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ET GEOMETRIE DIFFERENTIELLE

« ... zitti, ascoltando e guardando il rumore
che fa l'acqua a passare nel solco di luna. »
C. Pavese (La Cena Triste)

NATURAL ANADESES AND CATADESES ★

by Dominique BOURN

Firstly, we wished to find simple conditions on a 2-category to make possible the «Kleisli and the Eilenberg-Moore constructions» associated to a triple, as in the case of the 2-category $\mathfrak{N}$ of natural transformations. It appeared that the study of two kinds of morphisms between 2-functors allowed an easier access to this question. The first kind, which is called here a catadesis, was first defined by Gray in [8], but already used in some cases by Mac Lane («crossed homomorphism» [10]) and Ehresmann («homomorphismes croisés» in [6]); this suggests that the notion could be useful in Homological Algebra. The second kind, which is a dual possibility of generalization, is called an anadesis. (I changed Gray's terminology because I needed two symmetric names.)

In the first part, we obtain some results about commutations of the new kinds of limits associated to anadeses and catadeses, and about compatibilities of the 2-functors with these limits. In the second part, we study some interesting examples of that sort of limits: product of two categories, tensor and cotensor of an object e of a 2-category by a category (whence an application to the theory, due to Gray, of representable 2-categories), Kleisli and Eilenberg-Moore constructions (whence an application to the theory of triples). In the third part, we give conditions for existence of limits. In the fourth part, a generalization of catadeses leads to the definition of a tensor product for catadeses (foretold by Gray in [8]) and for anadeses. In the fifth part, we discuss the problem of Kan extensions in this theory. Finally, applications are given to the study of

★ Conférence donnée au Colloque d'Amiens.

structures defined as algebras in this setting.

This paper develops a thesis (Thèse de 3^e cycle, Paris, 1973 [4]). The Amiens Colloquium gave me the opportunity to know that Professor Gray was going to publish an important volume on the same subject in the Lecture Notes. After taking note of it, I must point out that, if some results of my paper are analogous to some of his work (tensor product), other ones are original (commutation of limits) and the methods and the proofs that I use are always different, and so may be of some interest. My terminology is different (and Gray also does not adopt the terminology of his previous paper [8]), and it would have been too difficult to change it, since my paper was already written. In order to facilitate the transcription, I add the following «lexicon».

LEXICON

In this paper	Gray	Street
catadesis	in *Categorical Comprehension Scheme* [8] 2-natural transformation in *Formal Category Theory (to appear)* quasi-natural transformation or, when it is necessary, quasi-up-natural transformation.	in *Two Constructions on Lax Functors* [11] left lax transformation
2-catadesis	modification	morphism of left lax transformation
catalimit	cartesian quasi_0-limit	
2-catalimit	cartesian quasi-limit	
anadesis	quasi-down-natural transformation	right lax transformation
2-anadesis	modification	morphism of right lax transformation

0. Notations.

If $C^{\cdot}$ is a category, $C^{\cdot}_o$ is the set of its objects, $\alpha^{\cdot}$ and $\beta^{\cdot}$ its domain and codomain mappings, and $C^{\cdot *}$ the «opposite category».

We denote by $\mathcal{M}$ the «category of sets» (and mappings), relative to a universe $\mathcal{U}$, by $\mathcal{F}$ the «category of categories» (and functors), by $p_{\mathcal{F}}$ the forgetful functor from $\mathcal{F}$ to $\mathcal{M}$ which associates to a category $C^{\cdot}$ the set of its morphisms.

There are several ways to describe 2-categories. We shall mention two of them :

- the way of structured categories [7] : a 2-category C is a $p_{\mathcal{F}}$-structured category (i.e. a double category) $(C^{\cdot}, C^{\square\square})$, in which $C^{\cdot}_o \subset C^{\square\square}_o$,
- the way of **V**-categories [8] ; a 2-category C is a $\mathcal{F}$-category, i.e. a set C_o $[= C^{\cdot}_o]$ equipped, for each pair (e', e) of objects of C_o, with a category $C(e', e)^{\square\square}$ which describes the «*hom*» between e and e', satisfying the known conditions.

Most often, we shall use the first point of view but we do not neglect the second one.

If $C = (C^{\cdot}, C^{\square\square})$ is a 2-category, we denote by $C^{\cdot}_o$ the set of objects of $C^{\cdot}$, by $C^{\square\square}_o$ the set of *1-morphisms* of C (which defines a subcategory of $C^{\cdot}$); the other elements of C (i.e. the morphisms of $C^{\square\square}$) are called *2-morphisms* (or 2-cells). In the same way, if F is a 2-functor from C to the 2-category $C' = (C'^{\cdot}, C'^{\square\square})$, we denote by $F^{\square\square}_o$ the functor from $C^{\square\square}_o$ to $C'^{\square\square}_o$, restriction of F. To C are associated the two «opposite 2-categories» :

$$C^* = (C^{\cdot *}, C^{\square\square}) \text{ and } C_* = (C^{\cdot}, C^{\square\square *}).$$

The 2-category $\mathcal{N} = (\mathcal{N}^{\cdot}, \mathcal{N}^{\square\square})$ is the 2-category of natural transformations associated to the cartesian closed category $\mathcal{F}$.

1. Definitions and first results.

Let C and C' be two 2-categories, F and F' two 2-functors from C toward C'.

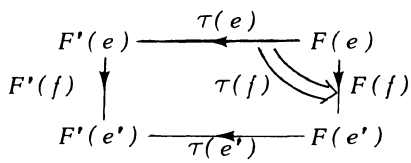
DEFINITION 1. By a *catadesis from F toward F'*, we mean a triple $t = (F', \tau, F)$, where τ is a mapping from $C_o^{\square\square}$ toward C', satisfying the following conditions:

a) $\tau(e)$ is a 1-morphism from $F(e)$ toward $F'(e)$ for each object e of C.

b) $\tau(f)$ is a 2-morphism from $F'(f).\tau(e)$ toward $\tau(e').F(f)$ for each 1-morphism f from e to e'.

c) Compatibility with lateral composition $(\square\square)$: if n is a 2-cell from f toward f', then: $\tau(f') \square\square F'(n).\tau(e) = \tau(e').F(n) \square\square \tau(f)$.

d) Compatibility with the main composition $(.)$: if $(\bar{f}, f)$ is a pair of composable 1-morphisms, then: $\tau(\bar{f}.f) = \tau(\bar{f}).F(f) \square\square F'(\bar{f}).\tau(f)$.



We can define a partial composition on the set of natural catadeses between 2-functors from C to C': if $\bar{t} = (F'', \bar{\tau}, F')$ is another catadesis, then $\bar{t}.t = (F'', \bar{\tau}.\tau, F)$, where

$$\bar{\tau}.\tau(e) = \bar{\tau}(e).\tau(e), \text{ for each object } e \text{ of } C,$$

$$\bar{\tau}.\tau(f) = \bar{\tau}(e').\tau(f) \square\square \bar{\tau}(f).\tau(e)$$

for each 1-morphism f from e toward e'; indeed, for each 2-cell n from f to f', we obtain:

$$\bar{\tau}.\tau(f') \square\square F''(n).\bar{\tau}.\tau(e) =$$

$$= \bar{\tau}(e').\tau(f') \square\square \bar{\tau}(f').\tau(e) \square\square F''(n).\bar{\tau}(e).\tau(e)$$

$$= \bar{\tau}(e').\tau(f') \square\square [\bar{\tau}(f') \square\square F''(n).\bar{\tau}(e)].\tau(e)$$

$$= \bar{\tau}(e').\tau(f') \square\square [\bar{\tau}(e').F'(n) \square\square \bar{\tau}(f)].\tau(e)$$

$$= \bar{\tau}(e').[\tau(f') \square\square F'(n).\tau(e)] \square\square \bar{\tau}(f).\tau(e)$$

$= \bar{\tau}(e').\ [\,\tau(e').F(n) \square\square \tau(f)\,] \square\square \bar{\tau}(f).\tau(e)$

$= \bar{\tau}.\tau(e').F(n) \square\square \bar{\tau}.\tau(f)$ (condition (c) for catadeses).

We would prove the condition (d) for catadeses in the same way.

Let $t = (F', \tau, F)$ and $t' = (F', \tau', F)$ be two natural catadeses.

DEFINITION 2. A *2-natural catadesis from* τ *toward* τ' is defined as a triple $d = (t', \delta, t)$, where δ is a mapping from C_0 to C' satisfying the conditions:

a) $\delta(e)$ is a 2-cell from $\tau(e)$ toward $\tau'(e)$ for each object e of C,

b) If f is a 1-morphism from e to e', then

$$\tau'(f) \square\square F'(f).\delta(e) = \delta(e').F(f) \square\square \tau(f).$$

If $d' = (t'', \delta', t')$ is another 2-natural catadesis, we define a 2-natural catadesis $d' \square\square d = (t'', \delta' \square\square \delta, t)$, where

$$\delta' \square\square \delta(e) = \delta'(e) \square\square \delta(e) \text{ for each object } e \text{ of } C.$$

Indeed, for each 1-morphism f from e to e', we have

$\tau''(f) \square\square F'(f).\delta' \square\square \delta(e) = \tau''(f) \square\square F'(f).\delta'(e) \square\square F'(f).\delta(e)$

$= \delta'(e').F(f) \square\square \tau'(f) \square\square F'(f).\delta(e)$

$= \delta'(e').F(f) \square\square \delta(e').F(f) \square\square \tau(f)$

$= \delta' \square\square \delta(e').F(f) \square\square \tau(f).$

We would prove by analogous computations that

$\bar{t}.d = (\bar{t}.t', \bar{\tau}.\delta, \bar{t}.t)$, where $\bar{\tau}.\delta(e) = \bar{\tau}(e).\delta(e)$ for each object e of C,

$\bar{d}.t = (\bar{t}'.t, \bar{\delta}.\tau, \bar{t}.t)$, where $\bar{\delta}.\tau(e) = \bar{\delta}(e).\tau(e)$ for each object e of C,

are 2-natural catadeses and that

$$\bar{t}'.d \square\square \bar{d}.t = \bar{d}.t' \square\square \bar{t}.d \quad (= \bar{d}.d \text{ by definition }).$$

Therefore, the set of 2-natural catadeses between 2-functors from C to C' has the structure of a 2-category; we denote it by $\overleftarrow{\mathfrak{N}}(C', C)$.

REMARK. Let $t = (F', \tau, F)$ be a catadesis. Let G and G' be 2-functors from C' to C'' and $s=(G',\sigma,G)$ a catadesis between them. Let us remark that the composition of G by t (we shall note it by Gt) is also a catadesis, as well as sF'. But the two catadeses $sF'.Gt$ and $G't.sF$ are not equal. There is a 2-catadesis between them, we shall denote it by st, by analogy with 2-categories.

Let $\overleftarrow{\Delta}^{C}_{C'}$ be the diagonal 2-functor from C' to $\overleftarrow{\mathfrak{N}}(C',C)$.

DEFINITION 3. We define a *catalimit functor* as a right adjoint to the functor $[\overleftarrow{\Delta}^{C}_{C'}]^{\square\square}_{o}$ and a *2-catalamit functor* as a 2-right adjoint to $\overleftarrow{\Delta}^{C}_{C'}$.

Clearly a 2-catalimit is a catalimit.

EXAMPLE. If $C'=\mathfrak{N}$, the 2-catalimit of a 2-functor F is the category $\overleftarrow{\mathfrak{N}}(\mathfrak{N},C)[F,\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(1)]^{\square\square}$ (where 1 is a terminal object of $\mathfrak{N}$).

DEFINITION 4. We define a *cocatalimit functor* as a left adjoint to the functor $[\overleftarrow{\Delta}^{C}_{C'}]^{\square\square}_{o}$, a *2-cocatalimit functor* as a 2-left adjoint to $\overleftarrow{\Delta}^{C}_{C'}$.

EXAMPLES. 1° If $C'=\mathfrak{N}$ and if $C^{\square\square}$ is discrete, a 2-functor F from C to $\mathfrak{N}$ is nothing else than a functor from $C^{\cdot}$ to $\mathfrak{F}$ (i.e. an «espèce de morphismes» [6]) and the 2-cocatalimit of F is the crossed-product category $P(F)$ associated to that «espèce de morphismes» [6] . (It has already been mentioned by Gray [8].)

Let us recall that the objects of $P(F)$ are the pairs (e,u) where e is an object of $C^{\cdot}$ and $u \in F(e)^{\cdot}_{o}$; its morphisms are defined as the (z,f,u), where f is a morphism of C from e to e', where $u \in F(e)^{\cdot}_{o}$ and $z \in F(e')$, the domain of z being $F(f)(u)$. The composition is given by :

$$(z',f',u').(z,f,u)=(z'.F(f)(z)\ ,f'.f,u)$$

if and only if the domain of f' is the codomain of f and the codomain of z is u'.

2° More generally, if $C^{\square\square}$ is not discrete, let us consider the following category $Q(F)$: its objects are the pairs (e,u), in which $u \in F(e)^{\cdot}_{o}$; its morphisms are the pairs (n,u) where $u \in F(\alpha^{\cdot}(n))_{o}$; the composition is given by :

$$(n',u').(n,u)=(n'.n,u)$$

if and only if $\alpha^{\cdot}(n')=\beta^{\cdot}(n)$ and $F(\beta^{\square}n)(u)=u'$.

We have the two following functors from $Q(F)$ to $P(F_o^{\square})$:

- the first one, $Q_1(F)$ is defined by:

$$Q_1(F)(e,u)=(e,u),\ Q_1(F)(n,u)=(F(n)(u),\alpha^{\square}(n),u);$$

- the second one, $Q_2(F)$, is defined by:

$$Q_2(F)(e,u)=(e,u),\quad Q_2(F)(n,u)=(F(\beta^{\square}n)(u),\beta^{\square}(n),u).$$

The 2-cocatalimit of F is the codomain of the cokernel of $Q_1(F)$ and $Q_2(F)$.

