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PULLBACK FUNCTORS AND CROSSED COMPLEXES

by James HOWIE

1. INTRODUCTION.

This paper contains a generalisation of a result which appeared in the authors thesis [8]. It is a classification of those morphisms in certain categories for which the corresponding «pullback functors» have coadjoints. Two categories were considered in [8], namely $\mathcal{G}pd$ (groupoids) and $\mathcal{DG}$ (double groupoids). Here we consider the category $\mathcal{CC}$ of crossed complexes, introduced by Brown and Higgins in [6], and an infinite ascending chain of full subcategories

$$\mathcal{CC}_0 \subset \mathcal{CC}_1 \subset \dots .$$

The first three terms of this chain are naturally equivalent to $\mathcal{S}et$ (the category of sets), $\mathcal{G}pd$ and $\mathcal{DG}$ respectively, so the result here contains the result in [8] as a special case.

I would like to thank Professor P.J. Higgins for suggesting the problem in the first place, and for his advice during the preparation of the manuscript.

2. CROSSED COMPLEXES.

The definition given below is due to Brown and Higgins [6]. Crossed complexes are a generalisation of the «group systems» of Blakers [1], the «homotopy systems» of Whitehead [11] and the «crossed resolutions» of Hübschmann [9,10].

DEFINITION. A *crossed complex* C consists of:

(a) a set C_0,

(b) a groupoid C_1 with vertex set C_0, source and target maps

$$\delta^0, \delta^1 : C_1 \longrightarrow C_0$$

and identities $e_1(x)$ $(x \in C_0)$,

(c) for each $n \geqslant 2$, a collection $\{C_n(x) \mid x \in C_0\}$ of groups, with «boundary maps»

$$\delta = \delta_n(x): C_n(x) \to C_{n-1}(x) \quad (n \geqslant 2,\ x \in C_0)$$

(here $C_1(x) = C_1(x, x)$ is just the vertex group of the groupoid C_1 at the vertex x),

(d) for each $n \geqslant 1$, an action of C_1 on the collection of groups

$$\{C_n(x)\}, \text{ written } (\alpha, a) \mapsto \alpha^a \in C_n(y)$$

$(\alpha \in C_n(x),\ a \in C_1(x, y))$, which in the case $n = 1$ coincides with action by conjugation in C_1, i.e., $b^a = a^{-1} b a$,

satisfying the following axioms:

(i) For $n \geqslant 3$ and all $x \in C_0$, $C_n(x)$ is abelian;

(ii) The maps δ are homomorphisms of groups, and respect the action of C_1;

(iii) $\delta\delta = 0$;

(iv) For $c \in C_2(x)$, δc acts trivially on $C_n(x)$ if $n \geqslant 3$, and δc acts on $C_2(x)$ as conjugation by c, i.e., $d^{\delta c} = c^{-1} d c$.

Let $e_n(x)$ denote the identity element of the group

$$C_n(x) \quad (n > 1,\ x \in C_0).$$

For $n \geqslant 2$, let C_n denote the groupoid

$$\amalg \{C_n(x) \mid x \in C_0\},$$

i.e., the disjoint union of the groups $\{C_n(x)\}$. Then

$$\{e_n(x) \mid x \in C_0\}$$

is the set of identities of C_n, and the boundary maps extend to groupoid homomorphisms $\delta = \delta_n: C_n \to C_{n-1}$.

DEFINITION. A *morphism of crossed complexes* $\phi: A \to B$ is a collection of maps $\phi_n: A_n \to B_n$ which preserve all the structure.

With the obvious definition of composition of morphisms, crossed complexes and their morphisms form a category $\mathcal{CC}$.

DEFINITION. The *rank of a crossed complex* C is the highest integer r such that C_r is not a discrete groupoid (i.e.,

$$C_r \neq \{ e_r(x) \mid x \in C_0 \}).$$

If no such highest r exists, C has rank ∞. If C_r is discrete for all $r \geqslant 1$, C has rank 0.

For any non-negative integer r, let $\mathcal{CC}_r$ denote the full subcategory of $\mathcal{CC}$ whose objects are all the crossed complexes of rank $\leqslant r$. We will also use the notation $\mathcal{CC}_\infty$ for $\mathcal{CC}$. One can show that, for n with $0 \leqslant n \leqslant r \leqslant \infty$, $\mathcal{CC}_n$ is both reflexive and coreflexive as a subcategory of $\mathcal{CC}_r$.

