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PRINCIPAL HOMOGENEOUS OBJECTS AS REPRESENTABLE FUNCTORS

by D. H. VAN OSDOL

INTRODUCTION.

Let R be a ring with identity and $\underline{A}$ the category of unitary left R-modules. Let X and Π be in $\underline{A}$; then $X \times \Pi$ represents the functor

$$\underline{A}(-, X) \times \underline{A}(-, \Pi): \underline{A}^{op} \to \underline{Sets} .$$

More generally, let

$$0 \to \Pi \xrightarrow{i} Y \xrightarrow{p} X \longrightarrow 0$$

be an exact sequence of R-modules. Does Y represent some functor, and if so what is it?

Let $G: \underline{A} \to \underline{A}$ be the free R-module functor with $\epsilon: G \to \underline{A}$ the natural projection. Since p is onto and GX is free, there is a homomorphism $s: GX \to Y$ such that $p \circ s = \epsilon X$. If $z: Z \to Y$ in $\underline{A}$ then

$$p \circ (z \circ \epsilon Z - s \circ G(p \circ z)) = 0$$

so there exists a unique $h: GZ \to \Pi$ such that

$$i \circ h = z \circ \epsilon Z - s \circ G(p \circ z).$$

Thus z gives rise to a pair of maps

$$p \circ z: Z \to X \quad \text{and} \quad h: GZ \to \Pi .$$

These are related in the following way. Since

$$p \circ (s \circ \epsilon GX - s \circ G\epsilon X) = 0 ,$$

there is a unique $f: G^2 X \to \Pi$ such that

$$i \circ f = s \circ \epsilon GX - s \circ G\epsilon X ,$$

and one can show that

$$f \circ G^2(p \circ z) = h \circ G\epsilon Z - h \circ \epsilon GZ .$$

In addition, f is a one-cocycle, i.e.

$$f \circ G\epsilon GX = f \circ G^2 \epsilon X + f \circ \epsilon G^2 X.$$

Thus if we define

$$D(Z,f) = \{ (g,h) \mid g: Z \to X,\ h: GZ \to \Pi,\ f \circ G^2 g = h \circ G\epsilon Z - h \circ \epsilon GZ \}$$

then there is a function (depending on s) $\underline{A}(Z,Y) \to D(Z,f)$. In fact, $D(-,f)$ is a functor $\underline{A}^{op} \to \underline{Sets}$ and $\underline{A}(-,Y) \to D(-,f)$ is a natural transformation. In this case, it is a natural equivalence (see I.5). Thus to define a homomorphism from Z into an extension of X by Π it is necessary and sufficient to give two homomorphisms $g: Z \to X$, $h: GZ \to \Pi$ such that

$$f \circ G^2 g = h \circ G\epsilon Z - h \circ \epsilon GZ .$$

If X is a topological space and Π is a topological abelian group, then $X \times \Pi$ represents

$$\underline{Top}(-,X) \times \underline{Top}(-,\Pi): \underline{Top}^{op} \to \underline{Sets} .$$

More generally let $Y \xrightarrow{p} X$ be a principal homogeneous fibre bundle with fibre Π. Then there are an open cover $\{U_i\}$ of X and homeomorphisms $\Phi_i: U_i \times \Pi \to p^{-1}(U_i)$ such that

$p \circ \Phi_i$ = the first projection p_1,

and there exist $f_{ij}: U_i \cap U_j \to \Pi$ such that

$$\Phi_j(x,a) = \Phi_i(x, f_{ij}(x) + a)$$

for all x in $U_i \cap U_j$ and all a in Π.

Then $f = \amalg f_{ij}$ represents a Čech one-cocycle. If $z: Z \to Y$, define

$$h: \amalg (p \circ z)^{-1} U_i \to \Pi$$

by taking the coproduct of the compositions

$$h_i: (p \circ z)^{-1} U_i \xrightarrow{z} p^{-1}(U_i) \xrightarrow{\Phi_i^{-1}} U_i \times \Pi \xrightarrow{p_2} \Pi .$$

One can show that for u in $(p \circ z)^{-1} U_i \cap (p \circ z)^{-1} U_j$ we have

$$h_i(u) - h_j(u) = f_{ij}((p \circ z)(u)).$$

Hence we have a function $\underline{Top}(Z, Y) \to D(Z, f)$ where

$$D(Z,f) = \{(g,h) \mid g: Z \to Y,\ h: \amalg g^{-1}(U_i) \to \Pi,$$
$$h_i(u) - h_j(u) = f_{ij}(g(u)) \text{ for } u \in g^{-1}(U_i) \cap g^{-1}(U_j)\}.$$

Once again this is a natural equivalence so that to give a map into the total space of a bundle, it is necessary and sufficient to give a map $Z \xrightarrow{g} X$ and maps $g^{-1}(U_i) \xrightarrow{h_i} \Pi$ such that

$$h_i(u) - h_j(u) = f_{ij}(g(u)) \quad \text{for all } u \text{ in } g^{-1}(U_i) \cap g^{-1}(U_j).$$

To complete the analogy between this example and that of R-modules, we leave it to the reader to define G and ϵ (see [2]).

It is the purpose of this paper to examine the relationships between principal homogeneous objects (extensions) defining a cocycle f and the functor $D(-, f)$. The main results are I.5, I.6, II.5 and II.6. Theorem II.6 is perhaps worthy of further consideration since it suggests a connection between realization of one-cohomology classes and generalized descent (for the standard theories of descent, see [3,4]). In addition, II.7 shows that tripleableness is a sufficient condition for interpretation of H^1 by principal homogeneous objects (a result of Beck [1]), while II.6 indicates that it is probably not a necessary condition. All of our results hold in case G is a cotriple arising from a tripleable adjoint pair.