Let F and F' be two 2-functors from C to C'.

DEFINITION 5. A *natural anadesis from F to F'* will be defined as a triple $r=(F',\rho,F)$, where ρ is a mapping defined on $C_o^{\square}$, to C', satisfying:

a) $\rho(e)$ is a 1-morphism from $F(e)$ toward $F'(e)$ for each object e of C.

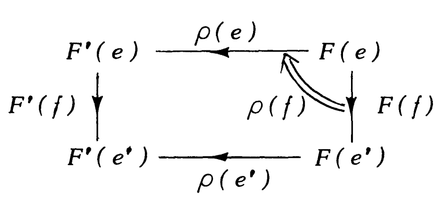
b) $\rho(f)$ is a 2-morphism from $\rho(e').F(f)$ toward $F'(f).\rho(e)$ for each 1-morphism f from e to e'.

c) compatibility with the lateral composition: if n is a 2-cell from f to f', then

$$\rho(f')\,\square\,\rho(e').F(n)=F'(n).\rho(e)\,\square\,\rho(f).$$

d) Compatibility with the main composition: if $(\bar{f},f)$ is a pair of composable 1-morphisms, then:

$$\rho(\bar{f}.f)=F'(\bar{f}).\rho(f)\,\square\,\rho(\bar{f}).F(f).$$



REMARK. The 2-functor F from C toward C' determines a 2-functor F_* between the «opposite» C_* and C'_*. So a natural anadesis r determines a natural catadesis r_* from F_* to F'_*.

DEFINITION 6. We define a *2-natural anadesis from r toward r'* as a triple $g=(r',\gamma,r)$, where γ is a mapping from C_0 to C', satisfying the conditions:

a) $\gamma(e)$ is a 2-cell from $\rho(e)$ toward $\rho'(e)$, for each object e of C.

b) If f is a 1-morphism from e to e', then

$$\rho'(f)\square\square\gamma(e').F(f)=F'(f).\gamma(e)\square\square\rho(f).$$

REMARK. A 2-natural anadesis from r toward r' determines a 2-natural catadesis from r'_* to r_*.

In other words, we can define a structure of a 2-category denoted by $\overrightarrow{\mathcal{N}}(C',C)$, on the set of 2-natural anadeses, by the isomorphism:

$$\overrightarrow{\mathcal{N}}(C',C)_*\simeq\overleftarrow{\mathcal{N}}(C'_*,C_*).$$

We have another diagonal functor $\overrightarrow{\Delta}^{C}_{C'}$ between C' and $\overrightarrow{\mathcal{N}}(C',C)$.

DEFINITION 7. We define an *analimit functor* as a right adjoint to the functor $[\overrightarrow{\Delta}^{C}_{C'}]^{\square\square}_{0}$ and a *2-analimit functor* as a 2-right adjoint to $\overrightarrow{\Delta}^{C}_{C'}$.

EXAMPLE. If $C'=\mathcal{N}$, the 2-analimit of a 2-functor F is the category $\overrightarrow{\mathcal{N}}(\mathcal{N},C)\,[F,\overrightarrow{\Delta}^{C}_{\mathcal{N}}(1)]^{\square\square}$.

DEFINITION 8. We define a *coanalimit functor* as a left adjoint to the functor $[\overrightarrow{\Delta}^{C}_{C'}]^{\square\square}_{0}$ and a *2-coanalimit functor* as a 2-left adjoint to $\overrightarrow{\Delta}^{C}_{C'}$.

EXAMPLE. We can deduce the 2-coanalimit of a 2-functor F, when $C'=\mathcal{N}$, from the construction of the 2-cocatalimit and from the isomorphism:

$$\overrightarrow{\mathcal{N}}(\mathcal{N},C)_*\simeq\overleftarrow{\mathcal{N}}(\mathcal{N}_*,C_*).$$

Let C, C' and C'' be three 2-categories.

PROPOSITION 1. *There is an isomorphism between the 2-category* $\overrightarrow{\mathcal{N}}(\overleftarrow{\mathcal{N}}(C,C''),C')$ *and the 2-category* $\overleftarrow{\mathcal{N}}(\overrightarrow{\mathcal{N}}(C,C'),C'')$.

PROOF. It is not difficult but rather tedious. The whole proof is given in [4]. Let us say only that, if F is a 2-functor from C' to $\overleftarrow{\mathcal{N}}(C,C'')$, the 2-functor $\bar{F}$ from C'' toward $\overrightarrow{\mathcal{N}}(C,C')$, associated to F by this isomorphism, is given by:

$\bar{F}(s)[-] = F(-)[s]$, for each object s of C'',

$\bar{F}(g)[-] = F(-)[g]$, for each 1-morphism g of C'',

$\bar{F}(r)[-] = F(-)[r]$, for each 2-cell r of C'';

if $t=(F',\tau,F)$ is a 1-morphism of $\overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C,C''),C')$, the 1-morphism $\bar{t}=(\bar{F}',\bar{\tau},\bar{F})$ of $\overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C,C'),C'')$, is defined by:

$\bar{\tau}(s)[-] = \tau(-)[s]$, for each object s of C'',

$\bar{\tau}(g)[-] = \tau(-)[g]$, for each 1-morphism g of C'';

if $d=(t',\delta,t)$ is a 2-morphism of $\overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C,C''),C')$, the 2-morphism $\bar{d}=(\bar{t}',\bar{\delta},\bar{t})$ of $\overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C,C'),C'')$, is defined by:

$$\bar{\delta}(s)[-] = \delta(-)[s], \text{ for each object } s \text{ of } C''. \blacksquare$$

REMARK. Let **2** be the 2-category $(2^{\cdot}, 2^{\square\square})$ where $2^{\cdot}$ is the usual category 2 and $2^{\square\square}$ the discrete category over 2. If we consider the isomorphism: $\overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C',C),\mathbf{2})^{\cdot}_{o} \sim \overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C',\mathbf{2}),C)^{\cdot}_{o}$, we assert that the set of natural catadeses between 2-functors from C to C' is isomorphic to the set of 2-functors from C toward $\overrightarrow{\mathfrak{N}}(C',\mathbf{2})$ (we call it the *2-category of anacylinders of* C'). Likewise, if we consider the isomorphism:

$$\overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C',C),\mathbf{2})^{\cdot}_{o} \simeq \overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C',\mathbf{2}),C)^{\cdot}_{o},$$

we assert that the set of natural anadeses between 2-functors from C to C' is isomorphic to the set of 2-functors from C toward $\overleftarrow{\mathfrak{N}}(C',\mathbf{2})$ (we call it the *2-category of catacylinders of* C').

Clearly, for each 2-category C, we can define two 2-endofunctors $\overleftarrow{\mathfrak{N}}(-,C)$ and $\overrightarrow{\mathfrak{N}}(-,C)$ on $\mathfrak{N}$. Therefore, for each adjoint pair (p,q) of 2-functors, there are two new pairs of adjoint 2-functors:

$$(\overleftarrow{\mathfrak{N}}(p,C),\overleftarrow{\mathfrak{N}}(q,C)) \text{ and } (\overrightarrow{\mathfrak{N}}(p,C),\overrightarrow{\mathfrak{N}}(q,C)).$$

PROPOSITION 2. *The 2-left-adjoints preserve coanalimits and cocatalimits, 2-coanalimits and 2-cocatalimits. The 2-right-adjoints preserve analimits and catalimits, 2-analimits and 2-catalimits.*

PROOF. For instance, let us show that a 2-right adjoint preserves the 2-catalimits. Let F be a 2-functor from C to C', and e its 2-catalimit. Let p be a 2-right adjoint (q being its 2-left adjoint) from C' to C''. We have successively, for each object e' of C'':

$$\overleftarrow{\mathfrak{N}}(C'', C)\,[p\,.\,F, \overleftarrow{\Delta}^{C}_{C''}(e')\,]^{\square\square} \simeq \overleftarrow{\mathfrak{N}}(C', C)\,[F, q\,.\,\overleftarrow{\Delta}^{C}_{C''}(e')\,]^{\square\square}$$

(since $\overleftarrow{\mathfrak{N}}(p, C)$ and $\overleftarrow{\mathfrak{N}}(q, C)$ are adjoints),

$$\overleftarrow{\mathfrak{N}}(C', C)\,[F, q\,.\,\overleftarrow{\Delta}^{C}_{C''}(e')\,]^{\square\square} = \overleftarrow{\mathfrak{N}}(C', C)\,[F, \overleftarrow{\Delta}^{C}_{C'}(q(e'))\,]^{\square\square},$$

$$\overleftarrow{\mathfrak{N}}(C', C)\,[F, \overleftarrow{\Delta}^{C}_{C'}(q(e'))\,]^{\square\square} \simeq C'(e, q(e'))^{\square\square}$$

(since e is a 2-catalimit of F),

$$C'(e, q(e'))^{\square\square} \simeq C''(p(e), e')^{\square\square}$$

(because p and q are adjoints).

The compatibility with catalimits is given by the restriction to the objects of these isomorphisms of categories. ■

PROPOSITION 3. *The two following diagrams commute:*

$$\begin{array}{ccc} \overleftarrow{\mathfrak{N}}(C, C'') & & \\ {\scriptstyle \overrightarrow{\Delta}^{C'}_{\overleftarrow{\mathfrak{N}}(C, C'')}} \downarrow & \searrow {\scriptstyle \overleftarrow{\mathfrak{N}}(\overrightarrow{\Delta}^{C'}_{C}, C'')} & \\ \overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C, C''), C') & \simeq & \overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C, C'), C'') \end{array}$$

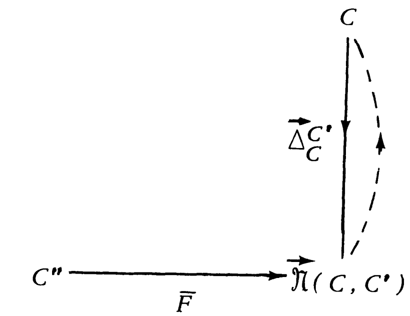
$$\begin{array}{ccc} \overrightarrow{\mathfrak{N}}(C, C') & & \\ {\scriptstyle \overleftarrow{\Delta}^{C''}_{\overrightarrow{\mathfrak{N}}(C, C')}} \downarrow & \searrow {\scriptstyle \overrightarrow{\mathfrak{N}}(\overleftarrow{\Delta}^{C''}_{C}, C')} & \\ \overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C, C'), C'') & \simeq & \overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C, C''), C') \end{array}$$

PROOF. It arises from the construction of the isomorphism of the proposition 1. ■

APPLICATION : *Construction of limits.*

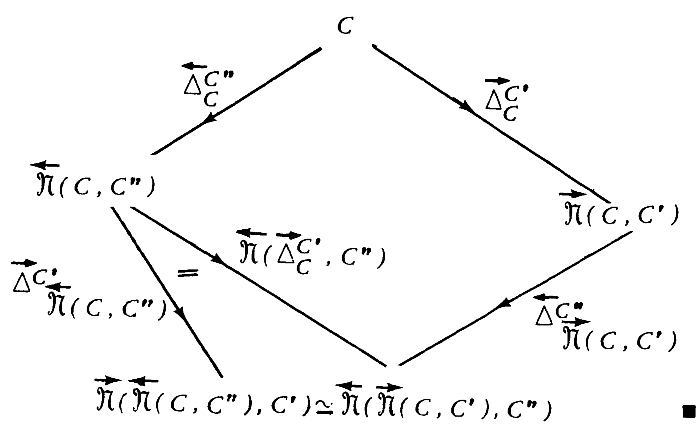
Let F be a 2-functor from C' to $\overleftarrow{\mathcal{N}}(C, C'')$.

If for each object e of C'', the 2-functor $\bar{F}(e)$ (see Prop. 1) has a 2-analimit (for instance), we can construct a 2-functor from C'' toward C, which is obviously a free costructure of $\bar{F}$ for the 2-functor $\overleftarrow{\mathcal{N}}(\overrightarrow{\Delta}_C^{C'}, C'')$, and therefore a 2-analimit of F by Proposition 3. In particular, if the 2-right adjoint $anl_C^{C'}$ of $\overrightarrow{\Delta}_C^{C'}$ exists, the 2-analimit of F is the 2-functor $\bar{F} . anl_C^{C'}$.



COROLLARY. *Analimits and catalimits, 2-analimits and 2-catalimits commute. The same holds for the analogous colimits.*

PROOF. It follows clearly from the consideration of the following diagram :

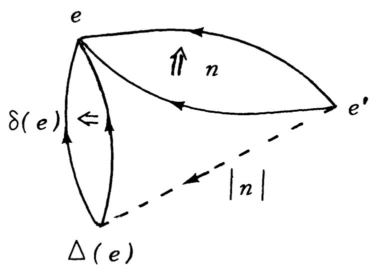


■

2. Applications and examples.

a) Representable and corepresentable 2-categories

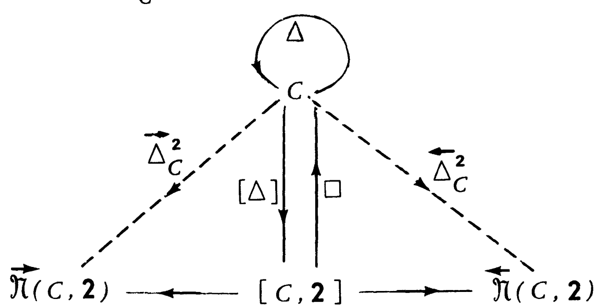
A 2-category C is said *representable* [9], when, for each object e of $C^{\cdot}$, there is a 2-cell $\delta(e)$ so that for each 2-cell n going from e' toward e there exists a unique 1-morphism $|n|$ satisfying: $n = \delta(e).|n|$.