Brown and Higgins [6] describe two other categories equivalent to $\mathcal{CC}$, one of which is the category of ω-groupoids. Using the notion of «thin elements» (éléments minces) described in [6], we can define an r-tuple groupoid to be an ω-groupoid G such that every element of G_n is thin for every $n > r$. Then for $0 \leqslant r < \infty$, the equivalence of $\mathcal{CC}$ with the category of ω-groupoids restricts to an equivalence of $\mathcal{CC}_r$ with the full subcategory of r-tuple groupoids. In particular, there are equivalences

$$\mathcal{S}et \approx \mathcal{CC}_0, \quad \mathcal{G}pd \approx \mathcal{CC}_1 \quad \text{and} \quad \mathcal{DG} \approx \mathcal{CC}_2.$$

3. FIBRATIONS.

Recall [4,6] that a morphism $\phi: A \to B$ of groupoids is a *fibration* if, whenever $x \in A_0$ and $b \in B_1$ with $\delta^0 b = \phi(x)$, there exists

$$a \in A_1 \quad \text{such that} \quad \phi(a) = b \quad \text{and} \quad \delta^0 a = x.$$

We can extend this notion to morphisms in $\mathcal{CC}$.

DEFINITION. A morphism $\phi: A \to B$ in $\mathcal{CC}$ is a *fibration* if each groupoid morphism $\phi_n: A_n \to B_n$ ($n \geqslant 1$) is a fibration of groupoids.

NOTES. 1° Every morphism in $\mathcal{CC}_0$ is a fibration for trivial reasons.

2° For $n \geqslant 2$, A_n and B_n are totally disconnected groupoids. Hence the condition that ϕ_n be a fibration of groupoids is equivalent to the condition that ϕ_n map each $A_n(x)$ surjectively onto $B_n(\phi x)$.

If $\phi: A \to B$ is a morphism, and $y \in B_0$, let F_n be the subset $\phi_n^{-1}(e_n(y))$ of A_n for each $n \geqslant 1$, and let F_0 be the subset $\phi_0^{-1}(y)$ of A_0. It is easy to show that the sets $\{F_n \mid n \geqslant 0\}$ form a sub-complex $F = F_\phi(y)$ of A. Call $F_\phi(y)$ *the fibre of* ϕ *over* y.

Fibrations of groupoids have been studied by Brown [5]. Many of the results of [5] may be generalised to apply to fibrations of crossed complexes. An example is Theorem 3.1 below, for which we must first introduce the notion of «homotopy groups» for crossed complexes.

Suppose $x \in A_0$. Define $\pi_0(A, x)$ to be the pointed set of components of the groupoid A_1, with base-point the component containing x. Define $\pi_1(A, x)$ to be the cokernel of $\delta_2(x): A_2(x) \to A_1(x)$. Finally, for $r \geqslant 2$, define $\pi_r(A, x)$ to be the subquotient

$$Ker\, \delta_r(x) / Im\, \delta_{r+1}(x) \text{ of } A_r(x).$$

THEOREM 3.1. *Let* $\phi: A \to B$ *be a fibration, and let*

$$x \in A_0, \quad y = \phi(x) \quad and \quad F = F_\phi(y).$$

Then there is a long exact sequence

$$\ldots \to \pi_r(F, x) \to \pi_r(A, x) \to \pi_r(B, y) \to \pi_{r-1}(F, x) \to \ldots$$
$$\ldots \to \pi_1(B, y) \to \pi_0(F, x) \to \pi_0(A, x) \to \pi_0(B, y)$$

of groups and pointed sets (in the sense of [5]*) such that*

$$\pi_1(F, x) \to \pi_1(A, x) \text{ is a crossed module.}$$

Furthermore, the whole construction is functorial in the obvious sense.

4. CATEGORICAL CONSIDERATIONS.

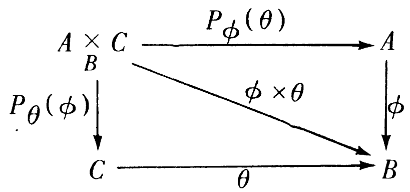
If A is an object of the category $\mathfrak{A}$, let $(\mathfrak{A}, A)$ denote the category of «$\mathfrak{A}$-objects over A», i.e., the category whose objects are $\mathfrak{A}$-morphisms with codomain A and whose morphisms are the obvious commutative triangles in $\mathfrak{A}$.

If $\phi : A \to B$ is an $\mathcal{A}$-morphism, let $\phi*$ denote the induced functor

$$\phi* = (\mathcal{A}, \phi) : (\mathcal{A}, A) \to (\mathcal{A}, B), \quad \psi \mapsto \phi \circ \psi .$$

If $\mathcal{A}$ admits pullbacks, then every induced functor $\phi*$ has a coadjoint (= right adjoint) $P_\phi : (\mathcal{A}, B) \to (\mathcal{A}, A)$. P_ϕ is the «pullback functor» determined by ϕ, i.e., if $\theta : C \to B$ is an object of $(\mathcal{A}, B)$, then : $P_\phi(\theta) : A \times_B C \to A$ is the pullback (in $\mathcal{A}$) of θ by ϕ.