The author would like to thank Michael Barr, Robert Paré and Jack Duskin (especially the latter for communicating Proposition 6.6.3 of [2]) for helpful conversations on the content of this paper. Some of the results were first announced at the 1975 winter meeting of the American Mathematical Society.

I. COCYCLES, HOMOGENEOUS OBJECTS AND REPRESENTABLE FUNCTORS.

We assume from the outset that $\underline{A}$ is a category, $G: \underline{A} \to \underline{A}$ is a functor and $\epsilon : G \to \underline{A}$ is a natural transformation such that ϵ is the coequalizer of ϵG and $G\epsilon$. Let X be an object of $\underline{A}$ and Π an abelian group object in $\underline{A}$, whose operations will be denoted additively.

I.1. DEFINITION. A *one-cocycle* (on X with values in Π) is a morphism $f: G^2 X \to \Pi$ such that

$$f \circ G\epsilon G X = f \circ G^2 \epsilon X + f \circ \epsilon G^2 X .$$

Given a one-cocycle f we define a functor $D(-, f): \underline{A}^{op} \to \underline{Sets}$ as follows. For Z an object of $\underline{A}$,

$$D(Z, f) = \{ (g, h) \mid g: Z \to X,\ h: GZ \to \Pi, \text{ and }$$
$$f \circ G^2 g + h \circ \epsilon GZ = h \circ G\epsilon Z \} .$$

Given $z : Z' \to Z$ in $\underline{A}$, $D(z, f)$ on (g, h) is $(g \circ z, h \circ Gz)$ in $D(Z', f)$. It is easy to check that this does indeed define a functor.

I.2. DEFINITION. A *G-trivial Π-principal homogeneous object over* X consists of an object Y in $\underline{A}$, a right action $\rho : Y \times \Pi \to Y$ of Π on Y (i.e. a morphism ρ such that $\rho \circ (z, 0) = z$ and

$$\rho \circ (\rho \circ (z, a_1), a_2) = \rho \circ (z, a_1 + a_2)$$

for morphisms $z : Z \to Y$, a_1, $a_2 : Z \to \Pi$), a morphism $p : Y \to X$, and a morphism $s : GX \to Y$ such that:

i) $Y \times \Pi \underset{\rho}{\overset{p_1}{\rightrightarrows}} Y \xrightarrow{p} X$ is a kernel pair diagram,

ii) $p \circ s = \epsilon X$.

I.3. PROPOSITION. *If* $(Y \xrightarrow{p} X, \rho, s)$ *is a G-trivial Π-principal homogeneous object over* X *then diagram I.2.i is a coequalizer, and there exists a unique* $t: GY \to \Pi$ *such that* $\rho \circ (s \circ Gp, t) = \epsilon Y$.

PROOF. Since

$$p \circ (s \circ Gp) = \epsilon X \circ Gp = p \circ \epsilon Y$$

and I.2.i is a kernel pair, the existence of t as asserted is guaranteed. Now suppose $z: Y \to Z$ has the property that $z \circ p_1 = z \circ \rho$. Then we have

$$(z \circ s) \circ \epsilon G X = z \circ \epsilon Y \circ G s = z \circ \rho \circ (s \circ G p, t) \circ G s =$$
$$= z \circ p_1 \circ (s \circ G p, t) \circ G s = z \circ s \circ G p \circ G s = (z \circ s) \circ G \epsilon X,$$

so there exists a unique

$$z': X \to Z \quad \text{such that} \quad z' \circ \epsilon X = z \circ s$$

(recall that ϵ is the coequalizer of ϵG and $G\epsilon$). A computation similar to that just given shows that

$$z' \circ p \circ \epsilon Y = z \circ \epsilon Y$$

and thus $z' \circ p = z$. Since $p \circ s = \epsilon X$ is a coequalizer, p is an epimorphism and hence z' is the unique morphism such that $z' \circ p = z$.

QED

I.4. PROPOSITION. *If $(Y \xrightarrow{p} X, \rho, s)$ is a G-trivial Π-principal homogeneous object over X then there is a unique $f: G^2 X \to \Pi$ such that*

$$\rho \circ (s \circ G\epsilon X, f) = s \circ \epsilon G X.$$

Moreover f is a cocycle and

$$f \circ G^2 p + t \circ \epsilon G Y = t \circ G\epsilon Y,$$

i. e. (p, t) is in $D(Y, f)$.

PROOF. The existence of f is assured by I.2.i and the fact that

$$p \circ (s \circ G\epsilon X) = p \circ (s \circ \epsilon G X).$$

That f is a cocycle follows from I.2.i and the verification that

$$\rho \circ (s \circ G\epsilon X \circ G^2 \epsilon X, f \circ G^2 \epsilon X + f \circ \epsilon G^2 X) =$$
$$\rho \circ (s \circ G\epsilon X \circ G^2 \epsilon X, f \circ G\epsilon G X).$$

The last assertion is proved analogously, since

$$\rho \circ (s \circ G\epsilon X \circ G^2 p, f \circ G^2 p + t \circ \epsilon G Y) =$$
$$\rho \circ (s \circ G\epsilon X \circ G^2 p, t \circ G\epsilon Y).$$

QED

I.5. THEOREM. *If $(Y \xrightarrow{p} X, \rho, s)$ is a G-trivial Π-principal homogeneous* *object over* X *and* $f\colon G^2X \to \Pi$ *is the cocycle constructed in I.4 then* Y *represents* $D(-,f)$.