In other words, C is representable if there is a catalimit of $\overrightarrow{\Delta}^2_C(e)$ (or an analimit of $\overleftarrow{\Delta}^2_C(e)$) for each object e of C. If we denote by $[C, \mathbf{2}]$ the full sub-2-category of $\overleftarrow{\mathfrak{N}}(C, \mathbf{2})$ which has the 2-functors $\overleftarrow{\Delta}^2_C(e)$ for objects and by $[\Delta]$ the factorization of $\overleftarrow{\Delta}^2_C$ through $[C, \mathbf{2}]$, the 2-category C is representable if and only if $[\Delta]^{\square\square}_0$ admits a right adjoint $\square^{\square\square}_0$.

Let us remark that $[C, \mathbf{2}]$ is equally the full sub-2-category of $\overrightarrow{\mathfrak{N}}(C, \mathbf{2})$ which has the 2-functors $\overrightarrow{\Delta}^2_C(e)$ for objects and that $[\Delta]$ is the factorization of $\overrightarrow{\Delta}^2_C$ through $[C, \mathbf{2}]$.

Then a 2-category is said *strongly representable* [9] if for each object e of C, the 2-functor $\overleftarrow{\Delta}^2_C(e)$ has a 2-catalimit (or what is the same: the 2-functor $\overrightarrow{\Delta}^2_C(e)$ has a 2-analimit). Therefore C is strongly



representable if and only if $[\Delta]$ has a 2-right adjoint $\Box$.

Let us call Δ the 2-functor $\Box . [\Delta]$. According to the corollary proposition 3, the 2-functor Δ is both the 2-catalimit of the 2-functor $\overleftarrow{\Delta}^{2}_{\overrightarrow{\mathfrak{N}}(C,C)}(idC)$ and the 2-analimit of the 2-functor $\overrightarrow{\Delta}^{2}_{\overleftarrow{\mathfrak{N}}(C,C)}(idC)$.

PROPOSITION 4. *If C is strongly representable , so are $\overleftarrow{\mathfrak{N}}(C,C')$ and $\overrightarrow{\mathfrak{N}}(C,C')$ (for any 2-category C').*

PROOF. Let us show that property for $\overleftarrow{\mathfrak{N}}(C,C')$. We must find a 2-analimit of $\overrightarrow{\Delta}^{2}_{\overleftarrow{\mathfrak{N}}(C,C')}[F]$, for each 2-functor F from C' to C.
According to proposition 3, the 2-functor associated to it by the isomorphism $\overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C,\mathbf{2}),C') \simeq \overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C,C'),\mathbf{2})$ is $\overleftarrow{\Delta}^{\mathbf{2}}_{C} . F$ and according to the application of page 11, its 2-analimit is $\Box . [\Delta] . F = \Delta . F$. ∎

REMARK (Due to J. Penon). Any kind of limit is the same kind of a 2-limit, when it is preserved by $[\Delta]$. There is no difficulty to prove this remark and it will greatly simplify our computations. Let us begin by:

PROPOSITION 5. *When C is strongly representable, cocatalimits and coanalimits are 2-cocatalimits and 2-coanalimits.*

PROOF. $[\Delta]$ has a 2-right adjoint and 2-left-adjoints preserve any kind of a colimit (prop. 2). ∎

A 2-category is *corepresentable* [9] when $[\Delta]^{\Box\Box}_{o}$ has a left adjoint and *strongly corepresentable* when $[\Delta]$ has a 2-left adjoint. Obviously we have the opposite results :

PROPOSITION 6. *If C is strongly corepresentable, so are $\overleftarrow{\mathfrak{N}}(C,C')$ and $\overrightarrow{\mathfrak{N}}(C,C')$ for any 2-category C'.*

PROPOSITION 7. *When C is strongly corepresentable, catalimits and analimits are 2-catalimits and 2-analimits.*

REMARK. Gray [9] defined strongly representable 2-categories as those 2-categories C in which the cotensor $Cot(e,2)$ of any object e with the category $2^{\cdot}$ exists, and strongly corepresentable 2-categories as those 2-categories in which the tensor of any object e with $2^{\cdot}$ exists.

Therefore $Cot(e,2)$ can be defined as a 2-catalimit and $e\times 2$ as a 2-cocatalimit. Then if D is a category, it is clear that $Cot(e,D)$ is the 2-catalimit of $\overleftarrow{\Delta}_C^D(e)$ (or a 2-analimit of $\overrightarrow{\Delta}_C^{D^*}(e)$), while the tensor $e\times D$ is the 2-cocatalimit of $\overleftarrow{\Delta}_C^D(e)$ (or the 2-coanalimit of $\overrightarrow{\Delta}_C^{D^*}(e)$).

In particular, if $C=\mathfrak{N}$ and if e is a category D', $\mathfrak{N}(D',D)$ is a 2-catalimit and $D\times D'$ is a 2-cocatalimit. We shall show later that a light adaptation of this remark will allow us to define a tensor product associated to anadeses and catadeses.

b) Triples (monads).

Let C be a 2-category.

A triple in C on an object e of C (the underlying object of the triple) is defined by the data (t,λ,μ) where $t\in C(e,e)_o^{\square\square}$, where λ and μ are two 2-morphisms from e to t, from t^2 to t respectively, satisfying:

$$\mu \square\square \lambda t = t = \mu \square\square t\lambda \quad \text{and} \quad \mu \square\square \mu t = \mu \square\square t\mu .$$

The simplicial 2-category S is the 2-category which has only one object (the empty set denoted by 0), integers (with usual addition) for 1-morphisms, non-decreasing mappings between integers for 2-morphisms (with usual composition).

Any 2-cell of S is generated by two families of 2-cells:

- increasing injections δ_n^i from n toward $n+1$ which do not take the value i,
- non-decreasing surjections σ_n^i from $n+1$ toward n which take twice the value i.

It is well known that the notion of a triple in C is equivalent to that of a 2-functor from S to C:

If (t,λ,μ) is a triple on e in C, the associated 2-functor $\mathfrak{t}$ is defined by:

$\mathfrak{t}(0)=e$, $\mathfrak{t}(1)=t$, $\mathfrak{t}(n)=t^n$,

$\mathfrak{t}(\delta_0^0)=\lambda$, $\mathfrak{t}(\delta_n^i)=t^i\lambda t^{n-i}$,

$\mathfrak{t}(\sigma_1^0)=\mu$, $\mathfrak{t}(\sigma_n^i)=t^i\mu\, t^{n-(i+1)}$.

If $\mathfrak{t}$ is a 2-functor from S to C, $(\mathfrak{t}(1),\mathfrak{t}(\delta_0^0),\mathfrak{t}(\sigma_1^0))$ is a triple in C on $\mathbf{t}(0)$.

Henceforth we shall identify $\mathbf{t}$ and (t, λ, μ).

There are two 2-categories associated to triples in C:

- The first one (denoted by $\overleftarrow{Trip}_C$ in [11]) has its 1-morphisms of the form $\mathbf{T} = (\mathbf{t}', (f, v), \mathbf{t})$, where $\mathbf{t}$ and $\mathbf{t}'$ are triples in C, $f \in C(e', e)_0^{\square\square}$ (if $\mathbf{t}(0) = e$ and $\mathbf{t}'(0) = e'$), v being a 2-cell between $t'.f$ and $f.t$ satisfying:

$$f.\lambda = v \square\square \lambda'.f \quad \text{and} \quad f.\mu \square\square v.t \square\square t'.v = v \square\square \mu'.f.$$

The 2-morphisms are of the form $(\mathbf{T}', \pi, \mathbf{T})$ where $\mathbf{T}' = (\mathbf{t}', (f', v'), \mathbf{t})$ and $\mathbf{T}$ are 1-morphisms of $\overleftarrow{Trip}_C$ and where π is a 2-cell of C from f to f' satisfying:

$$\pi.t \square\square v = v' \square\square t'.\pi.$$

- The second one (denoted by $\overrightarrow{Trip}_C$ in [11]) having its 1-morphisms of the form $\underline{\mathbf{T}} = (\mathbf{t}', (v, f), \mathbf{t})$ where $f \in C(e', e)_0^{\square\square}$, v being a 2-cell from $f.t$ to $t'.f$ satisfying:

$$v \square\square f.\lambda = \lambda'.f \quad \text{and} \quad v \square\square f.\mu = \mu'.f \square\square t'.v \square\square v.t.$$

The 2-morphisms of $\overrightarrow{Trip}_C$ being of the form $(\underline{\mathbf{T}}', \underline{\pi}, \underline{\mathbf{T}})$, where $\underline{\mathbf{T}}$ and $\underline{\mathbf{T}}' = (\mathbf{t}', (v', f'), \mathbf{t})$ are 1-morphisms of $\overrightarrow{Trip}_C$ and where $\underline{\pi}$ is a 2-cell from f to f' satisfying:

$$v' \square\square \underline{\pi}.t = t'.\underline{\pi} \square\square v.$$

PROPOSITION 8. *The 2-category* $\overleftarrow{Trip}_C$ *is isomorphic to* $\overleftarrow{\mathfrak{N}}(C, S)$ *and the 2-category* $\overrightarrow{Trip}_C$ *is isomorphic to* $\overrightarrow{\mathfrak{N}}(C, S)$.

PROOF. We would prove by induction that the catadesis associated to $\mathbf{T} = (\mathbf{t}', (f, v), \mathbf{t})$ is given by:

$$\tau(0) = f \quad \text{and} \quad \tau(n) = \mathop{\square\square}_{i=0}^{n-1} t'^i . v . t^{n-i-1}.$$

The 2-catadesis associated to $(\mathbf{T}', \pi, \mathbf{T})$ is given by $\delta(0) = \pi$. ∎

This leads to the following definitions:

DEFINITION 9. We shall call *Kleisli preobjects of a triple* $\mathbf{t}$ *in* C the coanalimits of the 2-functor $\mathbf{t}$ from S to C and *Kleisli objects of* $\mathbf{t}$ the 2-coanalimits of $\mathbf{t}$.

DEFINITION 10. We shall call *E-M (Eilenberg-Moore) preobjects of* $\mathbf{t}$ the catalimits of the 2-functor $\mathbf{t}$ and *E-M objects of* $\mathbf{t}$ its 2-catalimits.

REMARK. The 2-cocatalimit and the 2-analimit of $\mathbf{t}$ always exist: they are its underlying object.

EXAMPLES. 1° The identity 2-functor of S determines a triple on S (denoted by $[0]$). The object 0 is at the same time a Kleisli preobject and an E-M preobject of $[0]$.

2° It is shown in [11] that the Kleisli construction in $\mathfrak{N}$ is a 2-left adjoint to the inclusion $\mathfrak{N} \to \overrightarrow{Trip}_{\mathfrak{N}}$, and the Eilenberg-Moore construction in $\mathfrak{N}$ is a 2-right adjoint to the inclusion $\mathfrak{N} \to \overleftarrow{Trip}_{\mathfrak{N}}$. Therefore the Kleisli category and the category of algebras associated to a triple in $\mathfrak{N}$ are Kleisli objects and E-M objects.

3° It is shown in [3] that Kleisli and E-M objects exist in the 2-category $\mathfrak{N}_p = (\mathfrak{N}^{\cdot}_p, \mathfrak{N}^{\square\square}_p)$ of p-structured natural transformations, when p is a «good» forgetful functor with pullbacks, from a category $H^{\cdot}$ towards the category $\mathfrak{M}$ of sets.

4° If $\mathbf{t}$ is a triple in $\overleftarrow{\mathfrak{N}}(C, C')$, its Kleisli preobject can be constructed by the corollary of proposition 3; the same is true for the E-M object associated to a triple in $\overrightarrow{\mathfrak{N}}(C, C')$.

5° This remark allows to classify some triples associated to a «distributive law» [2]. Indeed, a distributive law $(\mathbf{t}', d, \mathbf{t})$ may be considered as a triple $\bar{\mathbf{t}}$ on $\mathbf{t}'$ in $\overrightarrow{Trip}_{\mathfrak{N}}$ (and therefore as a triple $\bar{\mathbf{t}}'$ on $\mathbf{t}$ in $\overleftarrow{Trip}_{\mathfrak{N}}$ (proposition 1)). It determines the triples $Kl_{\mathfrak{N}}.\bar{\mathbf{t}}$ and $Alg_{\mathfrak{N}}.\bar{\mathbf{t}}'$ ($\simeq \tilde{\mathbf{t}}'$ in [2]). The triple $Kl_{\mathfrak{N}}.\bar{\mathbf{t}}$ is a Kleisli object of $\bar{\mathbf{t}}'$ and $Alg_{\mathfrak{N}}.\bar{\mathbf{t}}'$ an E-M object of $\bar{\mathbf{t}}$.

$$
\begin{array}{ccc}
S & \qquad\qquad & S \\
\downarrow{\scriptstyle \bar{\mathbf{t}}'} & & \downarrow{\scriptstyle \bar{\mathbf{t}}} \\
\overleftarrow{\mathfrak{N}}(\mathfrak{N}, S) & & \overrightarrow{\mathfrak{N}}(\mathfrak{N}, S) \\
\downarrow{\scriptstyle Alg_{\mathfrak{N}}} & & \downarrow{\scriptstyle Kl_{\mathfrak{N}}} \\
\mathfrak{N} & & \mathfrak{N}
\end{array}
$$

In [2], a triple $\mathbf{t}'.\mathbf{t}$ is associated to $(\mathbf{t}', d, \mathbf{t})$. It is easy to show that this construction is general and 2-functorial between $\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C,S),S)$ and $\overleftarrow{\mathfrak{N}}(C,S)$, and also between $\overrightarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C,S),S)$ and $\overrightarrow{\mathfrak{N}}(C,S)$. This $\mathbf{t}'.\mathbf{t}$ has not a universal simple property (it is neither the Kleisli object of $\bar{\mathbf{t}}$ nor the E-M object of $\bar{\mathbf{t}}'$). But we have the following result: *e' is an E-M object of $\mathbf{t}'.\mathbf{t}$ if and only if e' is a 2-free structure of $\bar{\mathbf{t}}'$ for $\overleftarrow{\Delta}^S_{\overleftarrow{\mathfrak{N}}(C,S)} . \overleftarrow{\Delta}^S_C$, and e'' is a Kleisli object of $\mathbf{t}'.\mathbf{t}$ if and only if e'' is a 2-free structure of $\bar{\mathbf{t}}$ for $\overrightarrow{\Delta}^S_{\overrightarrow{\mathfrak{N}}(C,S)} . \overrightarrow{\Delta}^S_C$.*

PROPOSITION 9. *If a triple $\mathbf{t}=(t,\lambda,\mu)$ in C has an E-M preobject e_A, then there is a factorization of t through e_A and any adjoint pair defining $\mathbf{t}$ has a factorization through the pair given by the factorization of t.* (But this is not an adjoint pair.)