$\mathcal{A}$ admits pullbacks iff, for every object B of $\mathcal{A}$, $(\mathcal{A}, B)$ admits products. If $\phi : A \to B$, $\theta : C \to B$ are objects of $(\mathcal{A}, B)$, then the product $\phi \times \theta$ in $(\mathcal{A}, B)$ is the diagonal arrow of the pullback square :



in $\mathcal{A}$, i.e.,

$$\phi \times \theta = \phi \circ P_\phi(\theta) = (\phi * \circ P_\phi)(\theta) = (\theta * \circ P_\theta)(\phi).$$

Hence, the functor $\phi \times (-) : (\mathcal{A}, B) \to (\mathcal{A}, B)$ is the composite $\phi * \circ P_\phi$. $\phi*$ has a coadjoint P_ϕ, so if P_ϕ has a coadjoint Q_ϕ then the composite $\phi \times (-)$ has a coadjoint $(-)^\phi = Q_\phi \circ P_\phi$. If this is true for any object ϕ of $(\mathcal{A}, B)$, then $(\mathcal{A}, B)$ is a cartesian closed category, and the following «exponential law» holds. Let ϕ, ψ, θ be objects of $(\mathcal{A}, B)$. Then

$$(\theta^\phi)^\psi \approx \theta^{(\phi \times \psi)} \approx (\theta^\psi)^\phi .$$

Booth [2, 3, 4] has studied exponential laws in $(\mathcal{A}, B)$ for various categories $\mathcal{A}$ of topological spaces, and sufficient conditions for the existence of $(-)^\phi$.

In general, the study of pullback functors and their adjoints is of interest in connection with topos theory. If $\mathcal{A}$ is a topos, then for every object A of $\mathcal{A}$, $(\mathcal{A}, A)$ is also a topos. For every morphism $\phi : A \to B$ in $\mathcal{A}$, the pullback functor $P_\phi : (\mathcal{A}, B) \to (\mathcal{A}, A)$ is logical, that is, it preserves the topos structure. Furthermore, Q_ϕ exists, and the adjoint pair (P_ϕ, Q_ϕ) forms a geometric morphism from $(\mathfrak{A}, A)$ to $(\mathfrak{A}, B)$ [14].

In particular, if B is the terminal object of $\mathfrak{A}$, and $\phi: A \to B$ is the unique morphism, then (P_ϕ, Q_ϕ) is a geometric morphism from $(\mathfrak{A}, A)$ to $\mathfrak{A}$. Among all geometric morphisms (f^*, f_*) with codomain $\mathfrak{A}$, those which arise in the above way may be classified (up to natural equivalence) by the property that f^* has an adjoint which preserves equalizers (see for example [13], 1.4).

In an arbitrary category with pullbacks $\mathfrak{A}$, the class of morphisms ϕ for which Q_ϕ exists depends to a large extent on properties of $\mathfrak{A}$. For example, if $\mathfrak{A}$ has a zero object Z (and so admits kernels), then the existence of Q_ϕ implies that $Ker\,\phi \approx Z$. If $\mathfrak{A}$ is a suitable category of abstract algebras, such as a variety of groups, or of groups with operators, then one can show that Q_ϕ exists only in the trivial case where ϕ is an isomorphism. On the other hand, if $\mathfrak{A}$ is a topos, such as $\mathcal{S}et$, then Q_ϕ exists for any ϕ.

If $\mathfrak{A}$ is one of the categories $\mathcal{CC}_r$ $(1 \leqslant r \leqslant \infty)$, there is a non-trivial classification of those morphisms ϕ for which Q_ϕ exists. They turn out to be precisely the fibrations. Note that this classification also holds for $r = 0$, since every morphism in $\mathcal{CC}_0$ is a fibration. This classification is our main result, and will be proved in the next section. The proof is essentially the same as that given in [8] for the case $\mathfrak{A} = \mathfrak{DG}$.

A corresponding classification result for the case $\mathfrak{A} = \mathcal{C}at$, the category of (small) categories, was proved by Conduché [12], who gave conditions on ϕ which are necessary and sufficient for the existence of Q_ϕ. Let $U: \mathcal{G}pd \to \mathcal{C}at$ denote the forgetful functor, and suppose $\phi: A \to B$ is a morphism of groupoids. Then $U(\phi)$ satisfies Conduché's conditions iff ϕ is a fibration. To this extent, our results agree with [12].

The following lemma, and its corollary, will be required in the next section.

LEMMA 4.1. *Let* $Dom: (\mathfrak{A}, B) \to \mathfrak{A}$ *denote the «domain» or forgetful func-*

tor, and let $\mathcal{C}$ *be any small category. Then* $\phi: A \to B$ *is the colimit of the diagram* $\mathfrak{D}: \mathcal{C} \to (\mathfrak{A}, B)$ *iff* $A = Dom(\phi)$ *is the colimit of the diagram* $Dom \circ \mathfrak{D}: \mathcal{C} \to \mathfrak{A}$.