PROOF. Given z in $\underline{A}(Z,Y)$ define $\alpha(z) = (p\circ z,\ t\circ Gz)$. Then $\alpha(z)$ is in $D(Z,f)$ by I.4, and α is obviously a natural transformation

$$\alpha : \underline{A}(-,Y) \to D(-,f).$$

Given (g,h) in $D(Z,f)$ we have:

$$\begin{aligned}
&\rho\circ(s\circ Gg,\ h)\circ G\epsilon Z = \rho\circ(s\circ G\epsilon X\circ G^2 g,\ h\circ G\epsilon Z) = \\
&= \rho\circ(s\circ G\epsilon X\circ G^2 g,\ f\circ G^2 g + h\circ\epsilon GZ) = \\
&= \rho\circ(\rho\circ(s\circ G\epsilon X,\ f)\circ G^2 g,\ h\circ\epsilon GZ) = \\
&= \rho\circ(s\circ\epsilon GX\circ G^2 g,\ h\circ\epsilon GZ) = \rho\circ(s\circ Gg,\ h)\circ\epsilon GZ,
\end{aligned}$$

so there exists a unique

$$z : Z \to Y \quad \text{such that} \quad z\circ\epsilon Z = \rho\circ(s\circ Gg,\ h).$$

Define $\beta(g,h) = z$; then $\beta : D(Z,f) \to \underline{A}(Z,Y)$. A simple computation shows that

$$p\circ z\circ\epsilon Z = g\circ\epsilon Z, \quad \text{so} \quad p\circ\beta(g,h) = g.$$

Thus to prove that $\alpha\circ\beta(g,h) = (g,h)$ it suffices to see that $t\circ Gz = h$. But

$$\begin{aligned}
\rho\circ(s\circ Gg,\ t\circ Gz) &= \rho\circ(s\circ Gp,t)\circ Gz = \epsilon Y\circ Gz = \\
&= z\circ\epsilon Z = \rho\circ(s\circ Gg,\ h),
\end{aligned}$$

so I.2.i implies $t\circ Gz = h$. Finally

$$\begin{aligned}
(\beta\circ\alpha(z))\circ\epsilon Z &= \beta(p\circ z,\ t\circ Gz)\circ\epsilon Z = \\
&= \rho\circ(s\circ Gp\circ Gz,\ t\circ Gz) = \epsilon Y\circ Gz = z\circ\epsilon Z,
\end{aligned}$$

so $\beta\circ\alpha(z) = z$.

QED

I.6. THEOREM. *If* $f\colon G^2X \to \Pi$ *is a cocycle and* Y *represents* $D(-,f)$, *then there exist*

$$\rho : Y\times\Pi \to Y, \quad p : Y \to X \quad \textit{and} \quad s : GX \to Y$$

such that (p,ρ,s) *is a G-trivial* Π*-principal homogeneous object over* X.

Moreover the cocycle which it defines is exactly f.

PROOF. Let $[-,-]: D(-,f) \to \underline{A}(-,Y)$ be a natural equivalence. There exist $p: Y \to X$, $t: GY \to \Pi$ such that $[p,t]$ is the identity on Y and we get the following computational rules:

i) if $z: Z' \to Z$ then, for any (g,h) in $D(Z,f)$:

$$[g,h] \circ z = [g \circ z,\ h \circ Gz] ;$$

ii) if (g,h) is in $D(Z,f)$ then $g = p \circ [g,h]$ and $h = t \circ G[g,h]$;

iii) if $y: Z \to Y$ then $y = [p \circ y,\ t \circ Gy]$.

Moreover since f is a cocycle, $(\epsilon X, f)$ is in $D(GX, f)$ and, by ii,

$$p \circ [\epsilon X, f] = \epsilon X .$$

Hence for $s = [\epsilon X, f]$, I.2.ii is satisfied. Next notice that

$$(p \circ p_1,\ t \circ Gp_1 + \epsilon\Pi \circ Gp_2)$$

is in $D(Y \times \Pi, f)$, and let

$$\rho = [p \circ p_1,\ t \circ Gp_1 + \epsilon\Pi \circ Gp_2] .$$

By ii, $p \circ \rho = p \circ p_1$, so suppose $z, z': Z \to Y$ are such that $p \circ z = p \circ z'$; we want to show that there is a unique $w: Z \to Y \times \Pi$ such that

$$p_1 \circ w = z' \quad \text{and} \quad \rho \circ w = z$$

(this is I.2.i). Let $g, g': Z \to X$ and $h, h': GZ \to \Pi$ be the unique maps such that

$$[g,h] = z \quad \text{and} \quad [g',h'] = z' .$$

Then $p \circ z = p \circ z'$ means $g = g'$, and we have

$$\begin{aligned}
&(h-h') \circ \epsilon GZ = h \circ \epsilon GZ - h' \circ \epsilon GZ = \\
&= -f \circ G^2 g + h \circ G\epsilon Z - (-f \circ G^2 g' + h' \circ G\epsilon Z) \\
&= h \circ G\epsilon Z - f \circ G^2 g + f \circ G^2 g - h' \circ G\epsilon Z = \\
&= (h-h') \circ G\epsilon Z .
\end{aligned}$$

Thus there exists a unique

$$k: Z \to \Pi \quad \text{such that} \quad k \circ \epsilon Z = h - h'$$

and $([g,h'],k): Z \to Y\times\Pi$. Now by the above

$$\rho\circ([g,h'],k) = [p\circ[g,h'],\ t\circ G[g,h'] + \epsilon\Pi\circ Gk] =$$
$$= [g, h'+k\circ\epsilon Z] = [g,\ h'+h-h'] = [g,h].$$