PROOF. Let us denote by $(\mathbf{t}, p_A, \gamma_A, \overleftarrow{\Delta}^S_C(e_A))$ the 1-morphism projection. Then the pair (t,μ) determines a 1-morphism $(\mathbf{t}, t, \mu, \overleftarrow{\Delta}^S_C(e))$, whence a 1-morphism g_A between e and e_A satisfying:

$$(\mathbf{t}, t, \mu, \overleftarrow{\Delta}^S_C(e)) = (\mathbf{t}, p_A, \gamma_A, \overleftarrow{\Delta}^S_C(e_A)).\overleftarrow{\Delta}^S_C(g_A),$$

id est:

$$p_A \cdot g_A = t \quad \text{and} \quad \gamma_A \cdot g_A = \mu .$$

Let (p,g) be an adjoint pair defining $\mathbf{t}$, let ν be the 2-cell naturalisation from $g \cdot p$ toward e' (if e' is the domain of p). The pair $(p, p.\nu)$ determines a 1-morphism $(\mathbf{t}, p, p.\nu, \overleftarrow{\Delta}^S_C(e'))$, whence a 1-morphism p' so that

$$(\mathbf{t}, p, p.\nu, \overleftarrow{\Delta}^S_C(e')) = (\mathbf{t}, p_A, \gamma_A, \overleftarrow{\Delta}^S_C(e_A)).\overleftarrow{\Delta}^S_C(p'),$$

id est:

$$(1) \quad p_A \cdot p' = p \quad \text{and} \quad \gamma_A \cdot p' = p \cdot \nu .$$

Besides:

$$(\mathbf{t}, p_A, \gamma_A, \overleftarrow{\Delta}^S_C(e_A)).\overleftarrow{\Delta}^S_C(p').\overleftarrow{\Delta}^S_C(g)$$
$$= (\mathbf{t}, p, p.\nu, \overleftarrow{\Delta}^S_C(e')).\overleftarrow{\Delta}^S_C(g) = (\mathbf{t}, p \cdot g, p.\nu.g, \overleftarrow{\Delta}^S_C(e))$$
$$= (\mathbf{t}, t, \mu, \overleftarrow{\Delta}^S_C(e)) = (\mathbf{t}, p_A, \gamma_A, \overleftarrow{\Delta}^S_C(e_A)).\overleftarrow{\Delta}^S_C(g_A),$$

which gives: $p' \cdot g = g_A$ (2). ■

PROPOSITION 10. *If* $\mathbf{t}$ *has an E-M object, the pair* (p_A, g_A) *is an adjoint pair* (we shall denote by ν_A the naturalisation, from $g_A \cdot p_A$, to e_A). *If* (p, g) *is an adjoint pair defining* $\mathbf{t}$ *then:* $p' . \nu = \nu_A . p'$.

PROOF. The 2-cell γ_A from $t . p_A = p_A \cdot g_A \cdot p_A$ toward p_A determines a 2-morphism from

$$(\mathbf{t}, p_A, \gamma_A, \overleftarrow{\Delta}^S_C(e_A)).\overleftarrow{\Delta}^S_C(g_A).\overleftarrow{\Delta}^S_C(p_A) \text{ to } (\mathbf{t}, p_A, \gamma_A, \overleftarrow{\Delta}^S_C(e_A))$$

since:

$$\gamma_A \square\square\, p_A \cdot g_A \cdot \gamma_A = \gamma_A \square\square\, t . \gamma_A = \gamma_A \square\square\, \mu . p_A = \gamma_A \square\square\, \gamma_A \cdot g_A \cdot p_A .$$

Therefore there is a unique 2-cell ν_A from $g_A \cdot p_A$ toward e_A satisfying $p_A \cdot \nu_A = \gamma_A$. So

$$p_A \cdot \nu_A \square\square\, \lambda . p_A = \gamma_A \square\square\, \lambda . p_A = p_A$$

and

$$p_A . [\nu_A \cdot g_A \square\square\, g_A . \lambda] = p_A \cdot \nu_A \cdot g_A \square\square\, p_A \cdot g_A . \lambda$$
$$= \gamma_A . g_A \square\square\, t . \lambda = \mu \square\square\, t . \lambda = t = p_A . g_A ;$$

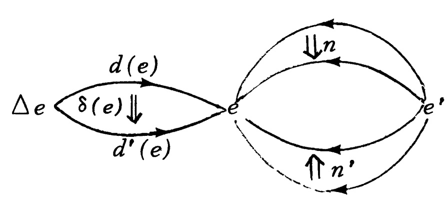
therefore $\nu_A . g_A \square\square\, g_A . \lambda = g_A$. Then (p_A, g_A) is an adjoint pair. At last if (p, q) is an adjoint pair defining $\mathbf{t}$, we get:

$$p_A . (p' . \nu) = p . \nu = \gamma_A \cdot p' = p_A . (\nu_A . p') \text{ and } p' . \nu = \nu_A . p' . \blacksquare$$

Obviously we have opposite results. Proposition 10 and its opposite give back the well known result in $\mathfrak{N}$.

3. Existence of (co)-cata-(ana) limits.

Let us recall two results on representable categories. Let C be a representable 2-category and $\delta(e)$ the 2-cell from $d(e)$ to $d'(e)$ defining Δe. If n and n' are 2-cells from e toward e' and k a 1-morphism,



then:

$n.k = n'.k \iff \delta(e).|n|.k = \delta(e).|n'|.k \iff |n|.k = |n'|.k.$

Therefore, if C is representable and if $C_o^{\square\square}$ admits kernels of pairs we can speak of the kernel of two 2-cells.

It is proved in [9] that, if C is representable, there exist comma objects. In particular the comma object of a 1-morphism f from e toward e' and its codomain e' is given by the pullback of f and $d(e')$, while the comma object of e' and f is given by the pullback of $d'(e')$ and f.

PROPOSITION 11. *If C is a representable 2-category and if $C_o^{\square\square}$ admits limits, the 2-category C admits catalimits and analimits.*(*)

PROOF. Let G be a 2-functor from a 2-category D to C. If f is a 1-morphism of D from e toward e', we will denote by $\partial(f)$ the comma object of $G(f)$ and $G(e')$, by $a(f)$ and $b(f)$ its projections toward $G(e)$ and $G(e')$ and by $c(f)$ the 2-cell from $G(f).a(f)$ to $b(f)$. If n is a 2-cell from f toward f', the 2-cell $c(f') \square\square G(n).a(f')$ determines a unique 1-morphism (denoted by $\partial(n)$) from $\partial(f')$ to $\partial(f)$, such that

$$a(f).\partial(n) = a(f'), \quad b(f).\partial(n) = b(f'),$$

$$c(f).\partial(n) = c(f') \square\square G(n).a(f').$$

In fact, this construction defines a functor ∂ from $D(e',e)^{\square\square *}$ toward $C_o^{\square\square}$.

Let $\bar{D}^{\cdot}$ be the following category: $\bar{D}^{\cdot}_o = D^{\cdot}_o \cup (D^{\cdot}_o \times D^{\cdot}_o)$; for each pair (e', e), there is only one morphism (e, e', e) from (e', e) toward e and only one (e', e', e) from $(e'.e)$ toward e'. Therefore there is no composition except between objects and morphisms. We can define a functor $\bar{d}$ from $\bar{D}^{\cdot}$ to $\mathfrak{N}$ (an «espèce de morphismes» [6]), by: $\bar{D}^{\cdot}(e) = I$ for each object e of D, $\bar{D}^{\cdot}(e', e) = D(e', e)^{\square\square *}$, and $D(e, e', e)$ being the unique functor from $D(e', e)^{\square\square *}$ to I.

Let $P(\bar{d})$ be the 2-cocatalimit («produit croisé» category) of $\bar{d}$. We are going to define a functor $\bar{\bar{d}}$ from $P(\bar{d})$ to $C_o^{\square\square}$ by:

$\bar{\bar{d}}((e', e), n) = \partial(n), \quad \bar{\bar{d}}((e, e', e), f) = a(f), \quad \bar{\bar{d}}((e', e', e), f) = b(f).$

(*) This result is given in [9] for functors toward C whose domain is a usual category.

Let a' be the limit of $\bar{\bar{d}}$ and b the natural transformation from a' to $\bar{\bar{d}}$.

We observe now that, for each pair $(\bar{f}, f)$ of composable 1-morphisms of D ($\bar{f}$ going from e' toward e'', f from e to e'), we have:

$$G(\bar{f}).\,a(\bar{f}).\,b((e'', e'), \bar{f}) = G(\bar{f}).\,b(e') = G(\bar{f}).\,b(f).\,b((e', e), f),$$

so that the 2-cells $c(\bar{f}).\,b((e'', e'), \bar{f})$ and $G(\bar{f}).\,c(f).\,b((e', e), f)$ are composable. Let $k(\bar{f}, f)$ be the kernel of:

$$c(\bar{f}.f).\,b((e'',e), \bar{f}.f) \text{ and } c(\bar{f}).\,b((e'',e'), \bar{f}) \square\!\square\, G(\bar{f}).\,c(f).\,b((e', e), f).$$

Let u be the pullback of all these $k(\bar{f}, f)$ (through $r(\bar{f}, f)$), and $j = k(\bar{f}, f).\,r(\bar{f}, f)$. We assert that u is the catalimit of G, the catadesis from $\overleftarrow{\Delta}^D_C(u)$ to G being defined by:

$$\tau(e) = b(e).\,j, \quad \tau(f) = c(f).\,b((e', e), f).\,j\,.$$

Indeed:

$$\begin{aligned}
\tau(f') \square\!\square\, G(n).\,\tau(e) &= c(f').\,b((e', e), f').\,j \square\!\square\, G(n).\,b(e).\,j \\
&= [\,c(f').\,b((e', e), f') \square\!\square\, G(n).\,b(e)\,]\,.\,j \\
&= (c(f').\,b((e', e), f') \square\!\square\, G(n).\,a(f').\,b((e', e), f')).\,j \\
&= (c(f') \square\!\square\, G(n).\,a(f')).\,b((e', e), f').\,j = c(f).\,\partial(n).\,b((e', e), f').\,j \\
&= c(f).\,b((e', e), f).\,j = \tau(f),
\end{aligned}$$

and

$$\begin{aligned}
\tau(\bar{f}) \square\!\square\, G(\bar{f}).\,\tau(f) &= c(\bar{f}).\,b((e'', e'), \bar{f}).\,j \square\!\square\, G(\bar{f}).\,c(f).\,b((e', e), f).\,j \\
&= (c(\bar{f}).b((e'', e'), \bar{f}) \square\!\square\, G(\bar{f}).\,c(f).\,b((e', e), f)).\,k(\bar{f}, f).\,r(\bar{f}, f) \\
&= c(\bar{f}.f).\,b((e'', e), \bar{f}.f),\,k(\bar{f}, f).\,r(\bar{f}, f) \\
&= c(\bar{f}.f).\,b((e'', e), \bar{f}.f).\,j = \tau(\bar{f}.f).
\end{aligned}$$

Let $(G, \tau', \overleftarrow{\Delta}^D_C(v))$ be a catadesis. For each 1-morphism f of D from e toward e' there is a unique 1-morphism $h'(f)$ in C such that:

$$\tau'(e) = a(f).\,h'(f), \quad \tau'(e') = b(f).\,h'(f), \quad \tau'(f) = c(f).\,h'(f).$$

