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ON A SEQUENCE OF K.A. HARDIE

by Klaus Heiner KAMPS

The aim of this Note is to show how an exact sequence in pair-homotopy due to K. A. Hardie ([3], Theorem 1) and generalisations thereof in semicubical homotopy theory can be derived from the theory of fibrations of groupoids.

0. For the convenience of the reader we recall some of the basic definitions of semicubical homotopy theory (see [4, 5, 6, 8]).

0.1. DEFINITION. *A semicubical set* Q consists of a sequence of sets Q_n, $n = 0, 1, 2, \ldots$, and of maps

$$0^i_n,\ 1^i_n : Q_n \to Q_{n-1},\quad i = 1, \ldots, n \qquad (\textit{face operators}),$$

$$\zeta^j_n : Q_n \to Q_{n+1},\qquad j = 1, \ldots, n+1 \qquad (\textit{degeneracy operators}),$$

satisfying (1)-(5):

(1) $\epsilon^i_{n-1}\omega^j_n = \omega^{j-1}_{n-1}\epsilon^i_n,\qquad i < j,$

(2) $\zeta^i_{n+1}\zeta^j_n = \zeta^{j+1}_{n+1}\zeta^i_n,\qquad i \leqslant j,$

(3) $\epsilon^i_{n+1}\zeta^j_n = \zeta^{j-1}_{n-1}\epsilon^i_n,\qquad i < j,$

(4) $\epsilon^i_{n+1}\zeta^i_n = 1_{Q_n},$

(5) $\epsilon^i_{n+1}\zeta^j_n = \zeta^j_{n-1}\epsilon^{i-1}_n,\qquad i > j,$

where $\epsilon = 0, 1$, $\omega = 0, 1$.

The elements of Q_n are called *n-cubes*. If $\psi \in Q_n$,

$$D\psi = (\psi^1_0, \psi^1_1, \ldots, \psi^n_0, \psi^n_1),\ \text{ where }\ \psi^i_\epsilon = \epsilon^i_n\psi,$$

denotes the boundary of ψ. If $\psi \in Q_1$, we use the notation $D\psi = (\psi_0, \psi_1)$.

A semicubical map $f: Q \to Q'$ between semicubical sets is a sequence of maps $f_n: Q_n \to Q'_n$ commuting with face and degeneracy operators.

We get a category $\mathcal{K}$, the category of semicubical sets.

0.2. DEFINITION. Let Q be a semicubical set, (n,ν,k) a triple of integers such that

$$n \geqslant 1, \quad \nu \in \{0,1\}, \quad k \in \{1,\dots,n\}.$$

A *(n,ν,k)-equation γ in Q* is a family

$$\gamma = (\gamma_\epsilon^i \mid \epsilon = 0,1, \; i = 1,\dots,n, \; (\epsilon,i) \neq (\nu,k))$$

such that

$$\gamma_\epsilon^i \in Q_{n-1} \quad \text{and} \quad \epsilon_{n-1}^i \gamma_\omega^j = \omega_{n-1}^{j-1} \gamma_\epsilon^i$$

for $i < j$ and $(\epsilon,i), (\omega,j) \neq (\nu,k)$.

Q satisfies the Kan-condition $E(n,\nu,k)$ if for each (n,ν,k)-equation γ there exists a *solution* λ *of* γ, i.e. an element $\lambda \in Q_n$ such that

$$D\lambda = (\gamma_0^1, \gamma_1^1, \dots, \lambda_\nu^k, \dots, \gamma_0^n, \gamma_1^n):$$

Q satisfies the Kan-condition $E(n)$ if it satisfies

$$E(n,\nu,k) \quad \text{for all } \nu \in \{0,1\}, \; k \in \{1,\dots,n\}.$$

For each integer $n = 0,1,2,\dots$, we have the functor *n-skeleton* $sk^n: \mathcal{K} \to \mathcal{S}$ into the category $\mathcal{S}$ of sets, defined by

$$Q \mapsto Q_n, \quad f \mapsto f_n,$$

where Q is a semicubical set and f a semicubical map.

0.3. DEFINITION. For a semicubical set Q which satisfies the Kan-conditions $E(2)$ and $E(3)$ we have the *fundamental groupoid* ΠQ *of* Q. The objects are the 0-cubes of Q. If f, g are 0-cubes of Q the morphisms $f \to g$ in ΠQ are the classes $[\alpha]$ of those elements $\alpha \in Q_1$ with $D\alpha = (f,g)$ with respect to the following relation $\sim$:

$\alpha \sim \beta$ if and only if there exists $\psi \in Q_2$ with $D\psi = (\alpha, \beta, \zeta_0^1 f, \zeta_0^1 g)$.