If also $([x,y],z): Z \to Y\times\Pi$ satisfies

$$p_1\circ([x,y],z) = [g,h'] \quad\text{and}\quad \rho\circ([x,y],z) = [g,h],$$

then

$$[x,y] = [g,h'] \quad\text{and}\quad [x,\ y+\epsilon\Pi\circ Gz] = [g,h]$$

so

$$x = g,\quad y = h',\quad y+\epsilon\Pi\circ Gz = h.$$

Thus $h-h' = z\circ\epsilon Z$, which implies

$$z = k \quad\text{and}\quad ([x,y],z) = ([g,h'],k).$$

Therefore I.2.i is verified. It remains to check that ρ is an action of Π on Y. This follows easily from iii. Hence (p,ρ,s) is a G-trivial Π-principal homogeneous object over X. For the final sentence we use I.4, together with i and ii:

$$\rho\circ(s\circ G\epsilon X, f) = [p\circ p_1,\ t\circ Gp_1 + \epsilon\Pi\circ Gp_2]\circ([\epsilon X,f]\circ G\epsilon X, f) =$$
$$= [p\circ[\epsilon X,f]\circ G\epsilon X,\ t\circ G[\epsilon X,f]\circ G^2\epsilon X + \epsilon\Pi\circ Gf] =$$
$$= [\epsilon X\circ G\epsilon X,\ f\circ G^2\epsilon X + f\circ\epsilon G^2X] =$$
$$= [\epsilon X\circ\epsilon GX,\ f\circ G\epsilon GX] = [\epsilon X,f]\circ\epsilon GX = s\circ\epsilon GX.$$

QED

I.7. THEOREM. *Let $(Y \xrightarrow{p} X, \rho, s)$ be a G-trivial Π-principal homogeneous object over X and f the cocycle that it induces (see I.4). If $D(-,f)$ is represented by Y' then there exists an isomorphism $y: Y \to Y'$ such that*

$$\begin{tikzcd}
Y\times\Pi \arrow[r, shift left, "p_1"] \arrow[r, shift right, "\rho"'] \arrow[d, "y\times\Pi"'] & Y \arrow[r, "p"] \arrow[d, "y"] & X \arrow[d, equal] \\
Y'\times\Pi \arrow[r, shift left, "p_1"] \arrow[r, shift right, "\rho'"'] & Y' \arrow[r, "p'"] & X
\end{tikzcd}$$

commutes.

PROOF. The bottom row of the diagram was derived in I.6. Since Y represents $D(-,f)$ by I.5, there is an isomorphism $y\colon Y \to Y'$ such that

$$D(-,f) \xrightarrow{<-,->} \underline{A}(-,Y) \xrightarrow{\underline{A}(-,y)} \underline{A}(-,Y')$$

is equal to

$$D(-,f) \xrightarrow{[-,-]} \underline{A}(-,Y').$$

In particular, $y \circ <p,t> = [p,t]$; but $<p,t> = Y$ so

$$y = [p,t] \quad \text{and} \quad p' \circ y = p' \circ [p,t] = p$$

by I.6.ii. Now

$$\begin{aligned}\rho' \circ (y \times \Pi) &= [p' \circ p_1,\ t' \circ Gp_1 + \epsilon\Pi \circ Gp_2] \circ ([p,t] \times \Pi) = \\ &= [p' \circ [p,t] \circ p_1,\ t' \circ G[p,t] \circ Gp_1 + \epsilon\Pi \circ Gp_2] = \\ &= [p \circ p_1,\ t \circ Gp_1 + \epsilon\Pi \circ Gp_2]\end{aligned}$$

whereas

$$y \circ \rho = [p,t] \circ \rho = [p \circ \rho,\ t \circ G\rho] = [p \circ p_1,\ t \circ G\rho],$$

so it remains to show that

$$t \circ Gp_1 + \epsilon\Pi \circ Gp_2 = t \circ G\rho .$$

But

$$\begin{aligned}&\rho \circ (s \circ Gp \circ Gp_1,\ t \circ Gp_1 + \epsilon\Pi \circ Gp_2) = \\ &= \rho \circ (\rho \circ (s \circ Gp,\ t) \circ Gp_1,\ \epsilon\Pi \circ Gp_2) = \\ &= \rho \circ (\epsilon Y \circ Gp_1,\ \epsilon\Pi \circ Gp_2) = \rho \circ \epsilon(Y \times \Pi) = \\ &= \epsilon Y \circ G\rho = \rho \circ (s \circ Gp,\ t) \circ G\rho = \\ &= \rho \circ (s \circ Gp \circ G\rho,\ t \circ G\rho) = \rho \circ (s \circ Gp \circ Gp_1,\ t \circ G\rho).\end{aligned}$$

QED

I.8. DEFINITION. A *morphism* of G-trivial Π-principal homogeneous objects over X is a map such that the diagram in I.7 commutes.

I.9. PROPOSITION. *If*

$$(Y \xrightarrow{p} X, \rho, s) \quad \textit{and} \quad (Y' \xrightarrow{p'} X, \rho', s')$$

are G-trivial Π*-principal homogeneous objects over* X*, with corresponding cocycles* f, f'*, and* $y: Y \to Y'$ *is a morphism between them, then there exists* $u: GX \to \Pi$ *such that* $f - f' = u \circ G\epsilon X - u \circ \epsilon GX$.