COROLLARY. *Let* $\phi: A \to B$ *be a morphism in* $\mathfrak{A}$. *Then the object* θ *of* $(\mathfrak{A}, A)$ *is the colimit of the diagram* $\mathfrak{D}: \mathcal{C} \to (\mathfrak{A}, A)$ *iff* $\phi_*(\theta) = \phi \circ \theta$ *is the colimit of the diagram* $\phi_* \circ \mathfrak{D}: \mathcal{C} \to (\mathfrak{A}, B)$.

5. RESULTS.

THEOREM 5.1. *Fix* r $(0 \leqslant r \leqslant \infty)$, *and let* $\phi: A \to B$ *be a morphism in* $\mathcal{CC}_r$. *Then the following are equivalent:*

(i) ϕ *is a fibration.*

(ii) P_ϕ *has a coadjoint* $Q_\phi: (\mathcal{CC}_r, A) \to (\mathcal{CC}_r, B)$.

(iii) $\phi \times (-)$ *has a coadjoint* $(-)^\phi: (\mathcal{CC}_r, B) \to (\mathcal{CC}_r, B)$.

REMARK. The equivalence of (ii) and (iii) may be deduced from Lemma 4.1 using the Adjoint functor Theorem. However it is just as easy to prove it directly.

PROOF. (ii) $\Rightarrow$ (iii). As remarked in Section 4, $(-)^\phi = Q_\phi \circ P_\phi$ is coadjoint to $\phi \times (-)$.

(iii) $\Rightarrow$ (i). If $\phi \times (-) = \phi_* \circ P_\phi$ has a coadjoint, it preserves colimits. By the Corollary to Lemma 4.1, so does P_ϕ. In particular, P_ϕ preserves pushouts. It is sufficient to prove that:

a) ϕ_1 is a fibration of groupoids; and

b) For $x \in A_0$ and $2 \leqslant n \leqslant r$, the group homomorphism

$$\phi_n(x): A_n(x) \to B_n(\phi x)$$

is surjective. (For $n > r$, this is trivial.)

a) Let T be the tree groupoid with three vertices x_0, x_1, x_2, i.e. the free groupoid on the tree

$$x_0 \xrightarrow{g_1} x_1 \xrightarrow{g_2} x_2$$

Let T^0, T^1, T^2 be the full subgroupoids with vertex sets

$$\{x_1\},\quad \{x_0, x_1\},\quad \{x_1, x_2\}$$

respectively. Then

$$\begin{array}{ccc} T^0 & \hookrightarrow & T^1 \\ \downarrow & & \downarrow \\ T^2 & \hookrightarrow & T \end{array}$$

is a pushout in $\mathcal{G}pd$, and so also in $\mathcal{CC}_n$, if we identify $\mathcal{G}pd$ with the coreflexive subcategory $\mathcal{CC}_1$ of $\mathcal{CC}_n$. Suppose

$$x \in A_0,\quad b \in B_1 \quad \text{are such that} \quad \delta^0 b = \phi(x).$$

Define a morphism $\theta: T \to B$ by

$$\theta(g_1) = b,\quad \theta(g_2) = b^{-1}.$$

Writing $\hat{T}$ for $T \times_B A$ and $\hat{T}^i$ for $T^i \times_B A$ $(i = 0, 1, 2)$, we have

$$P_\phi(\theta): \hat{T} \to A \quad \text{and} \quad P_\theta(\phi): \hat{T} \to T.$$

By two applications of Lemma 4.1, together with the fact that P_ϕ preserves pushouts, if follows that

$$\begin{array}{ccc} \hat{T}^0 & \hookrightarrow & \hat{T}^1 \\ \downarrow & & \downarrow \\ \hat{T}^2 & \hookrightarrow & \hat{T} \end{array}$$

is a pushout in $\mathcal{CC}_n$. Hence $\hat{T}$ is generated by its subcomplexes $\hat{T}^1$, $\hat{T}^2$. Now $\hat{T} = T \times_B A$ is a subcomplex of $T \times A$, and $k = (g_1 g_2, e_1(x))$ is an element of $\hat{T}_1$, since

$$\theta(g_1 g_2) = b b^{-1} = e_1(\phi x) = \phi(e_1(x)).$$

Hence k can be written as a composite $k = k_1 k_2 \dots k_n$ with, for $1 \leqslant i \leqslant n$, either $k_i \in \hat{T}^1_1$ or $k_i \in \hat{T}^2_1$. Thus

$$g_1 g_2 = P_\theta(\phi)(k_1 \dots k_n) = P_\theta(\phi)(k_1) \dots P_\theta(\phi)(k_n) =$$
$$= h_1 \dots h_n, \text{ say.}$$

