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SOME INVARIANT THEORY AMONG QUASI-PROJECTIVES

by Paul CHERENACK

0. Introduction.

Coequalizers and, in particular, group quotients exist in the category of affine schemes of a countable type over a field k. See the author's paper [2].

The calculation of group quotients in the category of quasi-projective schemes (almost always of finite type over a field k) is an important facet of geometric invariant theory [8].

The starting point for this paper consists of the above two remarks. The appropriate notions are now presented. We assume that k is an algebraically closed field.

DEFINITION.1 *A quasi-projective scheme of countable type over k* is an irreducible, reduced open subscheme of some $\mathbf{P}roj\,A$, where A is a countably generated graded k-algebra such that the elements of degree 0 in A are precisely the elements of k. **QP** will denote the category whose objects are quasi-projective schemes of a countable type over k and whose maps are maps of schemes.

The basic sources for our definitions and notation are Grothendieck [3,4], and Artin and Mazur [1]. For convenience, we refer to the objects of **QP** as quasi-projectives and to the maps of **QP** as quasi-projective maps.

Let $f, g: X \rightrightarrows Y$ be two fixed maps of k-schemes, where X is irreducible and reduced and Y is quasi-projective. In the process of determining the nature of the coequalizer to f and g, we will first consider the local situation. Suppose that Y is an open subscheme of $\mathbf{P}roj\,A$. Y can be covered by affine open sets of the form $Y_b = Spec\,A_{(b)}$, where b is a homogeneous element of A^+. Here A^+ denotes the set of objects in A which have no constant term. See [4] , 2.3.4.

Let Y_b be an affine open subset of Y and restrict the maps f and g so that the maps

$$f_b, g_b : f^{-1}(Y_b) \cap g^{-1}(Y_b) \rightrightarrows Y_b$$

are obtained. There correspond to f_b and g_b two maps of k-algebras

$$f_b^*, g_b^* : A_{(b)} \rightrightarrows \mathcal{O}_X(f^{-1}(Y_b) \cap g^{-1}(Y_b)).$$

Suppose that $f^{-1}(Y_b) \cap g^{-1}(Y_b) \neq \emptyset$. We define

$$E_{(b)} = \{ x \in A_{(b)} \mid f_b^*(x) = g_b^*(x) \}.$$

Then, if $f^{-1}(Y_b) \cap g^{-1}(Y_b)$ is affine, and thus not empty, we know from [2] that $Spec E_{(b)}$ is the coequalizer of f_b and g_b in the category of affine schemes of a countable type over k.

If $Spec E_{(b)}$ is a k-valued point, i.e. $E_{(b)}$ is k, it may be the case that f and g are too rigid from an algebraic standpoint to allow a meaningful coequalizer from a topological point of view. Consider example 1, below. We wish to associate to a homogeneous element b in A^+ and the corresponding k-algebra $E_{(b)}$ (the local case) a graded k-subalgebra (E^b) of A (the global case) such that the coequalizer C of f and g is, as an object, an open subscheme of $\mathbf{P}roj(E^b)$. In this process, we will always assume that $E_{(b)} \neq k$.

Lemma 1 of n° 1 shows that we can choose a countable covering $\{ Y_{b_i} \}_{i=1,2,\dots}$ of Y such that

$$f^{-1}(Y_{b_i}) \cap g^{-1}(Y_{b_i}) \neq \emptyset \quad \text{for} \quad i = 1, 2, \dots .$$

Also the empty set $\emptyset$ is not an affine scheme (our rings are commutative with unity). From a topological point of view, it makes sense to write

$$E_{(b)} = A_{(b)} \quad \text{if, for some homogeneous element } b \in A^+, \quad f^{-1}(Y_b) \cap g^{-1}(Y_b) = \emptyset .$$

However, because we want and are able to eliminate the situation

$$f^{-1}(Y_b) \cap g^{-1}(Y_b) = \emptyset ,$$

if $f^{-1}(Y_b) \cap g^{-1}(Y_b) = \emptyset$ for a homogeneous element $b \in A^+$, we write

$$E_{(b)} = 0 .$$

DEFINITION 2. Let b be a homogeneous element of A^+, and suppose that $E_{(b)} \neq k$ or 0. The set

$$E^b = \{ a \in A \mid a \text{ is homogeneous, } a/b^n \in E_{(b)},\ n > 0 \}$$