For each 2-cell n from f to f':

$$\begin{aligned}
c(f).\,\partial(n).\,h'(f') &= (c(f') \square\!\square\, G(n).\,a(f')).\,h'(f') \\
&= c(f').\,h'(f') \square\!\square\, G(n).\,a(f').\,h'(f') = \tau'(f') \square\!\square\, G(n).\,\tau'(e)
\end{aligned}$$

$$= \tau'(f) = c(f) \,.\, h'(f).$$

Whence $\partial(n).h'(f') = h'(f)$. So, if we write $h'(e) = \tau'(e)$, the mapping h' defines a natural transformation from v to $\bar{\bar{d}}$. Whence a unique 1-morphism t' such that:

$$\tau'(e) = h(e).t', \quad h'(f) = h((e', e), f).t'$$

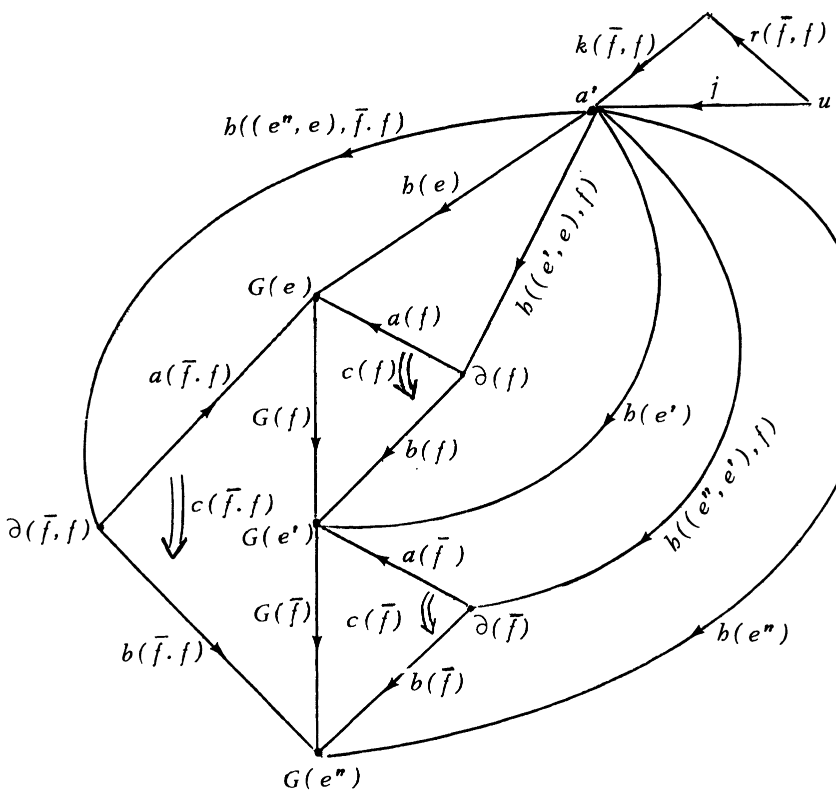
$$(\Longleftrightarrow c(f).h'(f) = c(f).h((e', e), f).t'$$

$$\Longleftrightarrow \tau'(f) = c(f).h((e', e), f).t' \,).$$

Then, for each pair $(\bar{f}, f)$ of composable 1-morphisms of D:

$$c(\bar{f}.f).h((e'', e), \bar{f}.f).t' = \tau'(\bar{f}.f) = \tau'(\bar{f}) \,\square\square\, G(\bar{f}).\tau'(f)$$

$$= c(\bar{f}).h((e'', e'), \bar{f}).t' \,\square\square\, G(\bar{f}).c(f).h((e', e), f).t'$$



$$= (c(\bar{f}).h((e'', e'), \bar{f}) \boxminus G(\bar{f}).c(f).h((e', e), f)).t',$$

and there exists a unique 1-morphism $r'(\bar{f}, f)$ so that $k(\bar{f}, f).r'(\bar{f}, f) = t'$, and therefore a 1-morphism t such that $r'(\bar{f}, f) = r(\bar{f}, f).t$. Clearly we have:

$$\tau'(f) = \tau(f).t \quad \text{and} \quad \tau'(e) = \tau(e).t.$$

Let $\bar{t}$ be another 1-morphism satisfying

$$\tau'(f) = \tau(f).\bar{t} \quad \text{and} \quad \tau'(e) = \tau(e).\bar{t}.$$

We get $\bar{t} = t$, since

$$\bar{t} = t \iff r(\bar{f}, f).\bar{t} = r(\bar{f}, f).t \iff$$

$$k(\bar{f}, f).r(\bar{f}, f).\bar{t} = k(\bar{f}.f).r(\bar{f}, f).t \iff j.\bar{t} = j.t \iff$$

$$h(e).j.\bar{t} = h(e).j.t \text{ and } h((e', e), f).j.\bar{t} = h((e', e), f).j.t \iff$$

$$\tau(e).\bar{t} = \tau(e).t \text{ and } \tau(f).\bar{t} = \tau(f).t,$$

and these last equalities are satisfied. ■

We would prove the existence of analimits in the same way by taking the comma object of $G(e')$ and $G(f)$ instead of the comma object of $G(f)$ and $G(e')$.

REMARK. In [4], it is shown that the condition «$C_o^{\square\square}$ has pullbacks of two morphisms» is sufficient for the existence of limits when the domain of G is the simplicial 2-category S.

COROLLARY. *A 2-functor F from C to C' preserves catalimits and analimits when F preserves the representations and $F_o^{\square\square}$ preserves limits.*

Let i_C be the inclusion functor between $C_o^{\square\square}$ and $C^{\cdot}$.

PROPOSITION 12. *If C is strongly representable, if $C_o^{\square\square}$ admits limits, if i_C preserves these limits, the 2-category C admits 2-analimits and 2-catalimits.*

PROOF. If C is strongly representable, $[\Delta]$ preserves the representation; since $i_C = [\Delta]_o^{\square\square}$, the functor $[\Delta]_o^{\square\square}$ preserves limits and therefore $[\Delta]$ preserves catalimits and analimits. Thus every catalimit (or analimit) is a 2-catalimit (or a 2-analimit). (See remark page 11.) ■

Obviously we have the opposite results:

PROPOSITION 13. *If C is a corepresentable 2-category and if $C_o^{\square\square}$ has colimits, C admits cocatalimits and coanalimits.*

PROPOSITION 14. *If C is strongly corepresentable, if $C_o^{\square\square}$ has colimits and if i_C preserves colimits, C admits 2-cocatalimits and 2-coanalimits.*

4. Tensor product.

Let *2-Cat* be the 2-category whose objects are the 2-categories, 1-morphisms the 2-functors, 2-morphisms the 2-natural transformations (i.e. the **V**-natural transformations where $\mathbf{V} = \mathfrak{F}$, since 2-categories are $\mathfrak{F}$-categories). The 2-category *2-Cat* is 2-cartesian closed. We shall call *3-category* a *2-Cat*-category. It is given by $(C, C^{\cdot}, C^{\square\square})$, where the pairs $(C, C^{\cdot})$, $(C^{\cdot}, C^{\square\square})$ and $(C, C^{\square\square})$ are 2-categories; the first law is the concatenation, the second law is defined by $\cdot$ and the third by $\square\square$.

In this section, F and F' will be 3-functors from a 3-category C toward *2-Cat* (considered as a 3-category).

DEFINITION 1. We say that $t = (F', \tau, F)$ is a *catanatural catadesis* from F to F' if

1° $\tau(e)$ is a 2-functor from $F(e)$ to $F'(e)$ for each object e of C,

2° $\tau(f)$ is a natural catadesis from $F'(f).\tau(e)$ to $\tau(e').F(f)$, for each 1-morphism f from e to e',

3° $\tau(n)$ is a 2-catadesis from $\tau(e')F(n).\tau(f)$ to $\tau(f').F(n)\tau(e)$, for each 2-morphism n of C from f to f',

satisfying:

1° for each 3-morphism q from n to n':

$$(\tau f'.F'q\tau e) \square\square \tau(n) = \tau(n') \square\square (\tau e' F q.\tau f),$$

2° for each pair (g, f) of composable 1-morphisms of C:

$$\tau(gf) = \tau(g)F(f).F'(g)\tau(f),$$

3° for each pair (n', n) of composable 2-morphisms of C:

$$\tau(n'.n) = (\tau(n').F'(n)\tau(e)) \square\square (\tau(e')F(n').\tau(n)),$$

4° for each pair (m, f), where m is a 2-morphism from g to g':

$$\tau(mf)=\tau(m)F(f).F'(g)\tau(f),$$

5° for each pair (g, n):

$$\tau(gn)=(\tau(g)F(f').F'(g)\tau(n))\boxminus(\tau(g)F(n).F'(g)\tau(f)).$$

DEFINITION 12. Let t and t' be catanatural catadeses from F to F'. Then, $d=(t', \delta, t)$ is a *2-catanatural catadesis from t to t'* if:

1° $\delta(e)$ is a catadesis from $\tau(e)$ to $\tau'(e)$, for each object e of C,

2° $\delta(f)$ a 2-catadesis from $\tau'(f).F'(f)\delta(e)$ to $\delta(e')F(f).\tau(f)$ for each 1-morphism f of C from e to e', satisfying for each 2-morphism n from f to f':

$$(\delta(f').F'(n)\tau(e))\boxminus(\tau'(n).F'(f)\delta(e))=$$

$$=(\delta(e')F(f').\tau(n))\boxminus(\delta(e')F(n).\tau(f))\boxminus(\tau'(e')F(n).\delta(f)).$$

We compose two 2-catanatural catadeses as follows:

$$(t'', \delta', \bar{t}').(t', \delta, t)=(t'', \delta'.\delta, t)$$

if and only if $\bar{t}'=t'$, where

$$\delta'.\delta(e)=\delta'(e).\delta(e),$$

$$\delta'.\delta(f)=(\delta'(e')F(f).\delta(f))\boxminus(\delta(f).F'(f)\delta(e)).$$

DEFINITION 13. We say that (d', γ, d) is a *2-catanatural 2-catadesis* if $d=(t', \delta, t)$ and $d'=(t', \delta', t)$ are two 2-catanatural catadeses and if $\gamma(e)$ is a 2-catadesis from $\delta(e)$ to $\delta'(e)$ for each object e of C, satisfying:

$$\delta'(f)\boxminus(\tau'(f).F'(f)\gamma(e))=(\gamma(e')F(f).\tau(f))\boxminus\delta(f).$$

We compose two 2-catanatural catadeses as follows:

$$(d'', \gamma', \bar{d}')\boxminus(d', \gamma, d)=(d'', \gamma'\boxminus\gamma, d),$$

if and only if $\bar{d}'=d'$, where

$$\gamma'\boxminus\gamma(e)=\gamma'(e)\boxminus\gamma(e) \text{ for each object } e \text{ of } C.$$

Then we can define:

$$\bar{d}.(d', \gamma, d)=(\bar{d}.d', \bar{\delta}.\gamma, \bar{d}.d),$$

if and only if $\bar{d}$ and d are composable, where

$$\bar{\delta}.\gamma(e) = \bar{\delta}(e).\gamma(e) \text{ for each object } e \text{ of } C;$$

$$(\bar{d}', \bar{\gamma}, \bar{d}).d = (\bar{d}'.d, \bar{\gamma}.\delta, \bar{d}.d),$$

if and only if $\bar{d}'$ and d are composable, where:

$$\bar{\gamma}.\delta(e) = \bar{\gamma}(e).\delta(e) \text{ for each object } e \text{ of } C.$$

Finally, we prove that

$$(\bar{d}', \bar{\gamma}, \bar{d}).d' \,\square\square\, \bar{d}.(d', \gamma, d) = \bar{d}'.(d', \gamma, d) \,\square\square\, (\bar{d}', \bar{\gamma}, \bar{d}).d.$$

In this way, we equip the set of 2-catanatural catadeses from F to F' with the structure of a 2-category, denoted by $\overset{\Leftarrow}{\mathfrak{N}}(2\text{-}Cat, C)[F', F]$.

We can define in the «opposite» way $\overset{\Rightarrow}{\mathfrak{N}}(2\text{-}Cat, C)[F', F]$.

DEFINITION 14. A *3-strong catalimit of a 3-functor* F from C to *2-Cat* is defined as a 2-category $\hat{F}$ such that, for each 2-category A,

$$\overset{\Leftarrow}{\mathfrak{N}}(2\text{-}Cat, C)\ [F, \hat{A}] \simeq \overset{\leftarrow}{\mathfrak{N}}(\hat{F}, A)$$

(we denote by $\hat{A}$ the 3-functor from C to *2-Cat* constant on A).

PROPOSITION 15. *F admits a 3-strong catalimit* $\hat{F}$.

PROOF. $\hat{F}$ is the 2-category $\overset{\Rightarrow}{\mathfrak{N}}(2\text{-}Cat, C)[F, \hat{1}]$. ∎

EXAMPLE. If C is given by the 2-simplicial category S (the third law being discrete), a 3-functor from C to *2-Cat* is a 2-triple (2-monad). The objects of the 3-strong catalimit are exactly what Burroni called 2-algebras over this 2-triple [5].

DEFINITION 15. A *3-strong cocatalimit of a 3-functor* F from C to *2-Cat* is defined as a 2-category $\hat{\hat{F}}$ such that:

$$\overset{\Leftarrow}{\mathfrak{N}}(2\text{-}Cat, C)[\hat{A}, F] \simeq \overset{\leftarrow}{\mathfrak{N}}(A, \hat{\hat{F}}) \text{ for each 2-category } A.$$

PROPOSITION 16. *F admits a 3-strong cocatalimit.*

PROOF. In general, it is more difficult to prove the existence of colimits than that of limits. It is the same here.

First by adapting the proposition 5 to the remark of the proposition 1 it is sufficient to prove that

$$\overset{\Leftarrow}{\mathfrak{N}}(\mathit{2\text{-}Cat}, C)\,[\hat{A}, F]_0 \simeq \overleftarrow{\mathfrak{N}}(A, \hat{\hat{F}})_0 .$$

Then we apply twice a slight adaptation of the proof of the proposition 13. For that we use the following two facts:

- If F and F' are two 2-functors from C to C' and if $t=(F', \tau, F)$ is a catadesis, there is a 2-functor G such that Gt is a 2-functor, and G has an obvious universal property: this is true, since *2-Cat* admits quasi-quotients [6].

- If H and H' are two 2-functors with the same domain, there is a dual notion of a comma object, relative to catadeses. Indeed, we shall see later (remark, proposition 17) that it is sufficient to prove that, for each 2-category A, there exists a 3-strong cocatalimit for the 3-functor from the 3-category *2* (where the two last laws are discrete) constant on A. And this is not difficult to construct.

We shall denote by $A \otimes B$ the 3-strong cocatalimit of the 3-functor from the 2-category A (considered as a 3-category whose third law is discrete) constant on B.