Composition in ΠQ which is written additively is given as follows: if

$$[\alpha]: f \to g, \quad [\beta]: g \to h$$

are morphisms of ΠQ and if $\lambda \in Q_2$ is such that $D\lambda = (\alpha, \lambda_1^1, \zeta_0^1 f, \beta)$ (λ exists by $E(2)$), then $[\beta] + [\alpha] = [\lambda_1^1]$.

0.4. DEFINITION. *A semicubical homotopy system* in a category $\mathcal{C}$ is a functor $Q: \mathcal{C} \times \mathcal{C} \to \mathcal{K}$, contravariant in the first variable, covariant in the second one, such that the 0-skeleton $Q_0 = sk^0 \circ Q: \mathcal{C} \times \mathcal{C} \to \mathcal{S}$ is the morphism functor $\mathcal{C}(-,-)$.

0.5. EXAMPLE. Let (I, j_0, j_1, s) be a *homotopy system* in a category $\mathcal{C}$, i.e. $I: \mathcal{C} \to \mathcal{C}$ is a functor *(cylinder functor)* and

$$j_0, j_1 : 1_{\mathcal{C}} \to I, \quad s: I \to 1_{\mathcal{C}}$$

are natural transformations with $s j_0 = s j_1 = 1$. Then we obtain a semicubical homotopy system in $\mathcal{C}$ with n-cubes $Q_n(X, Y) = \mathcal{C}(I^n X, Y)$, face operators

$$\epsilon_n^i = \mathcal{C}(I^{i-1} j_\epsilon I^{n-i} X, 1_Y)$$

and degeneracy operators

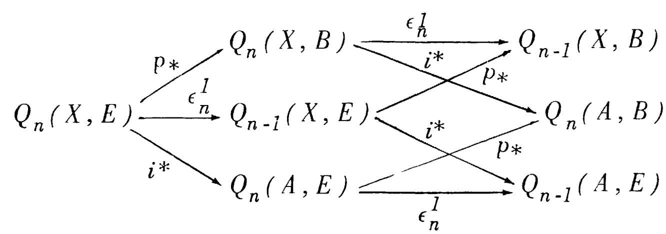
$$\zeta_n^j = \mathcal{C}(I^{j-1} s I^{n+1-j} X, 1_Y),$$

where X, Y are objects of $\mathcal{C}$.

0.6. DEFINITION. A semicubical homotopy system Q in a category $\mathcal{C}$ *satisfies the Kan-condition $E(n)$* if for all objects X, Y of $\mathcal{C}$ the semicubical set $Q(X, Y)$ satisfies the Kan-condition $E(n)$.

If solutions of equations can be chosen naturally with respect to X and Y, then Q is said to satisfy the Kan-condition $NE(n)$. If in addition degenerate equations give rise to degenerate solutions, Q is said to satisfy $DNE(n)$. For details we refer to [4], 3.

0.7. DEFINITION. Let Q be a semicubical homotopy system in a category $\mathcal{C}$, let $i: A \to X$, $p: E \to B$ be morphisms of $\mathcal{C}$, $n \geqslant 1$ an integer. *(i, p) satisfies $CHEP(n)$ ($CHEP$ = covering homotopy extension property)* if for $\epsilon = 0, 1$ the diagram



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where $p_* = Q(1, p)$, $i^* = Q(i, 1)$, is a weak limit in $\mathcal{S}$. This means: for

$$\beta \in Q_n(X, B),\quad f \in Q_{n-1}(X, E),\quad \alpha \in Q_n(A, E)$$

with

$$\epsilon_n^1 \beta = p_* f,\quad \epsilon_n^1 \alpha = i^* f,\quad i^* \beta = p_* \alpha ,$$

there exists $\phi \in Q_n(X, E)$ with

$$p_* \phi = \beta ,\quad \epsilon_n^1 \phi = f,\quad i^* \phi = \alpha .$$

REMARK. For an extensive discussion of examples of the *CHEP* we refer the reader to [1], 2.2.