is called *the collection of hypersurfaces invariant in* Y_b. (E^b) will denote the graded k-subalgebra of A generated by E^b. A hypersurface a is called *locally invariant* if, for every point P lying on a and Y, there is an affine open neighborhood Y_b of P such that a is invariant in Y_b. Note that: $Y_b \subset Y$. A point $P \in Y$ is *locally stable* if it lies in the complement of a locally invariant hypersurface. A point $P \in Y$ is *stable in* Y_b if $P \in Y_b$ and there is a hypersurface a invariant in Y_b such that

$$P \in Y_a \quad \text{and} \quad Y_a \neq Y_b .$$

If a point $P \in Y$ is stable in some Y_b, we call it *weakly locally stable.*

It is almost immediate that the localization of (E^b) at b is $E_{(b)}$. Therefore, at least locally on Y_b, an open piece of $\mathbf{P}roj(E^b)$ defines the coequalizer of f and g. The global solution to the problem of finding a coequalizer of f and g entails these considerations:

a) Assume that $\{ b_m \}_{m \in M}$ is a collection of locally invariant hypersurfaces of Y. Every closed invariant subset of Y is the intersection of hypersurfaces in $\{ b_m \}_{m \in M}$. $\{ b_m \}_{m \in M}$ is closed under products and nonzero linear combinations of elements of the same degree.

b) (β) $Y = \bigcup_{m \in M} Y_{b_m}$

c) The local coequalizers must determine the same global coequalizers, i.e.

$$(\gamma) \qquad \mathbf{P}roj(E^{b_i}) = \mathbf{P}roj(E^{b_j})$$

for b_i, b_j appearing in a. From Grothendieck [4], we see that γ is implied (important implication) by:

Stability condition (relative to a collection $\{h_m\}_{m \in M}$ *of locally invariant hypersurfaces):* There are integers s and t larger than zero such that

$$(E^{h_i})^{(s)} = (E^{h_j})^{(t)}$$

as subsets of A.

Here, $(E^h)^{(s)}$ denotes, as in Grothendieck [4], the graded k-subalgebra of (E^h) whose elements of degree ν are precisely the elements of degree $s\nu$ in (E^h). If $h_i = h_j$, clearly $s = t$. Also, $s, t \neq 0$.

Theorem 1, below, shows that the conditions a and b and the stability condition (relative to a) lead to a coequalizer in **QP**. There is no reason to believe that conditions a, b and c are always true. Lacking these conditions, we introduce the notion of a homogeneous quasi-projective and of a relative coequalizer. In the process, we formalize the notion of a coequalizer in **QP**.

DEFINITION 3. $\underline{\mathbf{QP}}$ will denote the subcategory of **QP** whose objects X are open subschemes of some $\mathbf{P}roj\, A$, where A is a countably generated homogeneous ring, and whose maps $f: X \to Y$ have a homogeneous representation $f = (a_0, a_1, \dots)$ where $a_0, a_1, \dots$ are homogeneous elements of A of the same degree.

DEFINITION 4. Let $\underline{\mathbf{QP}}$ be a subcategory of a category $\underline{D}$, Y be an element of $\underline{\mathbf{QP}}$, and $f, g: X \rightrightarrows Y$ be the maps described at the outset. *A coequalizer of f and g in $\underline{D}$ relative to* $\underline{\mathbf{QP}}$ is a map $c: Y \to C$ in $\underline{D}$ such that $c \circ f = c \circ g$ and such that, for any map

$$e: Y \to W \text{ in } \underline{\mathbf{QP}} \text{ with } e \circ f = e \circ g,$$

there is a unique map

$$e': C \to W \text{ in } \underline{D} \text{ with } e' \circ c = e.$$

If one replaces $\underline{D}$ and $\underline{\mathbf{QP}}$ in this definition by **QP**, then c is called a *coequalizer in* **QP**, or simply a coequalizer.

One cannot infer from the definition that a coequalizer in $\underline{D}$ relative to $\underline{\mathbf{QP}}$ is uniquely determined. The objects of $\underline{\mathbf{QP}}$ are called *homogeneous quasi-projectives.*

We can now begin to state our results.

THEOREM 1. *If Y contains a point which is not weakly locally stable, the coequalizer of f and g (as an object) is Spec k. If conditions* a *and* b *and the stability condition (relative to* a*) hold, then the coequalizer of f and g in* **QP** *exists.*

Theorem 1 is proved in n° 1-3 (with help from n° 7). In n° 1 we show how to avoid coverings $\{ Y_{a_i} \}_{i \in J}$ such that $E_{(a_i)} = 0$ and, moreover, such that J is uncountable. We will construct the coequalizing map $c: Y \to C$ in n° 2. Before doing this, however, we eliminate the possibility that Y has a point which is not weakly locally stable. Finally, in n° 3, we prove that the map c constructed in n° 2 is indeed a coequalizer.