PROOF. Since

$$p' \circ (y \circ s) = p \circ s = \epsilon X = p' \circ s',$$

there exists a unique $u: GX \to \Pi$ such that $\rho' \circ (y \circ s, u) = s'$. The computation

$$\begin{aligned}
&\rho' \circ (y \circ s \circ G\epsilon X, u \circ G\epsilon X + f') = \rho' \circ (s' \circ G\epsilon X, f') = \\
&= s' \circ \epsilon GX = \rho' \circ (y \circ s \circ \epsilon GX, u \circ \epsilon GX) = \\
&= \rho' \circ (y \circ \rho \circ (s \circ G\epsilon X, f), u \circ \epsilon GX) = \\
&= \rho' \circ (\rho' \circ (y \times \Pi) \circ (s \circ G\epsilon X, f), u \circ \epsilon GX) = \\
&= \rho' \circ (\rho' \circ (y \circ s \circ G\epsilon X, f), u \circ \epsilon GX) = \\
&= \rho' \circ (y \circ s \circ G\epsilon X, u \circ \epsilon GX + f)
\end{aligned}$$

shows that the stated condition holds.

QED

I.10. DEFINITION. If two cocycles are related as in I.9 then they are said to be *cohomologous*.

I.11. PROPOSITION. *If* f *and* f' *are cohomologous, then* $D(-, f)$ *and* $D(-, f')$ *are naturally equivalent functors. If, in addition,* $D(-, f)$ *and* $D(-, f')$ *are representable, then there is a morphism between the associated homogeneous objects over* X.

PROOF. Let

$$f - f' = u \circ G\epsilon X - u \circ \epsilon GX.$$

If (g, h) is in $D(Z, f)$ then $(g, h - u \circ Gg)$ is in $D(Z, f')$, and this defines a natural transformation $D(-, f) \to D(-, f')$. The inverse is given by sending (g, h) to $(g, h + u \circ Gg)$. The second sentence follows from the first and I.7.

QED

It follows from all the above that if $f: G^2 X \to \Pi$ is a cocycle and

$D(-,f)$ is representable, then there is a G-trivial Π-principal homogeneous object over X associated to it. Conversely a G-trivial Π-principal homogeneous object over X gives rise to a cocycle. These two assignments are mutually inverse, provided we identify cohomologous cocycles on the one hand, and homogeneous objects if there is a morphism between them on the other. Since $H^1(X,\Pi)$ is by definition the abelian group of one-cocycles modulo the relation «is cohomologous to», we see that there is an interpretation of $H^1(X,\Pi)$ in terms of equivalence classes of G-trivial Π-principal homogeneous objects over X provided each $D(-,f)$ is representable.

In the next section we will give some necessary and sufficient conditions for a given $D(-,f)$ to be representable. For now, we offer the following problem:

Give necessary and sufficient conditions that a functor $F: \underline{A}^{op} \to \underline{Sets}$ be naturally equivalent to $D(-,f)$ for some cocycle $f: G^2X \to \Pi$.

II. NECESSARY AND SUFFICIENT CONDITIONS FOR $D(-,f)$ TO BE REPRESENTABLE.

Given a cocycle $f: G^2X \to \Pi$, under what conditions is $D(-,f)$ representable? The main purpose of this section is to provide two necessary and sufficient conditions for the representability of $D(-,f)$. The results of Section I serve as motivation for interest in this question. Throughout this section, let $f: G^2X \to \Pi$ be a cocycle and assume $G^nX\times\Pi$ exists for $0 \leqslant n \leqslant 3$.

II.1. PROPOSITION. *If $f = 0$, then $D(-,f)$ is represented by $X\times\Pi$.*

PROOF. Define $\underline{A}(Z, X\times\Pi) \to D(Z,f)$ by sending (z,a) to $(z, a\circ\epsilon Z)$. This obviously gives a natural transformation. For its inverse, if (g,h) is in $D(Z,f)$, then $h\circ G\epsilon Z = h\circ\epsilon GZ$, so there exists a unique

$$a: Z \to \Pi \quad \text{such that} \quad a\circ\epsilon Z = h;$$

thus sending (g,h) to (g,a) provides an inverse. This result also follows from II.6.

QED

II.2. DEFINITION. A three-tuple (G, ϵ, δ) is a *cotriple on* $\underline{A}$ if $G: \underline{A} \to \underline{A}$ is a functor, $\epsilon: G \to \underline{A}$ and $\delta: G \to G^2$ are natural transformations such that

$$\epsilon G \circ \delta = G = G\epsilon \circ \delta \quad \text{and} \quad \delta G \circ \delta = G\delta \circ \delta .$$

II.3. PROPOSITION. *If (G, ϵ) is part of a cotriple (G, ϵ, δ) on $\underline{A}$ and $X = GX_0$ for some X_0 in $\underline{A}$, then $D(-, f)$ is represented by $X \times \Pi$.*

PROOF. If (z, a) is in $\underline{A}(Z, X \times \Pi)$, then a short computation (using II.2 and the fact that f is a cocycle) shows that

$$(z,\ a \circ \epsilon Z + f \circ G\delta X \circ Gz)$$

is in $D(Z, f)$. Thus

$$\psi Z(z, a) = (z,\ a \circ \epsilon Z + f \circ G\delta X \circ Gz)$$

defines a function

$$\psi Z : \underline{A}(Z, X \times \Pi) \to D(Z, f),$$

and $\psi : \underline{A}(-, X \times \Pi) \to D(-, f)$ is obviously a natural transformation. Given (g, h) in $D(Z, f)$ one can see (for the same reasons as before) that

$$(h - f \circ G\delta X \circ Gg) \circ G\epsilon A = (h - f \circ G\delta X \circ Gg) \circ \epsilon GA .$$

Hence there is a unique

$$a: Z \to \Pi \quad \text{such that} \quad a \circ \epsilon Z = h - f \circ G\delta X \circ Gg .$$

It is easy to verify that the inverse of ψZ is given by sending (g, h) to (g, a).

