sd^2$ and $Ex^2 N$.

$$(4.6) \qquad \begin{tikzcd} c\,Sd^2 L \arrow[r] \arrow[d, "c\,Sd^2 i"'] & X \arrow[d, "p"] \\ c\,Sd^2 K \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

$$(4.7) \qquad \begin{tikzcd} L \arrow[r] \arrow[d, "i"'] & Ex^2 NX \arrow[d, "Ex^2 Np"] \\ K \arrow[r] \arrow[ur, dashed] & Ex^2 N Y \end{tikzcd}$$

LEMMA 4.7. *1° If $\mathrm{A} \to \mathrm{B}$ is a cofibration in Cat, and $\mathrm{A} \to \mathrm{C}$ is any morphism, the induced morphism into the pushout $\mathrm{C} \to \mathrm{B} \coprod_{\mathrm{A}} \mathrm{C}$ is a cofibration.*

2° If $\mathrm{A}_0 \to \mathrm{A}_1 \to \mathrm{A}_2 \to \dots$ is a sequence of cofibrations in Cat and $\mathrm{A}_\infty = \varinjlim \mathrm{A}_n$, the canonical morphism $\mathrm{A}_0 \to \mathrm{A}_\infty$ is a cofibration. If each

$A_i \to A_{i+1}$ is a weak equivalence, so is $A_0 \to A_\infty$.

PROOF. The assertions that maps are cofibrations follow from the definition by a mapping property and the universal mapping properties of pushouts and direct limits.

Suppose now each $A_i \to A_{i+1}$ is a weak equivalence. As N preserves direct limits, $NA_\infty = \varinjlim NA_i$. Using the fact that in *Simplicial Sets* the homotopy groups of a direct limit are the direct limits of the homotopy groups, one sees that as each $NA_i \to NA_{i+1}$ is a weak equivalence, so is

$$NA_0 \to \varinjlim NA_i = NA_\infty .$$

Thus $A_0 \to A_\infty$ is a weak equivalence.

LEMMA 4.8. *The categorical horns $c\,Sd^2\,\Delta[n,k] \to c\,Sd^2\,\Delta[n]$ are trivial cofibrations.*

PROOF. They are cofibrations by Proposition 4.6. That they are weak equivalences was noted just before Proposition 2.6.

I'm now ready to verify the factorization axiom CM5. Let $f: A \to B$ be any morphism in *Cat*. I will first show it factors as $f = pi$, with p a fibration and i a trivial cofibration. Consider the set of all diagrams (4.8) involving the categorical horns

$$\begin{tikzcd} c\,Sd^2\,\Delta[n,k] \arrow[r, "a"] \arrow[d, hook] & A \arrow[d, "f"] \\ c\,Sd^2\,\Delta[n] \arrow[r, "b"] & B \end{tikzcd} \tag{4.8}$$

Take the coproduct of all such categorical horns indexed by the set of all diagrams (4.8):

$$\coprod c\,Sd^2\,\Delta[n,k] \to \coprod c\,Sd^2\,\Delta[n] .$$

This map is a trivial cofibration as it is a coproduct of trivial cofibrations by Lemma 4.8. Further, it is a Dwyer map by Proposition 4.2.

The morphisms $a: c\,Sd^2\,\Delta[n,k] \to A$ in all the diagrams (4.8) induce a morphism $\coprod c\,Sd^2\,\Delta[n,k] \to A$. Let A_1 be the pushout of A and $\coprod c\,Sd^2\,\Delta[n]$ under $\coprod c\,Sd^2\,\Delta[n,k]$. Then the morphism $A \to A_1$ is a cofibration by Lemma 4.7 and a weak equivalence and a Dwyer map by Corollary 4.4 and Proposition 4.3. The morphisms $b: c\,Sd^2\,\Delta[n] \to B$ in the diagrams (4.8) and $f: A \to B$ induce a morphism $p_1: A_1 \to B$. The composition $A \to A_1 \to B$ is the original f. Further by construction any

$$b: c\,Sd^2\,\Delta[n] \to B$$

in any diagram (4.8) lifts through $A_1 \to B$ extending

$$c\,Sd^2\,\Delta[n,k] \xrightarrow{a} A \to A_1 .$$

Applying this entire construction to the new morphism $A_1 \to B$, one produces another factorization $A_1 \to A_2 \to B$ through $p_2: A_2 \to B$.