Let $f, g: X \rightrightarrows Y$ be arbitrary maps of k-schemes where X is irreducible and reduced and Y is quasi-projective. Consider the cases when Y and C are, respectively, open subschemes of projective k-varieties $\mathbf{P}roj\ A$ and $\mathbf{P}roj\ E$, i.e. Y and C are quasi-projectives of finite type over k. If $c: Y \to C$ is the coequalizer of f and g in **QP**, then c is dominant and, hence, rational. The rationality of c implies that there is a representation of c, in which the coordinates of c are homogeneous elements of A of the same degree. A graded k-algebra embedding $E \to A$ results from the composition of homogeneous elements of E with c. This argument shows that it is not, in general, unreasonable to expect that the coequalizer of f and g arises as an open subscheme of $\mathbf{P}roj\ E$, where E is a graded k-subalgebra of A.

The following example shows that, when Y contains a point which is not locally stable, it is likely that the coequalizer of f and g is Spec k.

EXAMPLE 1. Let $f, g: \mathbf{P}^1 \rightrightarrows \mathbf{P}^2$ be two k-scheme maps where f is constant with value P and g is a closed immersion with $P \in g(\mathbf{P}^1)$. The locally invariant hypersurfaces of $\mathbf{P}^2$ all contain P and, hence, P is not a locally stable point of $\mathbf{P}^2$. Note, however, that P is weakly locally stable. If $h: \mathbf{P}^2 \to V$ is a map of quasi-projectives of finite type over k and if $h \circ f = h \circ g$, writing

$$h(X_0, X_1, X_2) = (h_0(X_0, X_1, X_2), \ldots, h_n(X_0, X_1, X_2)),$$

a simple calculation shows that in $\mathbf{QP}$ the coequalizer of f and g is $Spec\, k$.

EXAMPLE 2. Not every locally invariant hypersurface is globally invariant. Consider $X = l_1 \amalg l_2$, the disjoint union of two lines. Let f be the map from X to k^2 which sends l_1 isomorphically onto $X_1 = 1$ and sends l_2 isomorphically onto $X_1 = 2$; and let g be the map which sends both l_1 and l_2 isomorphically onto $X_1 = 0$. Then $(X_1 - 1) X_1$ is locally invariant but not globally invariant.

EXAMPLE 3. Let $f, g: \mathbf{P}^1 \rightrightarrows \mathbf{P}^2$ be two closed immersions. Suppose that f and g have equations

$$f(X_0, X_1) = (X_0, X_1, 0) \quad \text{and} \quad g(X_0, X_1) = (X_0, 0, X_1).$$

Then

$$(X_1)^2 + (X_2)^2 - (X_0)^2, \quad (X_1)^2 + (X_2)^2 - 4(X_0)^2 \quad \text{and} \quad X_1 - X_2$$

are locally invariant hypersurfaces of $\mathbf{P}^2$. In fact, these hypersurfaces are globally invariant, and from this it follows that the stability condition holds when restricted to these three hypersurfaces. This result enables us to construct a coequalizer to f and g in $\mathbf{QP}$. If k is the complex number field, the real points of the coequalizer to f and g have the appearance of a double cone.

Often the assumptions of Theorem 1 can be simplified if Y is a quasi-projective of finite type over k. Also, one might replace the condition that $\{h_m\}_{m \in M}$ is closed under products and linear combinations of elements of the same degree by

$$f^{-1}(Y_{h_m}) \cap g^{-1}(Y_{h_m}) \neq \emptyset, \quad m \in M.$$

1. Affine coverings with non-empty pullbacks.

$f, g: X \rightrightarrows Y$ are maps of schemes with Y in **QP** in n° 1. Suppose that $P \in X$, $\{b_i\}_{i \in I}$ is a set of homogeneous elements of A^+, where Y is an open subset of $\mathbf{P}roj\, A$, and $Y = \bigcup_{i \in I} Y_{b_i}$.

LEMMA 1. *Y is covered by a countable number of affine open subsets*

$$Y_{a_i} = Spec\, A_{(a_i)},$$

where a_i is a homogeneous element of A^+, $g(P), f(P) \notin V((a_i))$ and a_i is a linear combination of powers of the b_i. Here $V((x))$ denotes the zeroes of x.