Therefore we have for each 2-category C:

$$\overset{\Leftarrow}{\mathfrak{N}}(\mathit{2\text{-}Cat}, A)\,[\hat{C}, \hat{B}] \simeq \overleftarrow{\mathfrak{N}}(C, A \otimes B).$$

But it is obvious that

$$\overset{\Leftarrow}{\mathfrak{N}}(\mathit{2\text{-}Cat}, A)\,[\hat{C}, \hat{B}] \simeq \overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C, B), A).$$

Whence

PROPOSITION 17. *There is a tensor product in 2-Cat such that:*

$$\overleftarrow{\mathfrak{N}}(C, A \otimes B) \simeq \overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C, B), A). \ \blacksquare$$

Of course, this isomorphism is natural in C.

REMARK. The restriction to the objects of

$$\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C, B), \mathbf{2}) \simeq \overleftarrow{\mathfrak{N}}(C, \mathbf{2} \otimes B)$$

shows us that a catadesis between 2-functors from B to C is a 2-functor from $\mathbf{2} \otimes B$ to C.

PROPOSITION 18. *This tensor product has a unit and is associative.*

PROOF. It is clear that $1\otimes A\simeq A\simeq A\otimes 1$. Moreover, for each 2-category D,

$$\overleftarrow{\mathfrak{N}}(D,(A\otimes B)\otimes C)\simeq\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(D,C),A\otimes B)\simeq\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(D,C),B),A)$$
$$\simeq\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(D,B\otimes C),A)\simeq\overleftarrow{\mathfrak{N}}(D,A\otimes(B\otimes C)).$$

These isomorphisms being natural in D, we have

$$(A\otimes B)\otimes C\simeq A\otimes(B\otimes C). \blacksquare$$

PROPOSITION 18. $A_*\otimes B_*\simeq(B\otimes A)_*$.

PROOF. $A_*\otimes B_*$ being a 3-strong cocatalimit it is sufficient (see Proposition 16) to show that, for each 2-category C, we have:

$$\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C,B_*),A_*)_o\simeq\overleftarrow{\mathfrak{N}}(C,(B\otimes A)_*)_o.$$

Indeed:

$$\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C,B_*),A_*)_o\simeq\overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C_*,B)_*,A_*)_o$$
$$\simeq\overrightarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C_*,B),A)_{*o}\simeq\overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C_*,B),A)_{*o}$$
$$\simeq\overrightarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C_*,A),B)_{*o}\simeq\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C_*,A),B)_{*o}$$
$$\simeq\overrightarrow{\mathfrak{N}}(C_*,B\otimes A)_{*o}\simeq\overrightarrow{\mathfrak{N}}(C,(B\otimes A)_*)_o$$
$$\simeq\overleftarrow{\mathfrak{N}}(C,(B\otimes A)_*)_o. \blacksquare$$

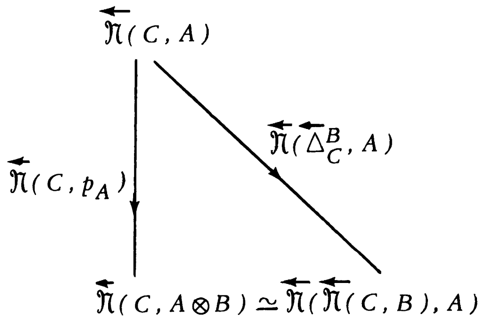
PROPOSITION 20. $\overrightarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C,B),A)\simeq\overrightarrow{\mathfrak{N}}(C,B\otimes A)$.

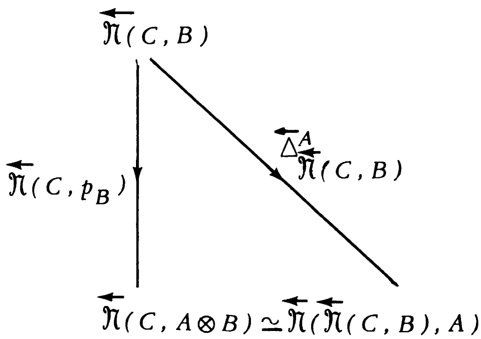
PROOF. We have:

$$\overrightarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C,B),A)\simeq\overleftarrow{\mathfrak{N}}(\overrightarrow{\mathfrak{N}}(C,B)_*,A_*)_*$$
$$\simeq\overleftarrow{\mathfrak{N}}(\overleftarrow{\mathfrak{N}}(C_*,B_*),A_*)_*\simeq\overleftarrow{\mathfrak{N}}(C_*,A_*\otimes B_*)_*$$
$$\simeq\overleftarrow{\mathfrak{N}}(C_*,(B\otimes A)_*)_*\simeq\overrightarrow{\mathfrak{N}}(C,B\otimes A). \blacksquare$$

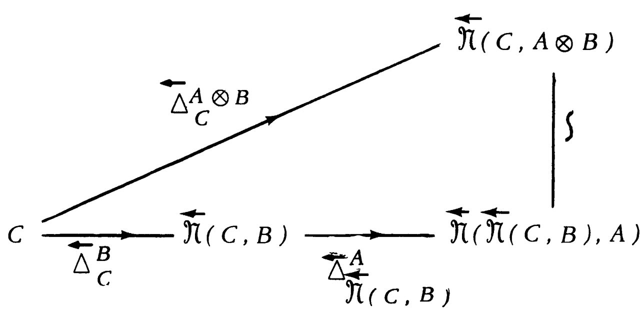
There are projections from $A\otimes B$ to B and to A. The projection p_B from $A\otimes B$ to B is given by the 2-functor from A to $\overleftarrow{\mathfrak{N}}(B,B)$ constant on idB, and the projection p_A from $A\otimes B$ to A by the 2-functor $\overleftarrow{\Delta}{}^B_A$ from A to $\overleftarrow{\mathfrak{N}}(A,B)$.

PROPOSITION 21. *The following diagrams commute:*



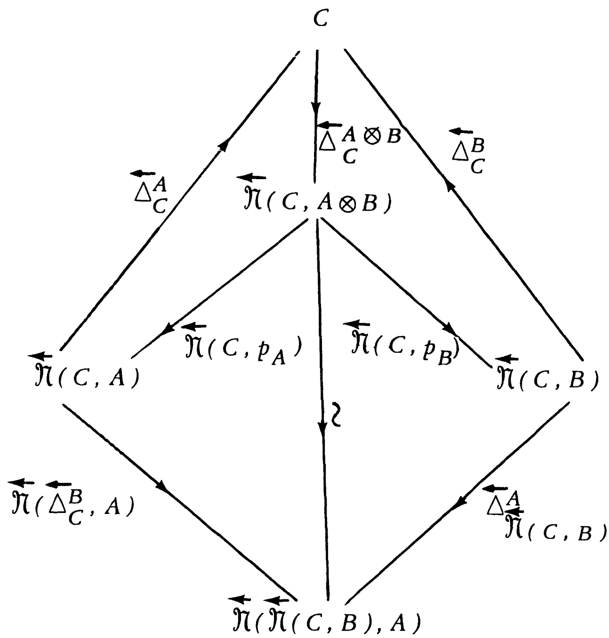


Furthermore it is clear that



commutes. ■

Then we get the following diagram; from it, we deduce all the usual consequences about calculus of limits and commutation between them.



4. 2-cata (ana) - Kan -(co) - extensions.

In a natural way, limits lead to the study of Kan extensions.

Let F be a 2-functor from A to B.

DEFINITION 16. We define a *cata-Kan-extension functor* for each 2-category C as a right adjoint to $\overleftarrow{\mathfrak{N}}(C, F)_o^{\square\square}$ and a 2-cata-Kan-extension functor as a 2-right adjoint to $\overleftarrow{\mathfrak{N}}(C, F)$. A *cata-Kan-coextension functor* will be a left adjoint to $\overleftarrow{\mathfrak{N}}(C, F)_o^{\square\square}$ and a *2-cata-Kan-coextension functor* a 2-left adjoint to $\overleftarrow{\mathfrak{N}}(C, F)$.

DEFINITION 17. We define an *ana-Kan-extension functor* as a right adjoint to $\overrightarrow{\mathfrak{N}}(C, F)_o^{\square\square}$ and a *2-ana-Kan-extension functor* as a 2-right adjoint to $\overrightarrow{\mathfrak{N}}(C, F)$. An *ana-Kan-coextension functor* will be a left adjoint to $\overrightarrow{\mathfrak{N}}(C, F)_o^{\square\square}$ and a *2-ana-Kan-coextension functor* a 2-left adjoint to $\overrightarrow{\mathfrak{N}}(C, F)$.

Even if $C = \mathfrak{N}$, the cata(ana)-Kan-(co)-extensions of a 2-functor may not exist. For instance if we consider $F: 1 \to \mathbf{2}$, where $F(0) = 0$, there is no cata-Kan-coextension. But:

$$\mathfrak{N}(\mathfrak{N},\mathbf{2}) \xhookrightarrow{\;i_2\;} \overleftarrow{\mathfrak{N}}(\mathfrak{N},\mathbf{2}) \xhookrightarrow{\;\overleftarrow{\mathfrak{N}}(\mathfrak{N},F)\;} \overleftarrow{\mathfrak{N}}(\mathfrak{N},1) \simeq \mathfrak{N}$$

has a 2-left adjoint given by the Yoneda lemma. Therefore, we shall call *2-weak cata-Kan-extension functor along* F a 2-right adjoint of:

$$\mathfrak{N}(C,B) \xhookrightarrow{\;i_B\;} \overleftarrow{\mathfrak{N}}(C,B) \xhookrightarrow{\;\overleftarrow{\mathfrak{N}}(C,F)\;} \overleftarrow{\mathfrak{N}}(C,A)$$

and *2-weak cata-Kan-coextension functor* a 2-left adjoint of the same 2-functor.

PROPOSITION 22. *If* C *admits catalimits,* C *admits weak cata-Kan-extensions; if* C *admits 2-catalimits, it admits 2-weak cata-Kan-extensions.*

PROOF. It is sufficient to study the usual proof of existence of usual Kan extensions. This proof proceeds as follows, where A, B, C are ordinary categories.

To the functor F from A to B are associated a functor $[F,\text{-}]$ from B^* to $\mathfrak{N}$ and a natural transformation from $[F,\text{-}]$ to the functor $\hat{A}$ from B^* constant on A. The functor $[F,\text{-}]$ is the following one:

- for each object e of B, the category $[F,e]$ is the comma object of F and $\hat{e}$ (where $\hat{e}$ is the functor from 1 to B defining e),

- for each morphism f of B then $[F,f]$ is the factorization generated by f.

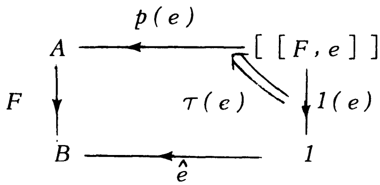
$$\begin{array}{ccc} A & \xleftarrow{\;p(e)\;} & [F,e] \\ {\scriptstyle F}\downarrow & \tau(e)\ \Nwarrow & \downarrow{\scriptstyle 1(e)} \\ B & \xleftarrow[\;\hat{e}\;]{} & 1 \end{array}$$

The natural transformation p is given by the naturalization $p(e)$ of the comma object $[F,e]$. Furthermore, we have:

$$\mathfrak{N}(C,A) \overset{\pi_F}{\simeq} \mathfrak{N}(\mathfrak{N},B^*)[\hat{C},[F,\text{-}]].$$

If C admits limits, there is a functor L from $\mathfrak{N}(\mathfrak{N},B^*)[C,[F,\text{-}]]$ to $\mathfrak{N}(C,B)$ so that $L(\tau)[e] = \lim \tau(e)$ for any natural transformation τ from $[F,\text{-}]$ to $\hat{C}$. Then L is a right adjoint to $\pi_F \cdot \mathfrak{N}(C,F)$.

We give a similar proof for the proposition 22 : we define a 2-functor $[\,[F,-]\,]$ from B^* to *2-Cat*, where $[\,[F,e]\,]$ is the strong comma object of F and $\hat{e}$ (strong means that $\tau(e)$ is a catadesis and satisfies a universal property for catadeses) and a 2-natural transformation from $[\,[F,-]\,]$ to the 2-functor $\hat{A}$ from B^* constant on A, given by the naturalization $p(e)$ of these strong comma objects.



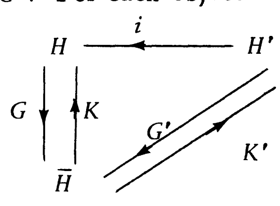
We have also :

$$\overleftarrow{\mathcal{N}}(C,A) \overset{\overline{\pi}_F}{\simeq} \overset{\Leftarrow}{\mathcal{N}}(2\text{-}Cat, B^*)[\hat{C}, [\,[F,-]\,]\,]^{\square},$$

$\overset{\Leftarrow}{\mathcal{N}}(2\text{-}Cat, H)(G', G)^{\square}$ being the full sub-2-category of $\overset{\Leftarrow}{\mathcal{N}}(2\text{-}Cat, H)(G', G)$ whose objects are 2-natural transformations.