Let j be the smallest subscript such that $\delta^1 h_j \neq x_0$. Then $\delta^0 h_j = x_0$, so $h_j \notin T^2$. Hence $k_j \notin \hat{T}^2$, so $k_j \in \hat{T}^1$, and $h_j \in T^1$. Since $\delta^1 h_j \neq x_0$, it follows that

$$\delta^1 h_j = x_1 \quad \text{and so} \quad h_1 \dots h_j = g_1 \,.$$

Let $a = P_\phi(\theta)(k_1 \dots k_j)$. Then $\delta^0 a = x$ and

$$\phi(a) = (\phi \times \theta)(k_1 \dots k_j) = \theta(h_1 \dots h_j) = b \,.$$

Hence ϕ_1 is a fibration of groupoids.

b) If $r \leqslant 1$, we are finished. Suppose $2 \leqslant n \leqslant r$, and let W be the following rank n crossed complex:

$$W_0 = \{ x_0 \},$$

$$W_n(x_0) = W_{n-1}(x_0) = \begin{cases} Z * Z & (n = 2) \\ Z \oplus Z & (n \geqslant 3), \end{cases}$$

$$W_m(x_0) = 0 \quad (m \neq n, n-1),$$

$$\delta = Id \colon W_n(x_0) \to W_{n-1}(x_0),$$

and in the case $n = 2$, $W_1(x_0)$ acts on $W_2(x_0)$ by conjugation. (For $n \geqslant 3$, there is no need to define an action, since $W_1(x_0)$ is trivial.) Let $\{\beta_1, \beta_2\}$ be a basis for $W_n(x_0)$ (either as a free group or as a free abelian group). Then $\{\delta\beta_1, \delta\beta_2\}$ is a basis for $W_{n-1}(x_0)$. Let W^i (for $i = 1, 2$) denote the subcomplex of W generated by β_i. That is

$$W^i_0 = W_0 = \{ x_0 \} \,, \quad \text{and}$$

$W^i_n(x_0)$, $W^i_{n-1}(x_0)$ are cyclic with generators β_i, $\delta\beta_i$ respectively. Let W^o denote the non-empty rank 0 subcomplex of W. That is,

$$W^o_0 = W_0 = \{ x_0 \} \quad \text{and} \quad W^o_m = \{ e_m(x_0) \} \quad (m \geqslant 1).$$

It is easy to show that the square

$$\begin{array}{ccc} W^o & \hookrightarrow & W^1 \\ \downarrow & & \downarrow \\ W^2 & \hookrightarrow & W \end{array}$$

is a pushout in $\mathcal{CC}_n$ (and hence also in $\mathcal{CC}_r$, since $\mathcal{CC}_n$ is coreflexive in $\mathcal{CC}_r$). Suppose $x \in A_0$ and $\beta \in B_n(\phi x)$. We must find

$$a \in A_n(x) \quad \text{such that} \quad \phi(a) = \beta \,.$$

Define a morphism $\theta \colon W \to B$ by

$$\theta(x_0) = \phi(x), \quad \theta(\beta_1) = \theta(\beta_2) = \beta,$$
$$\theta(\delta\beta_1) = \theta(\delta\beta_2) = \delta\beta .$$

Arguing as in Part a, and writing

$$\hat{W} \text{ for } W \underset{B}{\times} A, \quad \hat{W}^i \text{ for } W^i \underset{B}{\times} A \quad (i = 0, 1, 2),$$

we have that

$$\begin{array}{ccc} \hat{W}^0 & \hookrightarrow & \hat{W}^1 \\ \downarrow & & \downarrow \\ \hat{W}^2 & \hookrightarrow & \hat{W} \end{array}$$

is a pushout in $\mathcal{CC}_n$, and $\hat{W}$ is generated by its subcomplexes $\hat{W}^1$, $\hat{W}^2$. Regarding $\hat{W}$ as a subcomplex of $W \times A$ (and hence $\hat{W}_0$ as a subset of A_0), we have $\lambda = (\beta_1\beta_2^{-1}, e_n(x)) \in \hat{W}_n(x)$, since

$$\theta(\beta_1\beta_2^{-1}) = \beta\beta^{-1} = e_n(\phi x) = \phi(e_n(x)).$$

Hence λ may be written as a product

$$\lambda = \lambda_1^{h(1)} \lambda_2^{h(2)} \cdot \ldots \cdot \lambda_s^{h(s)}$$

in $\hat{W}_n(x)$ with, for $1 \leqslant i \leqslant s$, $h(i) \in \hat{W}_1(y_i, x)$ for some $y_i \in \hat{W}_0$, and either $\lambda_i \in \hat{W}_n^1(y_i)$ or $\lambda_i \in \hat{W}_n^2(y_i)$. For each i, $P_\theta(\phi)(\lambda_i^{h(i)})$ is $\beta_1^{q(i)}$ or $\beta_2^{q(i)}$ for some integer $q(i)$. Hence

$$(\theta \times \phi)(\lambda_i^{h(i)}) = \theta(P_\theta(\phi))(\lambda_i^{h(i)}) = \beta^{q(i)}.$$