PROOF. Y is covered by a countable number of Y_{b_i} where b_i is a homogeneous element of A^+, and $i \in I$. See n° 7. We can assume therefore that I is the collection of natural numbers. Unless Y is a point (when the theorem holds), there is some b_1 such that $g(P) \notin V((b_1))$ and some b_2 such that $f(P) \notin V((b_2))$. By taking an appropriate linear combination

$$a_1 = c_{11} b_1^{deg(b_2)} + c_{21} b_2^{deg(b_1)},$$

$f(P), g(P) \notin V((a_1))$. In a similar way, one constructs

$$a_2 = c_{12} b_1^{deg(b_2)} + c_{22} b_2^{deg(b_1)}$$

and, for $i > 2$,

$$a_i = c_{1i}\, b_{i-1}^{deg(b_i)} + c_{2i}\, b_i^{deg(b_{i-1})}$$

so that

$$f(P), g(P) \notin V((a_i)) \quad \text{and} \quad c_{2i} \neq 0 \quad \text{for each } i \in I.$$

Furthermore, it is clear that (c_{12}, c_{22}) and (c_{11}, c_{21}) can be chosen linearly independent. Then

$$\bigcap_{i \in I} V((a_i)) = \bigcap_{i \in I} V((b_i)) = \emptyset$$

and, hence, $Y = \bigcup_{i \in I} Y_{a_i}$.

2. Construction of the coequalizer.

The next lemma is an immediate consequence of our definitions.

LEMMA 2. *Let b be a homogeneous element of A^+. If $E_{(b)} \neq k$ or 0, then*

$$(E^b)_{(b)} = E_{(b)}$$

LEMMA 3. *If there is a point $P \in Y$ which is not weakly locally stable, the coequalizer of f and g in* **QP** *is $Spec\,k$.*

PROOF. Let $e: Y \to Z$ be a map in **QP** satisfying $e \circ f = e \circ g$. Suppose that $Z \neq Spec\,k$. We can assume that e is dominant and, hence, via Mumford [7] (Proposition 1, Chapter 1), that there is an induced map $e^*: k(Z) \to k(Y)$ sending the function field of Z into the function field of Y. Moreover, as k is algebraically closed,

$$dim\,Z = tr.\,deg_k(k(Z)) > 0.$$

Let Z be an open subscheme of **P**$roj\,D$ where D is a countably generated k-algebra. As $dim\,Z > 0$, it is not difficult finding two homogeneous elements $a, b \in D^+$ having the same degree and such that

α) $a(e(P)) \neq 0$.

β) $b(e(f(Q))), b(e(P)), b(e(g(Q))) \neq 0$ for some $Q \in X$.

γ) $t = a/b$ is not a constant.

$$e^*(t)(P) = t(e(P)) \neq 0$$

and $e^*(t)$ is not a constant. We write $e^*(t) = \frac{F}{H}$ where F and H are homogeneous elements of A^+. Because of our choice of b,

$$f^{-1}(Y_H) \cap g^{-1}(Y_H) \neq \emptyset.$$

It follows that $\frac{F}{H} \in E_{(H)}$ and, thus, that F is an invariant hypersurface in Y_H. As $F(P) \neq 0$, we have obtained a contradiction.

LEMMA 4. *Let α and α' be homogeneous elements in A^+. If*

$$E_{(\alpha)}, E_{(\alpha'\alpha)} \neq k \text{ or } 0,$$

then

$$\begin{array}{ccc} E_{(\alpha)} & \xrightarrow{\;e^*_\alpha\;} & A_{(\alpha)} \\ \gamma\downarrow & & \downarrow\delta \\ E_{(\alpha'\alpha)} & \xrightarrow{\;e^*_{\alpha'\alpha}\;} & A_{(\alpha'\alpha)} \end{array}$$

is a pullback diagram. Here, the map e^*_α *(resp.* $e^*_{\alpha'\alpha}$*) is the equalizer of the k-algebra maps* f^*_α *and* g^*_α *(resp.* $f^*_{\alpha'\alpha}$ *and* $g^*_{\alpha'\alpha}$*) defined at the outset of n° 0;* δ *is the localization at* α' *; and* γ *is the map induced because of the functoriality of equalizers.*