In the same way, if C admits 2-catalimits (resp. catalimits) we can define, from $\overset{\Leftarrow}{\mathcal{N}}(2\text{-}Cat, B^*)[\hat{C}, [\,[F,-]\,]\,]^{\square}$ to $\mathcal{N}(C,B)$, a 2-functor L (resp. a functor $L_o^{\square\square}$), taking for $L(\tau)(e)$ the 2-catalimit (resp. the catalimit) of $\tau(e)$ for each 2-natural transformation τ between $[\,[F,-]\,]$ and $\hat{C}$. Then the 2-functor L (resp. the functor $L_o^{\square\square}$) is a 2-right adjoint to $\overline{\pi}_F . \overleftarrow{\mathcal{N}}(C,F) . i_B$ (resp. a right adjoint to the functor $\overline{\pi}_{F\,o}^{\square\square} . \overleftarrow{\mathcal{N}}(C,F)_o^{\square\square} . i_{B\,o}^{\square\square}$). ■

REMARKS. 1° If a 2-cata-Kan-extension functor $\overline{L}$ exists and if G is a 2-functor from A to C, the 2-functor $\overline{L}(G)$ is a sub-2-functor of $L(G)$. From a general remark : let G be a functor from H to $\overline{H}$, K a right adjoint to G, H' a sub-category of H having the same objects than H and K' a right adjoint to $G.i = G'$. For each object $\bar{e}$ of the category $\overline{H}$, we can



consider $K'(\bar{e})$ and the ejection $\pi'(\bar{e})$ from $G'.K'(\bar{e})$ toward $\bar{e}$ and also $K(\bar{e})$ and the ejection $\pi(\bar{e})$ from $G.K(\bar{e})$. The morphism $\pi'(\bar{e})$ determines a morphism $\sigma'(\bar{e})$ in H from $K'(\bar{e})$ to $K(\bar{e})$ so that

$$\pi(\bar{e}).G(\sigma'(\bar{e}))=\pi'(\bar{e}).$$

The morphism $\pi(\bar{e})$ determines a morphism $\sigma(e)$ in H' from $K(\bar{e})$ to $K'(\bar{e})$ so that: $\pi'(\bar{e}).G(\sigma(\bar{e}))=\pi(\bar{e})$. Then the equalities

$$\pi(\bar{e}).G(\sigma'(\bar{e})).G(\sigma(\bar{e}))=\pi'(\bar{e}).G(\sigma(\bar{e}))=\pi(\bar{e})$$

imply that $\sigma(\bar{e})$ is a monomorphism.

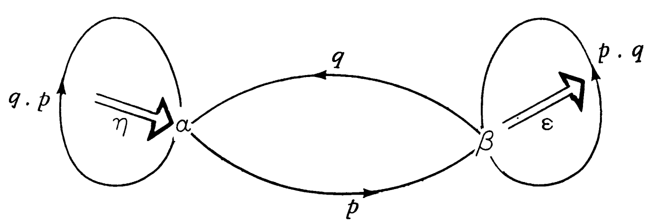
2° A 2-weak cata-Kan-extension of a 2-functor F by itself determines a 2-triple (as in the ordinary case).

APPLICATIONS. 1° If we take $F=idA$, we obtain that, if C admits 2-catalimits, the inclusion 2-functor $\mathfrak{N}(C,A) \xhookrightarrow{i_A} \overleftarrow{\mathfrak{N}}(C,A)$ has a 2-right adjoint. In particular this is true for the inclusion $\mathfrak{N}(\mathfrak{N},A) \hookrightarrow \overrightarrow{\mathfrak{N}}(\mathfrak{N},A)$. For instance the 2-cofree structure associated to the 2-functor from A to $\mathfrak{N}$, constant on 1, is the 2-functor H defined by $H(e)=A/e$, the 2-category of objects over e.

2° (With J. Penon). Let Adj be the 2-category such that a 2-functor from Adj «gives an adjunction»:

- Adj has two objects α and β;
- the 1-morphisms are generated by a pair (p,q) of 1-morphisms, where p goes from α to β and q from β to α;
- the 2-morphisms are generated by a pair (ε,η), where ε goes from β to $p.q$ and η from $q.p$ to α satisfying:

$$\eta q \square\square q\varepsilon = q \quad\text{and}\quad p\eta \square\square \varepsilon p = p.$$



Let H be a 2-category. We denote by $\overline{H}$ the push-out of the diagram:

$$\begin{array}{ccc} & \xrightarrow{\ \mu\ } & \coprod_{H_0} Adj \\ H_0 & & \\ & \longrightarrow & H \end{array}$$

where $\mu(e) = (e, \alpha)$ for each object e of H. The full sub-2-category of $\overline{H}$, whose objects are the (e, β)'s for all the objects e of H, will be denoted by $Relax\, H$. With the notations of R. Street [11], we have:

$$\overleftarrow{Lax}(C, H) \simeq \overleftarrow{\mathfrak{N}}(C, Relax\, H) \text{ and } \overrightarrow{Lax}(C, H) \simeq \overrightarrow{\mathfrak{N}}(C, Relax\, H).$$

Let $\overline{\overline{H}}$ be the push-out of the same diagram, but in which we take $\mu(e) = (e, \beta)$. Then $CoRelax\, H$ will be the full sub-2-category of $\overline{\overline{H}}$ whose objects are the (e, α)'s. If we denote by *Colax* what is defined as *Lax* by M. Bunge,

$$\overleftarrow{Colax}(C, H) \simeq \overleftarrow{\mathfrak{N}}(C, CoRelax\, H) \text{ and } \overrightarrow{Colax}(C, H) \simeq \overrightarrow{\mathfrak{N}}(C, CoRelax\, H).$$

The isomorphism of proposition 1 has obvious consequences here. Let us consider now the two constructions of R. Street [11]. He shows that the obvious 2-functor $\mathfrak{N}(\mathfrak{N}, H) \rightarrow \overleftarrow{Lax}(\mathfrak{N}, H)$ has a 2-right adjoint and that $\mathfrak{N}(\mathfrak{N}, H) \rightarrow Lax(\mathfrak{N}, H)$ has a 2-left adjoint. If we study these constructions, we can describe them in the following way. The 2-right adjoint is given by:

$$\overleftarrow{Lax}(\mathfrak{N}, H) \simeq \overleftarrow{\mathfrak{N}}(\mathfrak{N}, Relax\, H) \xrightarrow{\ \overline{h}\ } \mathfrak{N}(\mathfrak{N}, H);$$

$\overline{h}$ is the 2-weak cata-Kan-extension functor along the 2-functor h from $Relax\, H$ to H given by the identity lax functor $H \rightarrow H$. The 2-left adjoint is:

$$\overrightarrow{Lax}(\mathfrak{N}, H) \simeq \overrightarrow{\mathfrak{N}}(\mathfrak{N}, Relax\, H) \xrightarrow{\ \overline{\overline{h}}\ } \mathfrak{N}(\mathfrak{N}, H);$$

$\overline{\overline{h}}$ is the 2-weak ana-Kan-coextension functor along the same h. Therefore these constructions generalize to the case where we take, instead of $\mathfrak{N}$, a 2-category C admitting 2-catalimits (2-coanalimits).

6. Analgebras and catalgebras.

Gathering example (definition 3) and example 2 (definition 10), we get:

DEFINITION 18. If F is a 2-functor from C to $\mathfrak{N}$, a catadesis from the 2-functor $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(1)$ (from C to $\mathfrak{N}$ constant on 1) to F is called a *catalgebra over* F, and an anadesis from $\overrightarrow{\Delta}^{C}_{\mathfrak{N}}(1)$ to F is called an *analgebra over* F.

PROPOSITION 23. *If for each object e of C, the category $F(e)$ admits limits, the 2-catalimit of F (i.e. the category of catalgebras over F) admits limits.*

PROOF. Let h be a functor from H to the 2-catalimit of F. It determines a catadesis t from $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(H)$ to F, and therefore, for each object e of C a functor $\tau(e)$ from H to $F(e)$. Taking for each e the limit of $\tau(e)$, we construct a catadesis from $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(1)$ to F which is the limit of h in $\overleftarrow{\mathfrak{N}}(\mathfrak{N}, C)[F, \overleftarrow{\Delta}^{C}_{\mathfrak{N}}(1)]$. ∎

We have obviously the dual result:

PROPOSITION 24. *If, for each object e of C, the category $F(e)$ admits colimits, the 2-analimit of F (i.e. the category of analgebras over F) admits colimits.*

These propositions allow, in particular, to find anew the following well known result: if B and B' are categories, the category $\mathfrak{N}(B', B)$ of natural transformations between functors from B to B' admits the same limits than B': indeed $\mathfrak{N}(B', B)$ is at the same time a 2-catalimit of $\overleftarrow{\Delta}^{B}_{\mathfrak{N}}(B')$ and a 2-analimit of $\overrightarrow{\Delta}^{B^*}_{\mathfrak{N}}(B')$.

Catalgebras and analgebras over a 2-functor allow to describe structures (in the same way as algebras over a triple).

EXAMPLES.

1° Let $S_{\sigma^0_1}$ be the sub-2-category of S generated by the unique 2-morphism $u = \sigma^0_1$ between 2 and 1. Let G be the 2-functor to $\mathfrak{N}$ whose domain is $S_{\sigma^0_1}$ defined by:

- $G(0)$ is the category $\mathfrak{M}$ of maps between sets (associated to a universe belonging to $\mathfrak{U}$),
- $G(1)$ is the functor from $\mathfrak{M}$ to $\mathfrak{M}$ «product by 2» (where 2 is the set

with two elements),

- $G(u)$ associates to every set E the projection from $(E\times 2)\times 2$ to $E\times 2$.

The catalgebras over G identify with the maps $b: E \to E\times 2$ satisfying

$$b.\, b\times 2 = b.\, G(u)(E),$$

i.e. with the graphs.

Let us indicate that the analgebras over G, which are defined by the maps $b: E\times 2 \to E$ satisfying

$$b = G(u)\bar{(}E).\, b\times 2.\, b,$$

reduce to the data of a sub-set A of E and of an endomorphism f on E, idempotent and stable on A.

2° Let T be the 2-category with only one object (denoted by 0), genrated by a unique 1-morphism (denoted by 1, then every 1-morphism will be denoted by an integer), a 2-morphism i from 1 to 0 and a 2-morphism u from 2 to 1.

Let G be the 2-functor from T to $\mathfrak{N}$ defined by:

- $G(0) = \mathfrak{M}$, $G(1) = \bar{P}$, where $\bar{P}$ is the endofunctor of $\mathfrak{M}$ defined by:

$$\bar{P}(E) = \{(A, x) \mid A \in \mathcal{P}(E) \text{ and } x \in A\},$$

$\mathcal{P}$ being the «sub-set» functor;

- $G(i)$ associates to every set E, the projection from $\bar{P}(E)$ to E, and $G(u)$ is the map which associates to each pair $(\{(A_i, x_i)\}, (A, x))$ of $\bar{P}^2(E)$ the pair $(\bigcup_i A_i, x)$ of $\bar{P}(E)$.

The analgebras over G, which are defined by the maps b between E and $\bar{P}(E)$ satisfying

$$G(i)(E).\, b = E \quad\text{and}\quad G(u)(E).\, \bar{P}(b).\, b = b,$$

identify with the preordered sets, and the morphisms between analgebras «are» the maps f such that $f(x^<) = f(x)^<$, where $x^<$ is the set of the elements greater than x.

The catalgebras over G «are» obviously the sets (because of the 2-morphism i).

3° Let now G' be the 2-functor from T to $\mathfrak{N}$ defined by:

- $G'(0)=\mathfrak{M}$, $G'(1)=\bar{F}$, where $\bar{F}$ is the endofunctor on $\mathfrak{M}$ defined for every set E by

$$\bar{F}(E)=\{(\phi,x)\mid \phi\in F(E),\ \ x\in A\ \forall A\in\phi\},$$

F being the «filter functor»;

- $G'(i)$ associates to every set E the projection from $\bar{F}(E)$ to E, and $G'(u)(E)$ is the map which associates to every pair $(\Phi,(\phi,x))$ (therefore Φ is a filter on $\bar{F}(E)$) the pair $(\bigvee\Phi,x)$, where X belongs to $\bigvee\Phi$ if and only if there exists an element K of Φ such that X belongs to ϕ' for every pair (ϕ',x') of K.

The analgebras over G' identify with the topologies on E and the morphisms between analgebras «are» the maps f so that $F(f)(\mathcal{V}_x)=\mathcal{V}_{f(x)}$, if $\mathcal{V}_x$ is the filter of neighborhoods of x.

Finally, there is a cotriple on $\bar{P}$ whose counit associates to every set E the projection from $\bar{P}(E)$ to E and whose cocomposition associates to E the map which associates to (A,x) the pair

$$(\{(A,y)\mid y\in A\},(A,x))$$

of $\bar{P}^2(E)$; the analgebras (i.e. the coalgebras) «are» the equivalences.

4° Let us mention without detail that **T**-graphs [5] and pointed **T**-graphs [5] are analgebras over a 2-functor whose domain is a 2-category with two objects.

Conditionned catadeses, catalgebras, catalimits.

Let F be a 2-functor from C to C'. For each object e of C', we can consider the strong comma object $[[F,e]]$ of the 2-functor F and of the 2-functor from 1 to C' defining e. A catadesis τ from $\overleftarrow{\Delta}^{C}_{C'}(e)$ to F determines a 2-functor t from C to $[[F,e]]$. Let ϕ be a 2-functor from Φ to C, having a (usual) limit cone in C, denoted $lim\,\phi$, and whose vertex is $Lim\,\phi$.

DEFINITION 19. We say that τ is a *ϕ-conditionned catadesis* when t preserves the limit of ϕ.

When $C'=\mathfrak{N}$, a catalgebra will be said ϕ-conditionned when the catadesis from $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(1)$ to F defining it is ϕ-conditionned.

DEFINITION 20. A ϕ-conditionned catadesis will be called a *2-ϕ-conditionned catalimit* if it is 2-universal relative to ϕ-conditionned catadeses.

If Θ is a family of 2-functors, we define a *Θ-conditionned* catadesis, catalgebra, catalimit as a catadesis, catalgebra, catalimit which is ϕ-conditionned for every element ϕ of Θ.

EXAMPLE. If $\pi=(C,\Gamma)$ is a projective prototype [1], if $C'=\mathfrak{N}$ and if F is the 2-functor $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(D)$, the Θ-conditionned catalgebras over F are the π-structures in D (where Θ is the set of the bases of the elements of Γ).

REMARK. There is a weaker notion, where C is only a 2-neocategory [1] and where t associates a limit to a distinguished cone in C. The **T**-categories [5] give an example of this weak notion.

Let F be a 2-functor from C to $\mathfrak{N}$.