Iterating countably many times, one obtains a sequence of cofibrations and weak equivalences

$$A \to A_1 \to A_2 \to \dots,$$

and a family of morphisms $p_n: A_n \to B$ such that $A_n \to A_{n+1}$ composed with p_{n+1} is p_n. Let $A_\infty = \varinjlim A_n$. Then $i: A \to A_\infty$ is a cofibration and a weak equivalence by Lemma 4.7. The morphisms $p_n: A_n \to B$ induce a $p: A_\infty \to B$. I claim that p is a fibration. It satisfies statement 2 of Proposition 2.5.

The construction of the factorization required by the second half of CM5 is similar. Consider all diagrams

$$(4.9)\qquad \begin{array}{ccc} c\,Sd^2\,\dot\Delta[n] & \xrightarrow{a} & A \\ \downarrow & & \downarrow f \\ c\,Sd^2\,\Delta[n] & \xrightarrow{b} & B \end{array}$$

and let

$$\amalg\, c\,Sd^2\,\dot\Delta[n] \to \amalg\, c\,Sd^2\,\Delta[n]$$

be the coproduct indexed by all such diagrams. This morphism is a cofibration by Proposition 4.6. Let A_1 be the pushout of A and $\amalg c\,Sd^2\,\Delta[n]$ under $\amalg c\,Sd^2\,\dot\Delta[n]$. Then $A \to A_1$ is a cofibration by Lemma 4.7, and $f: A \to B$ factors as $A \to A_1 \to B$ with $A_1 \to B$ induced by

$$f: A \to B \quad\text{and}\quad \amalg b: \amalg\, c\,Sd^2\,\Delta[n] \to B .$$

Iterate this construction countably many times to produce $j: A \to A_\infty$ and

$q: A_\infty \to B$ such that f factors as qj. Then j is a cofibration by Lemma 4.7, and by construction, using the fact $c\,Sd^2\,\dot{\Delta}[n]$ is finite and so any morphism $c\,Sd^2\,\dot{\Delta}\,[n] \to A_\infty$ factors through some finite stage A_m, q has the lifting property mentioned in Proposition 2.6, and so is a trivial fibration. This completes the verification of CM5.

It remains to verify half of CM4 and the extra axiom of Propriety. To verify CM4, it must be shown that given a diagram (4.10) in *Cat*, with k a trivial cofibration and q a fibration, then the dotted arrow $B \to X$ exists. First note the lift $B \to X$ will exist if k is a categorical cofibration constructed by the process in the verification of the first half of CM5.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "k"'] & X \arrow[d, "q"] \\ B \arrow[r, "b"'] \arrow[ur, dashed] & Y \end{tikzcd} \tag{4.10}$$

Now take $k: A \to B$ any trivial cofibration, and use the above process to factor it as $k = pi$, p a fibration and i a trivial cofibration. Note p is also a weak equivalence by CM2, as k and i are. By the above observation on the cofibration $i: A \to A_\infty$, there is a lift $g: A_\infty \to X$ such that

$$gi = a, \quad qg = bp$$

for i has the required lifting property with respect to fibrations. If there is a section $s: B \to A_\infty$ of the fibration p such that $ps = 1$ and $sk = i$, then $gs: B \to X$ will serve as a dotted arrow in the original diagram (4.10).

Consider now the diagram (4.11). As k is a cofibration and p is a trivial fibration, the dotted arrow exists by definition of cofibration. It is the sought for section s.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "k"'] & A_\infty \arrow[d, "p"] \\ B \arrow[r, "1"'] \arrow[ur, dashed] & B \end{tikzcd} \tag{4.11}$$

This completes the verification of CM4, and so the proof of:

THEOREM 4.9. *Cat, the category of small categories, is a closed model category under the structure proposed by Definitions 2.1, 2.2, 2.3.*

LEMMA 4.10. *Let* $i: \mathbf{A} \to \mathbf{B}$ *be any cofibration. Let* $i = qj$, $j: \mathbf{A} \to \mathbf{A}_\infty$ *a cofibration, and* $q: \mathbf{A}_\infty \to \mathbf{B}$ *a trivial fibration, be the factorization constructed by the process used in verifying CM5 above. Thus* j *is the canonical map* $\mathbf{A} = \mathbf{A}_0 \to \varinjlim \mathbf{A}_n = \mathbf{A}_\infty$, *with each* $\mathbf{A}_n \to \mathbf{A}_{n+1}$ *induced by a coproduct of morphisms* $c\,Sd^2\,\dot{\Delta}[m] \to c\,Sd^2\,\Delta[m]$ *pushed out via a morphism* $\amalg c\,Sd^2\,\dot{\Delta}[m] \to \mathbf{A}_n$. *Then there is an*

$$s: \mathbf{B} \to \mathbf{A}_\infty \quad \textit{such that} \quad si = j \ \textit{and}\ qs = 1,$$

i.e., i *is a retract of* j.