QED

II.4. LEMMA. *Let $Z_\cdot : \Gamma \to \underline{A}$ be a functor which has a colimit C:*

$$\begin{tikzcd} Z_\alpha \arrow[d] \arrow[dr, "i_\alpha"] & \\ Z_\beta \arrow[r, "i_\beta"'] & C \end{tikzcd}$$

Then there is a function $\theta: D(C, f) \to \lim D(Z_\cdot, f)$ which is one-to-one.

PROOF. Define

$$\theta(g, h) = \text{the family } (g \circ i_\alpha,\ h \circ G i_\alpha) \text{ for } \alpha \text{ in } \Gamma ;$$

this clearly defines a function. To see that it is injective, suppose (g,h) and (g',h') are members of $D(C,f)$ such that

$$(g\circ i_\alpha,\ h\circ Gi_\alpha) = (g'\circ i_\alpha,\ h'\circ Gi_\alpha)$$

for each α in Γ. Then since

$$C = colim\, Z_- \quad\text{and}\quad g\circ i_\alpha = g'\circ i_\alpha\ ,$$

it follows that $g = g'$. Now

$$(h-h')\circ G\epsilon C = f\circ G^2 g + h\circ\epsilon GC - f\circ G^2 g' - h'\circ\epsilon GC = (h-h')\circ\epsilon GC$$

so there exists a unique

$$a: C\to\Pi \quad\text{such that}\quad a\circ\epsilon C = h-h'.$$

If $a = 0$ then we will be done. But for each α in Γ,

$$a\circ i_\alpha\circ\epsilon Z_\alpha = a\circ\epsilon C\circ Gi_\alpha = h\circ Gi_\alpha - h'\circ Gi_\alpha = 0$$

so $a\circ i_\alpha = 0$. Since $C = colim\, Z_-$, $a = 0$.

QED

II.5. THEOREM. *Suppose that $\underline{A}$ is cocomplete and $GX\times\Pi$ has only a set of regular quotients (i. e. quotients which are coequalizers). Then, $D(-,f)$ is representable if and only if the function θ of II.4 is onto for all functors Z_-: that is, if and only if $D(-,f)$ preserves limits.*

PROOF. Obviously if $D(-,f)$ is representable then it preserves limits. Conversely, it suffices to verify the solution set condition [5, V.6.3]. Let L be the class of all coequalizers of the form

$$G^2Z \underset{(Gg\circ\epsilon GZ,\ h\circ\epsilon GZ)}{\overset{(Gg\circ G\epsilon Z,\ h\circ G\epsilon Z)}{\rightrightarrows}} GX\times\Pi \xrightarrow{\ q\ } C$$

for all Z in $\underline{A}$, and all (g,h) in $D(Z,f)$. Since L is a subclass of the set of all regular quotients of $GX\times\Pi$, it is a set. We proceed to show that L is a solution set for $D(-,f)$. Since $D(-,f)$ preserves limits, if (g,h) is in $D(Z,f)$ then

$$D(C,f)\longrightarrow D(GX\times\Pi,f)\rightrightarrows D(G^2Z,f)$$

is an equalizer. Now

$$(\epsilon X \circ p_1,\ f \circ G p_1 + \epsilon \Pi \circ G p_2)$$

is in $D(GX \times \Pi, f)$ since f is a cocycle, and its two images in $D(G^2 Z, f)$ are

$$(\epsilon X \circ G g \circ G \epsilon Z,\ f \circ G^2 g \circ G^2 \epsilon Z + \epsilon \Pi \circ G h \circ G^2 \epsilon Z)$$

and

$$(\epsilon X \circ G g \circ \epsilon G Z,\ f \circ G^2 g \circ G \epsilon G Z + \epsilon \Pi \circ G h \circ G \epsilon G Z).$$

But these images are equal by the naturality of ϵ and the fact that (g, h) is in $D(Z, f)$. Hence there exists a unique (g', h') in $D(C, f)$ such that:

$$g' \circ q = \epsilon X \circ p_1 \quad \text{and} \quad h' \circ G q = f \circ G p_1 + \epsilon \Pi \circ G p_2 .$$

Noticing that

$$q \circ (G g, h) \circ \epsilon G Z = q \circ (G g, h) \circ G \epsilon Z,$$

we find a unique

$$k : Z \to C \quad \text{such that} \quad k \circ \epsilon Z = q \circ (G g, h).$$

If

$$g' \circ k = g \quad \text{and} \quad h' \circ G k = h,$$

then the solution set condition will be verified. But we have

$$\begin{aligned} D(\epsilon Z, f)(g' \circ k,\ h' \circ G k) &= (g' \circ k \circ \epsilon Z,\ h' \circ G k \circ G \epsilon Z) = \\ &= (g' \circ q \circ (G g, h),\ h' \circ G q \circ G(G g, h)) = \\ &= (\epsilon X \circ p_1 \circ (G g, h),\ (f \circ G p_1 + \epsilon \Pi \circ G p_2) \circ G(G g, h)) = \\ &= (\epsilon X \circ G g,\ f \circ G^2 g + \epsilon \Pi \circ G h) = (g \circ \epsilon Z,\ f \circ G^2 g + h \circ \epsilon G Z) = \\ &= (g \circ \epsilon Z,\ h \circ G \epsilon Z) = D(\epsilon Z, f)(g, h), \end{aligned}$$

and $D(\epsilon Z, f)$ is one-to-one since

$$D(Z, f) \longrightarrow D(GZ, f) \rightrightarrows D(G^2 Z, f)$$

is an equalizer.