1. Let $\mathcal{C}$ be a category, $Q : \mathcal{C} \times \mathcal{C} \to \mathcal{K}$ a semicubical homotopy system in $\mathcal{C}$, let $i : A \to X$ and $p : E \to B$ be morphisms of $\mathcal{C}$. Let Q' be the semicubical subset of $Q(A, E) \times Q(X, B)$ given by:

$$Q'_n = \{ (\phi, \psi) \in Q_n(A, E) \times Q_n(X, B) \mid p_* \phi = i^* \psi \},\quad n \geqslant 0 .$$

Thus we have a pullback in $\mathcal{K}$:

$$\begin{array}{ccc} Q' & \longrightarrow & Q(X, B) \\ \big\downarrow & & \big\downarrow i^* \\ Q(A, E) & \xrightarrow{\;p_*\;} & Q(A, B) \end{array} .$$

We assume that the functor Q satisfies the Kan-conditions $DNE(2)$ and $NE(3)$. Then the semicubical set Q' satisfies the Kan-condition $E(3)$. Let $H = \Pi Q'$ be the fundamental groupoid of Q', $G = \Pi(X, E)$ the fundamental groupoid of $Q(X, E)$. We define a functor (!) $\gamma : G \to H$ by

$\gamma(g) = (gi, pg)$ if $g \in \mathcal{C}(X, E)$ is an object of G, and
$\gamma[\phi] = [(i^* \phi, p_* \phi)]$ if $[\phi]$ is a morphism of G, $\phi \in Q_1(X, E)$.

1.1. PROPOSITION. *If (i, p) satisfies $CHEP(1)$, then $\gamma : G \to H$ is a fibration of groupoids.*

PROOF. Let $g \in \mathcal{C}(X, E)$ be an object of G,

$$[(\phi, \psi)] : (gi, pg) \to (k, r)$$

a morphism of H. Thus we have

$$k \in \mathcal{C}(A, E), \quad r \in \mathcal{C}(X, B) \text{ with } pk = ri,$$
$$\phi \in Q_1(A, E), \quad \psi \in Q_1(X, B) \text{ with}$$
$$D\phi = (gi, k), \quad D\psi = (pg, r) \text{ and } p_*\phi = i^*\psi .$$

Since (i,p) satisfies $CHEP(1)$, there exists $\Phi \in Q_1(X, E)$ such that:

$$\Phi_0 = g, \quad p_*\Phi = \psi, \quad i^*\Phi = \phi.$$

Then $[\Phi]$ is a morphism of G such that

$$\gamma[\Phi] = [(i^*\Phi, p_*\Phi)] = [(\phi, \psi)] .$$

This means that γ is star-surjective and hence a fibration of groupoids ([2], 2.1). //

[2], 4 yields:

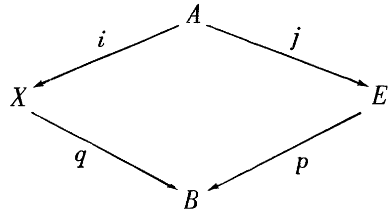
1.2. COROLLARY. *If (i,p) satisfies $CHEP(1)$ and $f: X \to E$ is a morphism of $\mathcal{C}$, then one has an exact sequence of group homomorphisms and pointed maps*

$$(1.3) \quad G\{f\} \to H\{(j,q)\} \to \pi_0 F \to \pi_0 G \to \pi_0 H,$$

where $j = fi$, $q = pf$.

$G\{f\}$ denotes the object group $G(f,f)$ of G, F the fibre $\gamma^{-1}\gamma(f)$ of γ over $\gamma(f) = (j,q)$ and π_0 the set of components. $\pi_0 F$ is pointed by the component of f in F, $\pi_0 G$ by the component of f in G, $\pi_0 H$ by the component of (j,q) in H. In (1.3) $\pi_0 G$ is the set $[X, E]$ of homotopy classes of morphisms of $\mathcal{C}$ from X to E, $\pi_0 H$ the set $[i, p]$ of pair-homotopy classes of pair-morphisms from i to p.

Our next aim is to give an interpretation of $\pi_0 F$ as the set, denoted by $[(i,q),(j,p)]^A_B$ of homotopy classes under A and over B (see [8], 2) for the diagram



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1.4. PROPOSITION. *If (i,q) satisfies $CHEP(2)$, then the map*

$$\theta: [(i,q),(j,p)]^A_B \to \pi_0 F, \quad \theta[g]^A_B = \hat{g},$$

where $\hat{g}$ denotes the component of g in F, is a bijection.

PROOF. It is easy to see that θ is well-defined and surjective. The injectivity of θ follows from [8], 2.1, since (i,p) is assumed to satisfy the condition $CHEP(2)$. //

1.5. COROLLARY. *If (i,p) satisfies $CHEP(2)$ and if $f: X \to E$ is a morphism of $\mathcal{C}$, then the sequence*

$$\Pi(X,E)\{f\} \xrightarrow{\gamma} \Pi Q'\{(j,q)\} \xrightarrow{\delta} [(i,q),(j,p)]^A_B \xrightarrow{V} [X,E] \xrightarrow{\alpha} [i,p]$$

is exact, where

$$j = fi, \quad q = pf, \quad V[g]^A_B = [g], \quad \alpha[g] = [(gi, pg)],$$

and δ is given as follows: if $[(\phi,\psi)]: (j,q) \to (j,q)$ is a morphism of $\Pi Q'$, Φ an element of $Q_1(X,E)$ with

$$\Phi_0 = f, \quad p_*\Phi = \psi, \quad i^*\Phi = \phi,$$

then $\delta[(\phi,\psi)] = [\Phi_1]^A_B$.