PROOF. Clearly,

$$E_{(\alpha)} \subset A_{(\alpha)} \cap E_{(\alpha'\alpha)} = A_{(\alpha)} \underset{A'}{\times} E_{(\alpha'\alpha)}$$

where $A' = A_{(\alpha'\alpha)}$. Suppose that $x \in A_{(\alpha)} \underset{A'}{\times} E_{(\alpha'\alpha)}$. Let

$$X_\xi = f^{-1}(Y_\xi) \cap g^{-1}(Y_\xi)$$

and θ be the restriction map from X_α to $X_{\alpha'\alpha}$. In that event,

$$\theta f^*_\alpha(x) = f^*_{\alpha'\alpha}(x) = g^*_{\alpha'\alpha}(x) = \theta g^*_\alpha(x).$$

As θ is a localization in an irreducible and reduced scheme, it is an injection. Therefore,

$$f^*_\alpha(x) = g^*_\alpha(x) \quad \text{and} \quad x \in E_{(\alpha)}. \qquad \text{Q. E. D.}$$

The proof of the first part of Theorem 1 is Lemma 3. Suppose $\underline{I} = \{h_m\}_{m \in \underline{M}}$ be a set of locally invariant hypersurfaces closed under products and non-zero linear combinations of elements of the same degree satisfying

$$(\nu) \qquad Y = \bigcup_{m \in \underline{M}} Y_{h_m}$$

and such that the stability condition holds with respect to the h_m, $m \in \underline{M}$. Applying Lemma 1, we can assume that for $\underline{M} = \{1, 2, \dots\}$, the set of natural numbers, (ν) and the stability condition with respect to the h_m, $m \in \underline{M}$, hold. Furthermore, we can assume that for h_i, $i \in \underline{M}$,

$$f^{-1}(Y_{b_i}) \cap g^{-1}(Y_{b_i}) \neq \emptyset .$$

Moreover, using Lemma 4, a non-constant element b^n/b_i where n is a strictly positive integer belongs to $E_{(b_i)}$. Therefore, since $E_{(b_i)} \neq k$, (E^{b_i}) makes sense (Definition 2).

Let $b_i, b_j \in \underline{I}$ and $b_i b_j$ denote their product in A.

$$f^{-1}(Y_{b_i b_j}) \cap g^{-1}(Y_{b_i b_j})$$

is equal to

$$(f^{-1}(Y_{b_i}) \cap g^{-1}(Y_{b_i})) \cap (f^{-1}(Y_{b_j}) \cap g^{-1}(Y_{b_j})).$$

As X is irreducible, the last set is not empty. Applying Lemma 4, as in the last paragraph, we discover that $(E^{b_i b_j})$ makes sense.

As the stability condition holds, for some integers s and t bigger than zero,

$$(E^{b_i})^{(s)} = (E^{b_i b_j})^{(t)}$$

From Lemma 2 and Grothendieck, [4] 2.4.7, it follows with

$$Spec\, E_{(b_i)} = Spec(E^{b_i})_{(b_i)} = (\mathbf{P}roj(E^{b_i}))_{b_i}$$

that

$$\mathbf{P}roj(E^{b_i}) = \mathbf{P}roj(E^{b_i b_j})$$

and

$$Spec\, E_{(b_i b_j)} = Spec(E^{b_i b_j})_{(b_i b_j)} = (\mathbf{P}roj(E^{b_i}))_{b_i b_j} .$$

Because of the functoriality of equalizers, we arrive at the following diagram :

$$\begin{array}{ccccc}
Y_{b_i b_j} & \longrightarrow & Spec\, E_{(b_i b_j)} & = & (\mathbf{P}roj(E^{b_i}))_{b_i b_j} \\
\theta \downarrow & & \xi \downarrow & & \downarrow \xi \\
Y_{b_i} & \longrightarrow & Spec\, E_{(b_i)} & = & (\mathbf{P}roj(E^{b_i}))_{b_i}
\end{array}$$

where the maps θ and ξ are open immersions.

This diagram together with the union (ν) above implies:

PROPOSITION 1. *There is a map* $c: Y \to \mathbf{P}roj(E^{b_i})$ *induced by the maps* $Y_{b_i} \to Spec\, E_{(b_i)}$ *which maps* Y *into an open subset*

$$C = \bigcup_{m=1}^{\infty} (\mathbf{P}roj(E^{b_i}))_{b_m}$$

of $\mathbf{P}roj(E^{b_i})$. *The map* $c: Y \to C$ *is dominant.*

We will show in n° 3 that $c: Y \to C$ is the coequalizer of f and g in **QP**. Clearly, $\underline{I}$ can be taken to be a subset of $I = \{ b_m \}_{m \in M}$ defined in condition a of n° 0. We assume this and that our covering of Y is effected through the countable in number elements of $\underline{I}$.