QED

II.6. THEOREM. *In order that $D(-, f)$ be representable it is necessary and sufficient that the following «descent-type» condition (see [3,4]) be fulfilled: For the diagram*

$$\begin{tikzcd}
G^2X\times\Pi \arrow[r, shift left, "(\epsilon GX\circ p_1,\, p_2)"] \arrow[r, shift right, "(G\epsilon X\circ p_1,\, f\circ p_1 + p_2)"'] \arrow[d, "p_1"'] & GX\times\Pi \arrow[d, "p_1"] & \\
G^2X \arrow[r, shift left, "\epsilon GX"] \arrow[r, shift right, "G\epsilon X"'] & GX \arrow[r, "\epsilon X"] & X
\end{tikzcd}$$

there should exist $GX\times\Pi \xrightarrow{q} Y \xrightarrow{p} X$ *such that*

i)

$$\begin{tikzcd}
GX\times\Pi \arrow[r, "q"'] \arrow[d, "p_1"'] & Y \arrow[d, "p"] \\
GX \arrow[r, "\epsilon X"] & X
\end{tikzcd}$$

is a pullback and

ii) $q\circ(\epsilon GX\circ p_1, p_2) = q\circ(G\epsilon X\circ p_1,\, f\circ p_1 + p_2)$.

PROOF. Suppose that Y represents $D(-,f)$. Then by I.6 there exists the structure of G-trivial Π-principal homogeneous object on Y, say

$$(\, Y \xrightarrow{p} X,\ \rho: Y\times\Pi \to Y,\ s: GX\to Y\,).$$

Let $q = \rho\circ(s\times\Pi): GX\times\Pi \to Y$. Condition ii follows from I.4 and the last sentence of I.6. For condition i, recall that in any category

$$\begin{tikzcd}
A\times C \arrow[r, "u\times C"] \arrow[d, "p_1"'] & B\times C \arrow[d, "p_1"] \\
A \arrow[r, "u"] & B
\end{tikzcd}$$

is a pullback. Applying this with $u = s$ and $C = \Pi$ and using I.2.i we see that each square in

$$\begin{tikzcd}
GX\times\Pi \arrow[r, "s\times\Pi"] \arrow[d, "p_1"'] & Y\times\Pi \arrow[r, "\rho"] \arrow[d, "p_1"'] & Y \arrow[d, "p"] \\
GX \arrow[r, "s"] & Y \arrow[r, "p"] & X
\end{tikzcd}$$

is a pullback. Now the juxtaposition of two pullbacks is a pullback, $p\circ s = = \epsilon X$ by I.2.ii, and $\rho\circ(s\times\Pi) = q$. Hence condition i has been verified.

Conversely assume conditions i and ii. Define $D(Z,f) \to \underline{A}(Z,Y)$ by sending (g,h) to $b: Z \to Y$, where b is the unique map such that

$$b \circ \epsilon Z = q \circ (Gg, h).$$

Such a morphism exists since

$$\begin{aligned} q\circ(Gg,h)\circ\epsilon GZ &= q\circ(\epsilon GX\circ p_1, p_2)\circ(G^2g, h\circ\epsilon GZ) = \\ &= q\circ(G\epsilon X\circ p_1, f\circ p_1 + p_2)\circ(G^2g, h\circ\epsilon GZ) = \\ &= q\circ(G\epsilon X\circ G^2g, f\circ G^2g + h\circ\epsilon GZ) = \\ &= q\circ(Gg\circ G\epsilon Z, h\circ G\epsilon Z) = q\circ(Gg,h)\circ G\epsilon Z. \end{aligned}$$

Then $D(-,f) \to \underline{A}(-,Y)$ so defined is clearly a natural transformation. Given $a: Z \to Y$, consider

$$\begin{tikzcd}
GZ \arrow[rr, "\epsilon Z"] \arrow[ddr, "G(p\circ a)"'] & & Z \arrow[dr, "a"] & \\
 & GX\times\Pi \arrow[rr, "q"] \arrow[d, "p_1"] & & Y \arrow[d, "p"] \\
 & GX \arrow[rr, "\epsilon X"] & & X
\end{tikzcd}$$

Since the outside diagram commutes and the inside is a pullback, there exists a unique

$$k: GZ \to \Pi \quad \text{such that} \quad q\circ(G(p\circ a), k) = a\circ\epsilon Z.$$

I claim that $(p\circ a, k)$ is in $D(Z,f)$, and this will be true provided

$$(G(p\circ a), k)\circ G\epsilon Z = (G\epsilon X\circ p_1, f\circ p_1 + p_2)\circ(G^2(p\circ a), k\circ\epsilon GZ).$$

These will be equal if their compositions with p_1, as well as q, are equal. The first components are obviously equal, and

$$\begin{aligned} q\circ(G\epsilon X\circ p_1, f\circ p_1 + p_2)\circ(G^2(p\circ a), k\circ\epsilon GZ) &= \\ = q\circ(\epsilon GX\circ p_1, p_2)\circ(G^2(p\circ a), k\circ\epsilon GZ) &= \\ = q\circ(G(p\circ a)\circ\epsilon GZ, k\circ\epsilon GZ) = q\circ(G(p\circ a), k)\circ\epsilon GZ &= \\ = a\circ\epsilon Z\circ\epsilon GZ = a\circ\epsilon Z\circ G\epsilon Z = q\circ(G(p\circ a), k)\circ G\epsilon Z. \end{aligned}$$