PROOF. The description of j may be read off the construction. The existence of s follows from axiom CM4.

5. It remains to show *Cat* satisfies the extra axiom of Propriety. To do this, I will show all cofibrations are Dwyer maps and invoke Corollary 4.4. I will also show all cofibrant categories are posets.

I need to use a characterization of Dwyer maps slightly different from Definition 4.1, and due to Fritsch and Latch.

DEFINITION 5.1. Let $\mathbf{A}$ be a subcategory of $\mathbf{B}$. Then $Z\,\mathbf{A}$ is the cosieve generated by $\mathbf{A}$ in $\mathbf{B}$; i.e., the full subcategory in $\mathbf{B}$ of objects B such that there exists a map $A \to B$ in $\mathbf{B}$ with A in $\mathbf{A}$. (Note the dependence of $Z\,\mathbf{A}$ on $\mathbf{B}$ is suppressed from the notation.)

LEMMA 5.2. *The following are equivalent for a morphism* $i: \mathbf{A} \to \mathbf{B}$ *in Cat:*

1° i is a Dwyer map;

2° i is a sieve and $\mathbf{A} \hookrightarrow Z\,\mathbf{A}$ *is a reflexion; i.e., there exists a retraction* $r: Z\,\mathbf{A} \to \mathbf{A}$ *which is right adjoint to the inclusion.*

PROOF. To see 1 implies 2, let $\mathbf{W} \subset \mathbf{B}$ be a cosieve as in Definition 4.1. As $\mathbf{A} \subset \mathbf{W}$ and $\mathbf{W}$ is a cosieve, $Z\,\mathbf{A}$ is contained in $\mathbf{W}$. On the other hand, for W in $\mathbf{W}$ the adjunction map $\eta: irW \to W$ has irW as an object of $\mathbf{A}$, so $\mathbf{W}$ is contained in $Z\,\mathbf{A}$. Thus $\mathbf{W} = Z\,\mathbf{A}$, and the conditions of Definition 4.1 imply those of statement 2.

To see 2 implies 1, let $\mathbf{W} = Z\,\mathbf{A}$. The conditions of Definition 4.1 are met.

LEMMA 5.3. *1° The composition of two Dwyer maps yields a Dwyer map.*

2° Let $\mathbf{A}_0 \to \mathbf{A}_1 \to \mathbf{A}_2 \to \dots$ *be a sequence with each* $\mathbf{A}_n \to \mathbf{A}_{n+1}$ *a Dwyer map. Then the induced* $\mathbf{A}_0 \to \varinjlim \mathbf{A}_n = \mathbf{A}_\infty$ *is a Dwyer map.*

3° Any retract of a Dwyer map is a Dwyer map.

PROOF. To prove statement 1, let $i: \mathbf{A} \to \mathbf{B}$ and $j: \mathbf{B} \to \mathbf{C}$ be Dwyer maps. It is easy to see that $ji: \mathbf{A} \to \mathbf{C}$ is a sieve, as i and j are. Let $Z\,\mathbf{A}$, $Z\,\mathbf{B}$ be the cosieves generated by $\mathbf{A}$ in $\mathbf{B}$, and by $\mathbf{B}$ in $\mathbf{C}$, respectively. Let $Z'\mathbf{A}$ be the cosieve generated by $\mathbf{A}$ in $\mathbf{C}$. Let $r: Z\,\mathbf{A} \to \mathbf{A}$, $s: Z\,\mathbf{B} \to \mathbf{B}$ be the retractions which exist as i and j are Dwyer maps.