3. Proof of Theorem 1.

Let $z: Y \to Z$ be a map of quasi-projective schemes such that $z \circ f = z \circ g$, and Z is an open subscheme of $\mathbf{P}roj\, B$, where B is a countably generated k-algebra. We wish to show that there is a unique map $z': C \to Z$ in **QP** such that $z' \circ c = z$. Clearly, as k is algebraically closed, we can assume that $dim\, Z > 0$. Let Y_{b_i} be an affine open subset of Y with $i \in \underline{M}$. We will first exhibit a map $Spec\, E_{(b_i)} \to Z$.

Homogeneous elements $b_1, b_2, \dots, b_n, \dots$ in B^+ which satisfy the following two conditions are chosen using Lemma 1:

I) $Z = \bigcup_{n=1}^{\infty} Z_{b_n}$;

II) For some Q in X, $z \circ f(Q) = z \circ g(Q)$ is in Z_{b_n} for every integer $n > 0$.

Then, for $n = 1, 2, 3, \dots$, the maps f and g restrict to maps

$$f_n, g_n : (z \circ f)^{-1}(Z_{b_n}) \rightrightarrows z^{-1}(Z_{b_n}).$$

Because we have assumed that condition a (0) is valid and $Y - z^{-1}(Z_{b_n})$

is an invariant closed subset of Y, $z^{-1}(Z_{b_n})$ is the union of affine opens $Y_{a_{np}}$, $p \in P(n)$, where a_{np} is a locally invariant hypersurface in I. Applying Lemma 1, one can determine an open affine covering (using the same notation) $\{Y_{a_{np}}\}_{p \in P(n)}$ of $z^{-1}(Z_{b_n})$ such that, for every integer $n > 0$ and $p \in P(n)$,

$$f^{-1}(Y_{a_{np}}) \cap g^{-1}(Y_{a_{np}}) \neq \emptyset .$$

From the selection of the b_i in n° 2 and the irreducibility of Y,

$$f^{-1}(Y_{b_i a_{np}}) \cap g^{-1}(Y_{b_i a_{np}}) \neq \emptyset$$

for $b_i \in \underline{I}$, an integer $n > 0$, and $p \in P(n)$. We fix a $b_i \in \underline{I}$. The map z restricts to a map

$$z_{np} : Y_{b_i a_{np}} \to Z_{b_n}$$

in **QP** which corresponds to a map

$$z^*_{np} : B_{(b_n)} \to A_{(b_i a_{np})}$$

of k-algebras. From the stability condition, we discover, as a_{np} is in I, that $(b_i a_{np})^s \in (E^{b_i})$ for some positive integer s bigger than zero. Since $z : Y \to Z$ satisfies $z \circ f = z \circ g$, z^*_{np} factors in an unique way through

$$E_{(b_i a_{np})} = (E^{b_i})_{((b_i a_{np})^s)} .$$

Here, the last equality follows from Lemma 2 and [4], 2.4.7.

One has

$$Y_{b_i} = \bigcup_{np} Y_{b_i a_{np}}$$

where n ranges over all integers bigger than zero and $p \in P(n)$. But Y_{b_i} is affine and, hence, compact. Therefore, there exists a finite set R such that

$$Y_{b_i} = \bigcup_{r \in R} Y_{b_i a_r} ,$$

and a_r is equal to a_{np} for some integer $n > 0$ and $p \in P(n)$. Also, one can choose s large enough so that

$$(b_i a_r)^s \in (E^{b_i}) \text{ for all } r \text{ in } R .$$

If $a_r = a_{np}$, we set $w(r) = n$ ..Suppose that z^*_{np} factors through $E_{(b_i a_{np})}$ via the map z'^*_r when $a_{np} = a_r$. Similarly, if

$$z^*_{npn'p'} : B_{(b_n b_{n'})} \to A_{(b_i a_{np} a_{n'p'})}$$

is the map of k-algebras corresponding to z, for some integer s bigger than zero, there is a factorization

$$z'^*_{rr'} : B_{(b_{w(r)} b_{w(r')})} \to (E^{b_i})_{((b_i a_r a_{r'})^s)}$$

of $z^*_{npn'p'}$ through

$$E_{(b_i a_r a_{r'})} = (E^{b_i})_{((b_i a_r a_{r'})^s)}$$

where $r, r' \in R$. For an integer s large enough, a commutative diagram, for $r, r' \in R$,

$$(\upsilon) \qquad \begin{array}{ccc} z'^*_r : B_{(b_{w(r)})} & \longrightarrow & (E^{b_i})_{((b_i a_r)^s)} \\ \big\downarrow & & \big\downarrow \\ z'^*_{rr'} : B_{(b_{w(r)} b_{w(r')})} & \longrightarrow & (E^{b_i})_{((b_i a_r a_{r'})^s)} \end{array}$$

follows from the functoriality of equalizer.