Hence we can map $\underline{A}(Z, Y) \to D(Z, f)$ by taking a to $(p \circ a, k)$, where k is uniquely determined by the condition

$$q \circ (G(p \circ a), k) = a \circ \epsilon Z .$$

We need to show, using the above notation, that

$$(g, h) = ((p \circ b), k) \quad \text{and} \quad a = b .$$

For the first,

$$p \circ b \circ \epsilon Z = p \circ q \circ (Gg, h) = \epsilon X \circ p_1 \circ (Gg, h) = g \circ \epsilon Z$$

so that $p \circ b = g$, and thus $k = h$ since

$$q \circ (G(p \circ b), k) = b \circ \epsilon Z = q \circ (Gg, h) = q \circ (G(p \circ b), h).$$

For the second,

$$b \circ \epsilon Z = q \circ (G(p \circ a), h) = a \circ \epsilon Z \quad \text{so} \quad b = a .$$

QED

II.7. COROLLARY *(Beck [1]). If $U: \underline{A} \to \underline{B}$ is tripleable (also called monadic in [5]) with left adjoint F, $G = FU$, and $UG^n X \times U\Pi$ exists for $0 \leqslant n \leqslant 2$, then $D(-, f)$ is represented by the coequalizer (which exists)*

$$\begin{tikzcd}[column sep=huge] G^2 X \times \Pi \arrow[r, shift left, "(\epsilon GX \circ p_1,\, p_2)"] \arrow[r, shift right, "(G\epsilon X \circ p_1,\, f \circ p_1 + p_2)"'] & GX \times \Pi \arrow[r, "q"] & Y . \end{tikzcd}$$

PROOF. We have the following U-split coequalizer diagram [5] :

$$\begin{tikzcd}[column sep=8em, row sep=6em] UG^2 X \times U\Pi \arrow[r, shift left, "(U\epsilon GX \circ Up_1,\, Up_2)"] \arrow[r, shift right, "(UG\epsilon X \circ Up_1,\, Uf \circ Up_1 + Up_2)"'] & UGX \times U\Pi \arrow[l, bend left=40, "(\eta UGX \circ Up_1,\, Up_2)"] \arrow[d, shift right, "(U\epsilon X \circ Up_1,\, Uf \circ \eta UGX \circ Up_1 + Up_2)"'] \\ & UX \times U\Pi \arrow[u, shift right, "(\eta UX \circ Up_1,\, Up_2)"'] \end{tikzcd}$$

where $\eta: \underline{B} \to UF$ is the unit for the adjunction. The only problem involved in the verification is that $f \circ F\eta UX = 0$, but

$$f \circ F\eta UX = f \circ G\epsilon GX \circ GF\eta UX \circ F\eta UX =$$
$$= f \circ G^2 \epsilon X \circ GF\eta UX \circ F\eta UX + f \circ \epsilon G^2 X \circ GF\eta UX \circ F\eta UX =$$
$$= f \circ F\eta UX + f \circ F\eta UX \circ \epsilon GX \circ F\eta UX = f \circ F\eta UX + f \circ F\eta UX.$$

Since U is tripleable, there exists $GX \times \Pi \xrightarrow{q} Y$ as asserted, and such that

$$Uq = (U\epsilon X \circ Up_1,\ Uf \circ \eta UGX \circ Up_1 + Up_2).$$

Since

$$\epsilon X \circ p_1 \circ (\epsilon GX \circ p_1,\ p_2) = \epsilon X \circ \epsilon GX \circ p_1 =$$
$$= \epsilon X \circ G\epsilon X \circ p_1 = \epsilon X \circ p_1 \circ (G\epsilon X \circ p_1,\ f \circ p_1 + p_2)$$

and q is a coequalizer, there exists a unique

$$p: Y \to X \quad \text{such that} \quad p \circ q = \epsilon X \circ p_1 .$$

By II.6 we need only see that $p \circ q = \epsilon X \circ p_1$ is a pullback diagram. But since U creates limits [5], it suffices to prove that

$$U\epsilon X \circ Up_1 = Up \circ Uq = Up_1 \circ (U\epsilon X \circ Up_1,\ Uf \circ \eta UGX \circ Up_1 + Up_2)$$

is a pullback. This was first noticed by Duskin and is proved in [2].

QED

We will end with two examples in which II.7 is not directly applicable but II.6 is. Let $\underline{A}$ be the category of torsion-free abelian groups and all homomorphisms. Let (G, ϵ, δ) be the free abelian group cotriple on $\underline{A}$. Then ϵ is the coequalizer of ϵG and $G\epsilon$ in $\underline{A}$. If $f: G^2 X \to \Pi$ is a cocycle in $\underline{A}$ then we can verify II.6 by using II.7 indirectly. Consider the diagram of II.6 in the category of abelian groups. By II.7, $D(-, f)$ is represented by an abelian group Y; if Y is in $\underline{A}$ then we will be done. But, by I.6,

$$0 \to \Pi \to Y \to X \to 0$$

is an exact sequence of abelian groups and hence Y is in $\underline{A}$.

An example in which the technique of the last paragraph is not available is that of «simplicially generated» spaces. Let G be the functor which assigns to a topological space the geometric realization of its singular simplicial set. Then there exist ϵ, δ making G a cotriple. Let $\underline{A}$ be the category of spaces X such that ϵX is the coequalizer of ϵGX and $G\epsilon X$, and all continuous maps. Then (G, ϵ, δ) is a cotriple on A and it is not (known to be) the cotriple of any tripleable adjoint pair. If $f: G^2 X \to \Pi$ is a cocycle and Π is discrete, then a space in $\underline{A}$ representing $D(-, f)$ would be a kind of simplicially generated simplicial covering space of X.

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