If d is the highest common factor of the non-zero $q(i)$'s, then

$$\beta_1\beta_2^{-1} = P_\theta(\phi)(\lambda)$$

is a product of conjugates of powers of β_1^d and β_2^d in $W_n(x_0)$. By the definition of $W_n(x_0)$, this is impossible for $d > 1$. Hence $d = 1$, and so there are integers $p(i)$ $(1 \leqslant i \leqslant s)$ such that

$$p(1)q(1) + \ldots + p(s)q(s) = 1.$$

Define

$$\mu = (\lambda_1^{p(1)})^{h(1)} \cdot \ldots \cdot (\lambda_s^{p(s)})^{h(s)} \in \hat{W}_n(x),$$

and $a = P_\phi(\theta)(\mu) \in A_n(x)$. Then

$$\phi(a) = (\theta \times \phi)(\mu) = \beta^{p(1)q(1) + \ldots + p(s)q(s)} = \beta,$$

and so $\phi_n(x): A_n(x) \to B_n(x)$ is surjective, as required.

(i) $\Rightarrow$ (ii). The actual description of a coadjoint Q_ϕ to the pullback functor P_ϕ depends on some rather unnatural-looking constructions. We will not give a detailed proof - merely describe the construction of $Q_\phi(\psi)$ for a given object ψ of $(\mathcal{CC}_n, A)$. The details are straightforward but tedious. We require two definitions.

DEFINITION. Let $\theta: C \to D$ be a morphism of crossed complexes. If G is a subcomplex of D, then a *section of* θ *on* G is a morphism $\xi: G \to C$, such that $\theta \circ \xi$ is the inclusion morphism $G \hookrightarrow D$.

DEFINITION. Let θ, C, D be as above. If X is a subset of D_n $(n \geqslant 0)$, then a *pseudo-section of* θ *on* X is a set-map $\eta: X \to C_n$ such that $\theta_n \circ \eta$ is the inclusion map $X \hookrightarrow D_n$.

Now suppose $\psi: D \to A$ is an object of $(\mathcal{CC}_n, A)$. We must define an object $Q_\phi(\psi): D' \to B$ of $(\mathcal{CC}_n, B)$. The trickiest part is the description of the crossed complex D'.

Define D'_0 to be the set of all pairs (y, ξ) with $y \in B_0$ and ξ a section of ψ on $F_\phi(y)$. If $(y_0, \xi_0), (y_1, \xi_1) \in D'_0$, define

$$D'_1((y_0, \xi_0), (y_1, \xi_1))$$

to be the set of all pairs (b, η) with $b \in B_1(y_0, y_1)$ and η a pseudo-section of ψ on $\phi_1^{-1}(b)$ satisfying the following two properties:

(K_1) Whenever $g, a, h \in A_1$ such that

$$\phi(g) = e_1(y_0), \quad \phi(a) = b, \quad \phi(h) = e_1(y_1)$$

and gah is defined in A_1, then

$$\eta(gah) = \xi_0(g)\ \eta(a)\ \xi_1(h) \text{ in } D_1.$$

(K'_1) Whenever $\alpha \in A_n(x_0)$ $(n \geqslant 2)$ and $a \in A_1(x_0, x_1)$ are such that

$$\phi(\alpha) = e_n(y_0) \text{ and } \phi(a) = b,$$

then $\xi_1(\alpha^a) = (\xi_0(\alpha))^{\eta(a)}$ in D_n.

If $n \geqslant 2$ and $(y, \xi) \in D'_0$, define $D'_n(y, \xi)$ to be the set of all pairs (β, ζ) with $\beta \in B_n(y)$ and ζ a pseudo-section of ψ on $\phi_n^{-1}(\beta)$ satisfying the following property:

(K_n) Whenever $\alpha_0 \in A_n(x_0)$, $\alpha_1 \in A_n(x_1)$ and $a \in A_1(x_0, x_1)$ are such that

$$\phi(\alpha_0) = \phi(\alpha_1) = \beta \quad \text{and} \quad \phi(a) = e_1(y),$$

then $\zeta(\alpha_0)^{\xi(a)} = \xi(\alpha_0^a \alpha_1^{-1})\ \zeta(\alpha_1)$ in D_n.