The inclusions $\mathbf{A} \subset \mathbf{B} \subset \mathbf{C}$ induce inclusions $Z\,\mathbf{A} \subset Z'\mathbf{A} \subset Z\,\mathbf{B}$. Further, $s(Z'\mathbf{A})$ is contained in $Z\,\mathbf{A}$. For if C is an object of $Z'\mathbf{A}$ there is a morphism $A \to C$ in $\mathbf{C}$ with A an object of $\mathbf{A}$, the morphism

$$A = sA \to sC \quad \text{in } \mathbf{B}$$

shows sC is in $Z\,\mathbf{A}$. As $s(Z'\mathbf{A}) \subset Z\,\mathbf{A}$,

$$t = rs: Z'\mathbf{A} \to \mathbf{A}$$

is defined. This $t = rs$ is right adjoint to ji as s is right adjoint to j and r is right adjoint to i. Further, t is a reflexion as

$$tji = rsji = ri = 1.$$

Now Lemma 5.2 shows ji is a Dwyer map.

The proofs of the other statements proceed in a similar spirit.

PROPOSITION 5.4. *Every cofibration in* Cat *is a Dwyer map.*

PROOF. Let $i: \mathbf{A} \to \mathbf{B}$ be a cofibration. Then by Lemma 4.10, i is a retract of a cofibration $j: \mathbf{A} \to \mathbf{A}_\infty$ with a structure as described in Lemma 4.10. By Lemma 5.3, 3 it suffices to show j is a Dwyer map. To do this, it suffices by Lemma 5.3, 2, to show each $\mathbf{A}_n \to \mathbf{A}_{n+1}$ occurring in the construction of j is a Dwyer map. As each $\mathbf{A}_n \to \mathbf{A}_{n+1}$ is the pushout of a coproduct of canonical morphisms

$$\amalg\, c\, Sd^2\, \dot{\Delta}[m] \to \amalg\, c\, Sd^2\, \Delta[m]$$

by a morphism $\amalg c\,Sd^2\dot{\Delta}[m] \to A_n$, it suffices to show by Proposition 4.3 the coproduct of canonical morphisms is a Dwyer map. But this is true by Proposition 4.2. Note this argument parallels the proof that j is a cofibration in the verification of CM5, using Lemma 5.3 and Proposition 4.3 in place of Lemma 4.7, and Proposition 4.2 in place of Proposition 4.6.

COROLLARY 5.5. *Cat is a proper closed model category.*

PROOF. Apply Proposition 5.4 and Proposition 4.3 to verify the axiom of Propriety. Then the result follows by Theorem 4.9.

LEMMA 5.6. *1° For any simplicial set K, $c\,Sd^2K$ is a poset.*

2° If $A_0 \to A_1 \to A_2 \to \ldots$ is a sequence of posets, then $A_\infty = \varinjlim A_n$ is a poset.

3° Any subcategory of a poset is a poset.

4° Let A, B, C be posets, and $i: A \to B$ a Dwyer map. Then for any morphism $A \to C$ the pushout D of B and C under A is a poset.

PROOF. Statement 1 holds for K a simplicial set coming from an ordered simplicial complex by the description of $c\,Sd^2K$ preceding Proposition 2.6. This is the only case used in the proof of Proposition 5.7 below; the general case then follows from Propositions 5.7 and 4.6.

The proofs of statements 2 and 3 are trivial.

To prove statement 4, refer back to the proof of Proposition 4.3. Let W, V, W', V' be as in that proof. One must show that if D, D' are objects of D, then $D(D, D')$ has at most one element, and that if there are morphisms $D \to D'$ and $D' \to D$, then $D = D'$. This may be done by a case by case check as D and D' range over objects in C and in V', using the facts that C and $V' \approx V$ are posets, that W' and V' are cosieves in D, that C is a sieve in D and that C is a reflexive subcategory of W'.

PROPOSITION 5.7. *Every cofibrant category is a poset.*

PROOF. Let C be cofibrant. Then $\emptyset \to C$ is a cofibration, where $\emptyset$ is the empty category. By Lemma 4.10, C is a retract, and hence a subcategory, of a category A_∞ constructed by the process described in Lemma 4.10. By Lemma 5.6, 3, it suffices to prove A_∞ is a poset. By Lemma 5.6, 2, it suffices to prove each A_n is a poset. One does this by induction. $A_0 = \emptyset$ is a poset. Suppose A_n is a poset. Then A_{n+1} will be a poset by Lemma 5.6, 4 and Proposition 4.2, provided $\amalg c\, Sd^2\, \dot{\Delta}[m]$ and $\amalg c\, Sd^2\, \Delta[m]$ are posets. But this is true by Lemma 5.6, 1.

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