Diagram (υ) implies that there is a map

$$z'_e : \bigcup_{r \in R} Spec(E^{b_i})_{((b_i a_r)^s)} \to \bigcup_{r \in R} Z_{b_{w(r)}} \subset Z .$$

To complete our first goal, the determination of a map $Spec\, E_{(b_i)} \to Z$ in **QP**, it remains to show that

$$(\Upsilon) \qquad Spec(E^{b_i})_{(b_i)} = Spec(E^{b_i})_{((b_i)^s)}$$
$$= \bigcup_{r \in R} Spec(E^{b_i})_{((b_i a_r)^s)} ,$$

where s is an integer chosen suitably large. Again, using the functoriality of equalizer, there is a diagram

$$\begin{array}{ccccc}
E_{(h_i)} & \longrightarrow & \prod_{r\in R} E_{((h_i a_r)^s)} & \rightrightarrows & \prod_{r,r'\in R} E_{((h_i a_r a_{r'})^s)} \\
\downarrow & & \downarrow & & \downarrow \\
A_{(h_i)} & \xrightarrow{\tau} & \prod_{r\in R} A_{(h_i a_r)} & \rightrightarrows & \prod_{r,r'\in R} A_{(h_i a_r a_{r'})}
\end{array}$$

where the horizontal arrow τ is an equalizer because it is one of the equalizers which define the sheaf structure on Y and where the two right vertical arrows are equalizers, as one sees, since equalizers commute with products.

To prove (Υ), we need the following lemma. The proof of this lemma is obtained by diagram chasing which we leave to the reader.

LEMMA 5. *If we have a commutative diagram*

$$\begin{array}{ccccc}
A_1 & \xrightarrow{\beta_2} & A_2 & \rightrightarrows & A_3 \\
{\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} \\
B_1 & \xrightarrow{\beta_1} & B_2 & \rightrightarrows & B_3 \\
\downarrow & & \downarrow & & \downarrow \\
C_1 & \xrightarrow{\beta_0} & C_2 & \rightrightarrows & C_3
\end{array}$$

where $\alpha_1, \alpha_2, \alpha_3$ and β_1 are equalizers and where β_0 is a monomorphism, then β_2 is an equalizer.

In our application,

$$C_1 = \mathcal{O}_X(f^{-1}(Y_{h_i}) \cap g^{-1}(Y_{h_i})), \quad etc.\ldots,$$

and β_0 is a monomorphism as localization in X is a monomorphism.

Let $h_j \in \underline{I}$ (n° 2) and associate R^j to h_j as we associated R to h_i. Then, by our choice of the set $\underline{I}$, R and R^j, we obtain

$$f^{-1}(Y_{h_i h_j a_r a_{\ddot r}}) \cap g^{-1}(Y_{h_i h_j a_r a_{\ddot r}}) \neq \emptyset$$

for $r \in R$ and $\ddot r \in R^j$,

$$Y_{b_i} = \bigcup_{r \in R} Y_{b_i a_r} \quad \text{and} \quad Y_{b_j} = \bigcup_{\ddot{r} \in R^j} Y_{b_j a_{\ddot{r}}} .$$

The diagram

$$\begin{array}{ccc} B_{(b_i)} & \xrightarrow{z_r^*} & A_{(b_i a_r)} \\ & \searrow & \downarrow \\ & & A_{(b_i b_j a_r a_{\ddot{r}})} \end{array}$$

induces, for an integer s large enough, a commutative diagram

$$\begin{array}{ccc} B_{(b_i)} & \xrightarrow{z_r'^*} & (E^{b_i})_{((b_i a_r)^s)} \\ & \searrow & \downarrow \\ & & (E^{b_i})_{((b_i b_j a_r a_{\ddot{r}})^s)} \end{array}$$

and, hence, a commutative diagram

$$\begin{array}{ccc} (\mathbf{P}roj(E^{b_i}))_{b_i b_j a_r a_{\ddot{r}}} & & \\ \downarrow & \searrow & \\ (\mathbf{P}roj(E^{b_i}))_{b_i a_r} & \longrightarrow & Z_{b_i} \end{array}$$

in **QP** . This diagram enables us to patch together the maps

$$z_e' : (\mathbf{P}roj(E^{b_i}))_{b_m} \to Z$$

to obtain a map $z' : C \to Z$ in **QP** satisfying $z' \circ c = z$.