The next stage is to define the groupoid structure on D'_n for $n \geqslant 1$. Suppose $(y_0, \xi_0), (y_1, \xi_1), (y_2, \xi_2) \in D'_0$ and

$$(\beta_1, \zeta_1) \in D'_n((y_0, \xi_0), (y_1, \xi_1)), \quad (\beta_2, \zeta_2) \in D'_n((y_1, \xi_1), (y_2, \xi_2)).$$

Note that for $n \geqslant 2$, this is possible only if the (y_i, ξ_i) all coincide. Define a pseudo-section $\zeta = \zeta_1\ \zeta_2$ of ψ on $\phi_n^{-1}(\beta_1 \beta_2)$ as follows. If $\alpha \in A_n(x_0, x_2)$ with $\phi(\alpha) = \beta_1 \beta_2$, then (since ϕ_n is a groupoid fibration) we can find $x_1 \in A_0$ and $\alpha_1 \in A_n(x_0, x_1)$ with $\phi(\alpha_1) = \beta_1$. Let

$$\alpha_2 = \alpha_1^{-1} \alpha \in A_n(x_1, x_2)$$

(so that $\phi(\alpha_2) = \beta_2$), and define $\zeta(\alpha) = \zeta_1(\alpha_1)\ \zeta_2(\alpha_2)$.

It follows from (K_1) and (K'_1) (for $n = 1$), or from (K_n) (for $n \geqslant 2$), that $\zeta_1\ \zeta_2$ is well-defined, and that

$$(\beta_1 \beta_2, \zeta_1 \zeta_2) \in D'_n((y_0, \xi_0), (y_2, \xi_2)).$$

Define

$$(\beta_1, \zeta_1)\ (\beta_2, \zeta_2) = (\beta_1\ \beta_2, \zeta_1\ \zeta_2).$$

We must also define boundary maps for D'. Suppose $n \geqslant 2$ and $(\beta, \zeta) \in D'_n(y, \xi)$ for some $(y, \xi) \in D'_0$. Define a pseudo-section $(\delta\zeta)$ of ψ on $\phi_{n-1}^{-1}(\delta\beta)$ as follows: Suppose

$$a \in A_{n-1}(x', x) \quad \text{with} \quad \phi(a) = \delta\beta.$$

(For $n \geqslant 3$, this implies $x' = x$). Since $\phi(x) = y$ and ϕ is a fibration, we can choose $\alpha \in A_n(x)$ with $\phi(\alpha) = \beta$. Then $c = a\ (\delta\alpha)^{-1}$ is defined in A_{n-1} with $\phi(c) = e_{n-1}(y)$. Define

$$(\delta\zeta)(a) = \xi(c)\ \delta(\zeta\alpha).$$

It follows from (K_n) that $(\delta\zeta)$ is well-defined, and that

$$(\delta\beta, \delta\zeta) \in D'_{n-1}(y, \xi).$$

Define $\delta(\beta,\zeta) = (\delta\beta, \delta\zeta)$.

Finally, we must define the action of D'_1 on the collection of groups D'_n for $n \geqslant 2$. (The action of D'_1 on its own vertex groups may be defined to be by conjugation in D'_1.) Suppose

$$(\beta,\zeta) \in D'_n(y_0,\xi_0) \quad \text{and} \quad (b,\eta) \in D'_1((y_0,\xi_0),(y_1,\xi_1)).$$

Define a pseudo-section $\zeta^{(b,\eta)}$ of ψ on $\phi_n^{-1}(\beta^b)$ as follows. If

$$\alpha \in A_n(x) \quad \text{with} \quad \phi(\alpha) = \beta^b,$$

then (since ϕ_1 is a groupoid fibration) we can find

$$a \in \phi_1^{-1}(b) \quad \text{with} \quad \delta^1 a = x.$$

Define

$$\zeta^{(b,\eta)}(\alpha) = \{\zeta(\alpha^{a^{-1}})\}^{\eta(a)}.$$

It follows from (K'_1) and (K_n) that $\zeta^{(b,\eta)}$ is well-defined, and that

$$(\beta^b, \zeta^{(b,\eta)}) \in D'_n(y_1,\xi_1).$$

Define $(\beta,\zeta)^{(b,\eta)} = (\beta^b, \zeta^{(b,\eta)})$,

The various crossed complex axioms for D' follow from those for A, B and D by virtue of the conditions (K_n) $(n \geqslant 1)$ and (K'_1), as does the fact that $\operatorname{rank} D' \leqslant r$. It is clear that the rules

$$\begin{array}{ll} (y,\xi) \longmapsto y & ((y,\xi) \in D'_0), \\ (b,\eta) \longmapsto b & ((b,\eta) \in D'_1), \\ (\beta,\zeta) \longmapsto \beta & ((\beta,\zeta) \in D'_n,\ n \geqslant 2) \end{array}$$

determine a morphism $Q_\phi(\psi): D' \to B$ of crossed complexes.

All that remains is to check that the definition of Q_ϕ extends to a functor $(\mathcal{CC}_n, A) \to (\mathcal{CC}_n, B)$, and that (P_ϕ, Q_ϕ) is indeed an adjoint pair. This is straightforward.

COROLLARY 1. *If B is a rank 0 crossed complex, then $(\mathcal{CC}_n, B)$ is a cartesian closed category.*

PROOF. This follows from the Theorem and the remark that any morphism with rank 0 codomain is a fibration.