2. In addition to the assumptions of 1 we are now requiring that $\mathcal{C}$ has a zero object 0. Then in an obvious way Q can be regarded as a functor into the category $\mathcal{K}^\circ$ of pointed semicubical sets, the skeleta $Q_n = sk^n \circ Q$ as functors into the category $\mathcal{S}^\circ$ of pointed sets.

Let us make the following assumptions:

$A \xrightarrow{i} X$ has a cokernel $X \xrightarrow{\pi} C$,

$E \xrightarrow{p} B$ has a kernel $K \xrightarrow{\iota} E$,

$Q_1(C,-)$, $Q_1(-,K)$ preserve zero objects,

$Q_1(X,-)$ preserves kernels, $Q_1(-,K)$ takes cokernels into kernels.

Let us consider the special case that f is the zero morphism

$$0_{X,E}: X \to E.$$

2.1. PROPOSITION. *The map*

$$\theta' : [C, K] \to [(i, 0_{X,B}), (0_{A,E}, p)]_B^A, \quad \theta'[v] = [\iota v \pi]_B^A,$$

is bijective.

The elementary proof is left to the reader.

2.2. COROLLARY. *If (i, p) satisfies CHEP(2), then*

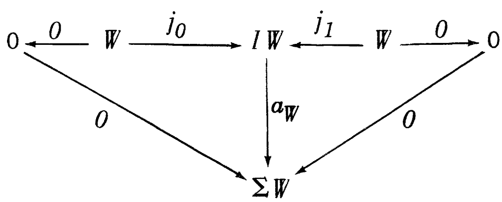
$$[C, K] \xrightarrow{\beta} [X, E] \xrightarrow{a} [i, p],$$

$$\beta[v] = [\iota v \pi], \quad a[g] = [(gi, pg)],$$

is an exact sequence of pointed sets.

Finally, let us deal with the case that the semicubical homotopy system Q is induced by a homotopy system (I, j_0, j_1, s) (see (0.5)). We assume that the cylinder functor $I: \mathcal{C} \to \mathcal{C}$ preserves zero objects and cokernels.

If W is an object of $\mathcal{C}$, we can choose a colimit diagram in $\mathcal{C}$:



ΣW is called a *suspension of* W. Any morphism $g: W \to W'$ of $\mathcal{C}$ induces

$$\Sigma g: \Sigma W \to \Sigma W' \text{ such that } (\Sigma g) \circ a_W = a_{W'} \circ Ig.$$

By [7], (1.11), we have:

2.3. PROPOSITION. *The map*

$$\theta'' : [\Sigma X, E] \to \Pi(X, E)\{0_{X,E}\}, \quad \theta''[v] = [v \circ a_X],$$

is a bijection of pointed sets.

By similar methods one can prove:

2.4. PROPOSITION. *The map*

$$\theta''' : [\Sigma i, p] \to \Pi Q'\{(0_{A,E}, 0_{X,B})\}, \quad \theta'''[(u, v)] = [(u \circ a_A, v \circ a_X)]$$

is a bijection of pointed sets.

θ'' and θ''' induce group structures on $[\Sigma X, E]$ and $[\Sigma i, p]$.

From 1.5, 2.2, 2.3, 2.4, we deduce:

2.5. THEOREM. *If (i, p) satisfies CHEP(2), then*

$$[\Sigma X, E] \xrightarrow{\alpha'} [\Sigma i, p] \xrightarrow{\delta'} [C, K] \xrightarrow{\beta} [X, E] \xrightarrow{\alpha} [i, p],$$

where $\alpha'[v] = [(v \circ \Sigma i,\ p \circ v)]$, is an exact sequence of groups and pointed sets.

δ' can be described as follows: given $[(u, v)] \epsilon [\Sigma i, p]$, choose $\phi : IX \to E$ with

$$\phi_0 = 0_{X,E}, \quad p\phi = v \circ a_X, \quad \phi(Ii) = u \circ a_A .$$

$\phi_1 : X \to E$ induces $g : C \to K$ with $\phi_1 = \iota g \pi$. Then $\delta'[(u, v)] = [g]$.

If $\mathcal{C}$ is the category of pointed topological spaces, i a closed cofibration, p a fibration, [9], Theorem 4, allows us to apply 2.5. We obtain [3], Theorem 1.

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