z' will be unique (i.e. $c: Y \to C$ is an epimorphism) if we show that the local maps

$$Y_{b_m} \to (\mathbf{P}roj(E^{b_i}))_{b_m}$$

are epimorphisms in **QP** . This result follows from the next lemma.

LEMMA 6. *Let A, B be two integral domains and k-algebras. Suppose that A is a k-subalgebra of B. If, in the diagram*

$$Spec\, B \xrightarrow{\mu} Spec\, A \underset{\kappa}{\overset{\eta}{\rightrightarrows}} \mathbf{P}roj\, C ,$$

where μ is the map corresponding to the inclusion $A \subset B$ and C is a graded k-algebra and an integral domain, $\eta \circ \mu = \kappa \circ \mu$, then $\eta = \kappa$.

PROOF. Suppose that

$$x \in Spec\, A \quad \text{and} \quad \eta(x) \neq \kappa(x).$$

Unless $\mathbf{P}roj\, C$ is a point, as in the proof of Lemma 1, one finds a homogeneous element $\psi \in C^{+}$ such that $\eta(x)$, $\kappa(x)$ belong to $Spec\, C_{(\psi)} = Z_{\psi}$. There are maps, the restrictions of μ, η and κ, in the diagram

$$\mu^{-1}(\eta^{-1}(Z_{\psi}) \cap \kappa^{-1}(Z_{\psi})) \xrightarrow{\ddot{\mu}} \eta^{-1}(Z_{\psi}) \cap \kappa^{-1}(Z_{\psi}) \underset{\ddot{\kappa}}{\overset{\ddot{\eta}}{\rightrightarrows}} Z_{\psi}$$

with $\ddot{\eta} \neq \ddot{\kappa}$ as the images of $\ddot{\eta}$ and $\ddot{\kappa}$ are distinct. Let $(Spec\, A)_a$ be an affine open subset of $Spec\, A$, containing x and contained in

$$\eta^{-1}(Z_{\psi}) \cap \kappa^{-1}(Z_{\psi}).$$

The maps in the above diagram restrict again yielding, this time,

$$(\dagger) \qquad (Spec\, B)_a \xrightarrow{\hat{\mu}} (Spec\, A)_a \underset{\hat{\kappa}}{\overset{\hat{\eta}}{\rightrightarrows}} Z_{\psi}$$

where

$$x \in (Spec\, A)_a \quad \text{and} \quad \hat{\eta}(x) \neq \hat{\kappa}(x).$$

$(\dagger)$ induces a dual diagram

$$C_{\psi} \underset{\kappa^*}{\overset{\eta^*}{\rightrightarrows}} A_a \xrightarrow{\mu^*} B_a$$

of k-algebras which are at the same time integral domains. The above procedure can be repeated if, for all x, $\eta(x) = \kappa(x)$. Clearly, μ^* is a monomorphism, and it follows that $\eta^* = \kappa^*$. This implies that $\eta = \kappa$ locally, and hence that $\eta = \kappa$.

The proof of the lemma is complete.

Theorem 1 has been proven.

4. Coequalizers of projectives are nearly projective.

PROPOSITION 2. *Let f and g be as in Theorem 1. Suppose that $c: Y \to C$ is the coequalizer of f and g and that $Y = \mathbf{P}roj\, A$. Then, $\mathbf{P}roj(E^{h_i}) - C$ is of codimension larger than 1 in $\mathbf{P}roj(E^{h_i})$.*

PROOF.

$$Y = \bigcup_{h_i, n, p} Y_{h_i a_{np}}$$

where $h_i \in \underline{I}$, n is an integer bigger than zero and $p \in P(n)$. See n° 3. One applies Lemma 5, n° 3, and finds that, as $\mathcal{O}_Y(Y) = k$, $\mathcal{O}_C(C) = k$. If C lies in the complement of a hypersurface of $\mathbf{P}roj(E^{h_i})$, unless $C = Spec\, k$, $\mathcal{O}_C(C) \neq k$.

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