COROLLARY 2. *$\mathcal{CC}_r$ is a cartesian closed category.*

PROOF. $\mathcal{CC}_r$ has a terminal object, namely the rank 0 crossed complex $*$ such that $*_0$ is a singleton. By Corollary 1, $(\mathcal{CC}_r, *)$ is cartesian closed. But since $*$ is terminal, $Dom: (\mathcal{CC}_r, *) \to \mathcal{CC}_r$ is an equivalence.

COROLLARY 3. *Any colimit diagram in $\mathcal{CC}_r$ can be «pulled back» along a fibration to another colimit diagram of the same type.*

PROOF. If B is the colimit of a diagram $\mathcal{D}: \mathcal{C} \to \mathcal{CC}_r$, let $(\mathcal{D}, B)$ denote the diagram $\mathcal{C} \to (\mathcal{CC}_r, B)$ which sends every object C of $\mathcal{C}$ to the canonical morphism $\mathcal{D}(C) \to B$ in $\mathcal{CC}_r$. Then

$$\mathcal{D} = Dom \circ (\mathcal{D}, B): \mathcal{C} \to \mathcal{CC}_r,$$

so by Lemma 4.1, Id_B is the colimit of $(\mathcal{D}, B)$.

If $\phi: A \to B$ is a fibration, then the «pullback» of the diagram $\mathcal{D}$ along ϕ is just the diagram $Dom \circ P_\phi \circ (\mathcal{D}, B)$. But both Dom and P_ϕ preserve colimits, by Lemma 4.1 and Theorem 5.1. Hence

$$colim(Dom \circ P_\phi \circ (\mathcal{D}, B)) = (Dom \circ P_\phi)(colim(\mathcal{D}, B))$$
$$= (Dom \circ P_\phi)(Id_B) = Dom(Id_A) = A.$$

NOTE. Corollary 3 contains as a special case the fact that colimit diagrams in $\mathcal{G}pd$ can be «pulled back» along coverings to colimit diagrams of the same type. For a direct proof of this, see Higgins [7].

6. REMARKS.

Corollary 2 to Theorem 5.1 states that, for any fixed r, $\mathcal{CC}_r$ is a cartesian closed category. That is, if A is any crossed complex of rank not greater than r, then $A \times (-): \mathcal{CC}_r \to \mathcal{CC}_r$ has a coadjoint $(-)^A$. Let C be any object of $\mathcal{CC}_r$, and let $*$ be the terminal object. Replacing ψ by the projection $A \times C \to A$ and ϕ by the canonical morphism $A \to *$ in the proof of Theorem 5.1, we obtain the following description of the crossed complex C^A:

$(C^A)_0$ is the set of morphisms $A \to C$ in $\mathcal{CC}_r$, i.e.,

$$(C^A)_0 = \mathcal{CC}_r(A, C).$$

If $\xi_0, \xi_1 : A \to C$ are morphisms, then $(C^A)_1(\xi_0, \xi_1)$ is the set of maps $\eta : A_0 \to C_1$ satisfying the properties:

$$\xi_1(a) = \eta(\delta^0 a)^{-1}\, \xi_0(a)\, \eta(\delta^1 a) \quad (a \in A_1),$$

$$\xi_1(\alpha) = \xi_0(\alpha)^{\eta(x)} \quad (\alpha \in A_n(x),\ n \geqslant 2,\ x \in A_0).$$

If $\xi : A \to C$ is a morphism and $n \geqslant 2$, $(C^A)_n(\xi)$ is the set of maps ζ satisfying the property:

$$\zeta(\delta^1 a) = \zeta(\delta^0 a)^{\xi(a)} \quad (a \in A_1).$$

Suppose A, B, C are objects of $\mathcal{CC}_n$. Let λ_A denote the map $\mathcal{CC}_n(B, C) \to \mathcal{CC}_n(B^A, C^A)$ determined by the functor $(-)^A$. One can also show the following

THEOREM 6.1. *There is a morphism* $\omega : C^B \to (C^A)^{(B^A)}$ *in* $\mathcal{CC}_n$ *such that* $\omega_0 = \lambda_A$.

If A is connected (i.e., if A_1 is a connected groupoid), then we can obtain the following alternative description of the groups $(C^A)_n(\xi)$ $(n \geqslant 1,\ \xi \in (C^A)_0)$:

Choose an arbitrary element x of A_0, and let $y = \xi(x) \in C_0$. Then for $n \geqslant 2$, $(C^A)_n(\xi)$ is isomorphic to the subgroup of $C_n(y)$ consisting of those elements left fixed by the action of $\xi(A_1(x))$. $(C^A)_1(\xi)$ is isomorphic to the subgroup of $C_1(y)$ consisting of those elements which act trivially on $\xi(A_n(x))$ for all $n \geqslant 1$. In particular, if either A or C has rank not greater than 1, $(C^A)_1(\xi)$ is isomorphic to the centraliser of $\xi(A_1(x))$ in $C_1(y)$.

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