PROPOSITION 25. *The category of ϕ-conditionned catalgebras over F is the 2-ϕ-conditionned catalimit of F, if the factorization of $F(lim\,\phi)$ through the 2-catalimit of $F.\phi$ has a right adjoint.*

PROOF. Let h be the factorization of $F(lim\,\phi)$ through the 2-catalimit H of $F.\phi$. A catalgebra τ over F determines obviously an object $\hat{\tau}$ in H. The catalgebra τ is ϕ-conditionned if and only if the value of $\tau(Lim\,\phi)$ is a cofree structure, for h, of $\hat{\tau}$.

If τ is a catadesis between $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(D)$ and F, it factorizes through H by a $\hat{\tau}$. Then τ is ϕ-conditionned if and only if the value of $\tau(Lim\,\phi)$ is a cofree structure, for $\mathfrak{N}(h,D)$, of $\hat{\tau}$.

Let $\bar{\tau}$ be the factorization of τ through the 2-catalimit of F. Then if h has a right adjoint, we compute termwise a right adjoint of $\mathfrak{N}(h,D)$ and therefore $\bar{\tau}(s)$ (for each object s of D) is not only a catalgebra but also a ϕ-conditionned catalgebra. ∎

PROPOSITION 26. *The category of ϕ-conditionned catalgebras over F admits limits in the conditions of proposition 23.*

PROOF. It comes from the fact that right adjoints preserve limits. ■

In particular, the category of π-structures (where π is a projective prototype) in a category admitting limits, admits limits.

We can also define ϕ-conditionned anadeses, analgebras, analimits. Thus we obtain :

PROPOSITION 27. *The category of ϕ-conditionned analgebras over F is a 2-ϕ-conditionned analimit of F if the factorization of $F(\lim \phi)$ through the 2-analimit of $F.\phi$ has a left adjoint.*

PROPOSITION 28. *The category of ϕ-conditionned analgebras over F admits colimits in the conditions of proposition 24.*

There is no difficulty to define co-ϕ-conditionned catalgebras (where ϕ has a colimit and t preserves it) and co-ϕ-conditionned analgebras.

Let us remark that if $F = \overrightarrow{\Delta}^{C}_{\mathfrak{N}}(D)$ and if C is a 2-category whose second law is discrete (i.e. an ordinary category), a ϕ-conditionned analgebra over F is a co-ϕ^*-conditionned catalgebra over $F' = \overleftarrow{\Delta}^{C^*}_{\mathfrak{N}}(D)$. Therefore the proposition 28 has for a corollary the (known) result that the category of π-structures (where π is an inductive prototype), in a category which admits colimits, admits colimits.

PROPOSITION 29. *The category of co-ϕ-conditionned catalgebras over F is a 2-co-ϕ-conditionned catalimit of F, if the category $F(coLim\,\phi)$ admits colimits.*

PROOF. A catalgebra τ over F determines the catalgebra $\tau.\phi$ over $F.\phi$. Denote by τ' the colimit cone $colim\,\phi$; then $F.\tau'$ is a catadesis from $F.\phi$ to $\Delta^{\Phi}_{\mathfrak{N}}(F(coLim\,\phi))$. The catadesis $F.\tau' \boxminus \tau.\phi$ from $\Delta^{\Phi}_{\mathfrak{N}}(1)$ to $\Delta^{\Phi}_{\mathfrak{N}}(F(coLim\,\phi))$ determines a 2-functor g from Φ to $F(coLim\,\phi)$. Then τ is co-ϕ-conditionned if and only if the value of $\tau(coLim\,\phi)$ is the colimit of g.

In the same way a catadesis τ between $\Delta^{C}_{\mathfrak{N}}(D)$ and F determines the catadesis $\tau.\phi$ from $\overleftarrow{\Delta}^{\Phi}_{\mathfrak{N}}(D)$ to $F.\phi$. The catadesis $F\tau' \boxminus \tau\phi$ going between $\overleftarrow{\Delta}^{\Phi}_{\mathfrak{N}}(D)$ and $\overleftarrow{\Delta}^{\Phi}_{\mathfrak{N}}(F(coLim\,\phi))$ determines a 2-functor g from Φ to $\mathfrak{N}(F(coLim\,\phi), D)$. Then τ is ϕ-conditionned if and only

$\tau(coLim\,\phi)$ is a colimit of g.

Let $\bar{\tau}$ be the factorization of τ through the 2-catalimit of F. Then if $F(coLim\,\phi)$ admits colimits, we compute termwise a colimit of g and therefore $\bar{\tau}(e)$ (for each object e of D) is not only a catalgebra but also a co-ϕ-conditionned catalgebra. ∎

To have an analogous of proposition 26, we need a commutation between limits and colimits in the category $F(coLim\,\phi)$. In the same way:

PROPOSITION 31. *The category of co-ϕ-conditionned analgebras over F is a 2-co-ϕ-conditionned analimit of F when $F(coLim\,\phi)$ admits limits.*

We have the same remark as after the proposition 28.

The notion of a ϕ-conditionned catalgebra has another interest: it generalizes the usual notion of a limit in a category. If ϕ is the inclusion functor from a category Φ' to the universally associated category with an initial object Φ^+, and if F is the 2-functor $\overleftarrow{\Delta}^{\Phi^+}_{\mathfrak{N}}(D)$, the ϕ-conditionned catalgebras correspond to the data of a functor from Φ to D and of a limit of this functor. If F is not a constant functor we have then a new concept of limits. Moreover the proofs of proposition 25 and 27 show that existence of this kind of limits (colimits) can be looked on as existence of right (left) adjoint as in the usual case.

Co-ϕ-relative catadeses and catalimits.

Let F be a 2-functor from C to C'. If we consider the strong comma object $[[e,F]]$ of $\overleftarrow{\Delta}^{1}_{C'}(e)$ and F, a catadesis τ from F to $\overleftarrow{\Delta}^{C}_{C'}(e)$ determines a 2-functor t from C to $[[e,F]]$. Let ϕ be a 2-functor from Φ to C, having a (usual) colimit in C (which we denote by $colim\,\phi$).

DEFINITION 21. We say that τ is a *co-ϕ-relative catadesis* if t preserves the colimit of ϕ.

DEFINITION 22. A co-ϕ-relative catadesis will be called a *2-co-ϕ-relative cocatalimit* if it is 2-universal for co-ϕ-relative catadeses.

More generally we could define ϕ-relative catadeses (if t preserves the limit of ϕ), 2-ϕ-relative catalimits, co-ϕ-relative and ϕ-relative anadeses, 2-co-ϕ-relative and 2-ϕ-relative analimits.

The situation here is not as good as for ϕ-conditionned catadeses. However we have the two following results:

Let F be a 2-functor from C to $\mathfrak{N}$ and τ be a catadesis from F to $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(D)$. Let H be the 2-cocatalimit of $F.\phi$, and h_1 the factorization of $\tau.\phi$ through H; if τ' is the colimit-cone defining $coLim\,\phi$, let h_2 be the factorization of $F.\tau'$ through H. Then *τ is a co-ϕ-relative catadesis if and only if $\tau(colim\,\phi)$ is a Kan-coextension of h_1 along h_2.*

If τ is always a catadesis from F to $\overleftarrow{\Delta}^{C}_{\mathfrak{N}}(D)$ and if τ'' is the limit-cone $lim\,\phi$, the catadesis $\tau.\phi \,\square\square\, F\tau''$ between $\overleftarrow{\Delta}^{\Phi}_{\mathfrak{N}}(F(Lim\,\phi))$ and $\overleftarrow{\Delta}^{\Phi}_{\mathfrak{N}}(D)$ determines a 2-functor g from Φ to $\mathfrak{N}(D, F(Lim\,\phi))$. Then *$\tau$ is a ϕ-relative catadesis if and only if $\tau(Lim\,\phi)$ is a limit of g.*

Of course, we have dual results concerning ϕ-relative and co-ϕ-relative anadeses.

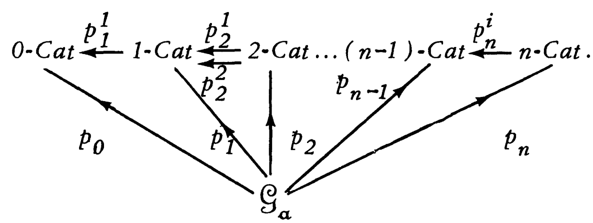
REMARKS ON COHOMOLOGY

Since I began to study catadeses, M[me] Bastiani told me that this notion is connected with the crossed homomorphisms defined by Ehresmann [6] in order to generalize Mac Lane's [10] cohomology constructions from groups to categories. Indeed, given a functor F from a category C to $\mathfrak{N}$, a crossed homomorphism [6] is nothing but a catalgebra over F. This leads to the following remarks about cohomology.

Firstly, let us set $0\text{-}Cat = \mathfrak{M}$, $1\text{-}Cat = Cat$. Denote by $n\text{-}Cat$ the category of the $(n-1)\text{-}Cat$-categories [or simply n-categories] . It is clear that a n-category is nothing but a n-uple $(C_1, C_2, \dots, C_n)$ of categories in which each pair (C_i, C_j) $(i<j)$ is a 2-category. We have, from $n\text{-}Cat$ to $(n\text{-}1)\text{-}Cat$, n interesting forgetful functors p_n^i defined by:

$$p_n^i(C_1, \dots, C_{i-1}, C_i, C_{i+1}, \dots, C_n) = (C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_n)$$

and $p_1^1 = p\mathfrak{F}$. Whence the diagram :



where $\mathcal{G}_a$ is the category of abelian groups. Indeed an abelian group A can be considered as a peculiar 2-category (A, A) (with only one object 0, one 1-morphism 0 and non-trivial 2-morphisms: the other elements). Therefore we can also consider A as the non-trivial (i.e. the n-th law is not discrete) n-category $(A_i)_{1 \leqslant i \leqslant n}$ where $A_i = A$ for each i. Thus this diagram commutes but $\mathcal{G}_a$ is not its limit.

A Π-module A can be looked upon as a functor p_A from the monoïd Π to $\mathcal{G}_a$. The underlying set of the 0-cohomology group

$$H^0(p_A) = \{ a \mid a \in A \text{ and } xa = a \quad \forall x \in \Pi \}$$

is clearly the limit of $p_0 \cdot p_A$ (we shall call its elements the algebras over $p_0 \cdot p_A$). The 1-cocycles of p_A [i.e. the maps from Π to A satisfying $f(xy)=f(x)+xf(y)$] are clearly the catalgebras over $p_1 \cdot p_A$. If f and f' are two catalgebras, a morphism between them is given by a morphism a in A such that

$$f'(x)+xa=a+f(x)$$

(in other words, $f(x)-f'(x)=xa-a$ is a principal crossed homomorphism [10]).
Thus the underlying set of the first cohomology group $H^1(p_A)$ is the set of the components of the catalimit (which is a groupoid) of $p_1 \cdot p_A$.

Let us define now a new kind of morphism between 3-functors by setting a 3-morphism where there is an equality in the definition of a catadesis and by imposing coherence conditions: there are obviously several ways to do it, we choose the best one for our purpose.

Let C and C' be two 3-categories, F and F' two 3-functors from C to C'.

DEFINITION. A *tetradesis from F to F'* will be defined as an element $\mathbf{t}=(F', \tau, F)$, where:

1° $\tau(e)$ is a 1-morphism from $F(e)$ to $F'(e)$, for each object e of C,

2° $\tau(f)$ is a 2-morphism from $F'(f)\tau(e)$ to $\tau(e')F(f)$, for each 1-morphism f from e to e',

3° $\tau(n)$ is a 3-morphism from $\tau(f').F'(n)\tau(e)$ to $\tau(e')F(n).\tau(f)$, for each 2-morphism n between f and f' satisfying the following conditions:

- for each 3-morphism t from n to n':

$$(\tau(e')F(t).\tau(f)) \boxminus \tau(n)=\tau(n') \boxminus (\tau(f').F'(t)\tau(e)),$$

- for each pair $(\bar{n}, n)$ of composable 2-morphisms:

$$\tau(\bar{n}.n)= (\tau(e')F(\bar{n}).\tau(n)) \boxminus (\tau(\bar{n}).F'(n)\tau(e));$$

4° $\tau(g,f)$ is a 3-morphism from $\tau(gf)$ to $\tau(g)F(f).F'(g)\tau(f)$, for each pair (g,f) of composable 1-morphisms, such that:

- for each triple (h,g,f) of composable 1-morphisms the diagram:

$$\begin{array}{ccc}
\tau(hgf) & \xRightarrow{\ \tau(hg,f)\ } & \tau(hg)F(f).F'(hg)\tau(f) \\
{\scriptstyle \tau(h,gf)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \tau(h,g)F(f).F'(hg)\tau(f)} \\
\tau(h)F(gf).F'(h)\tau(gf) & \xRightarrow[\ \tau(h)F(gf).F'(h)\tau(g,f)\ ]{} & \tau(h)F(gf).F'(h)\tau(g)F(f).F'(hg)\tau(f)
\end{array}$$

commutes;

- for each pair (m, f), where m is a 2-morphism from g to g', the diagram

$$\begin{array}{ccc}
\tau(g'f).F'(mf)\tau(e) & \xRightarrow{\ \tau(g',f').F'(mf)\tau(e)\ } & \tau(g')F(f).F'(g')\tau(f).F'(mf)\tau(e) \\
 & & \| \\
{\scriptstyle \tau(mf)}\Big\Downarrow & & \tau(g')F(f).F'(m)\tau(e')F(f).F'(g)\tau(f) \\
 & & \Big\Downarrow{\scriptstyle \tau(m)F(f).F'(g)\tau(f)} \\
\tau(e'')F(mf).\tau(gf) & \xRightarrow[\ \tau(e'')F(mf).\tau(g,f)\ ]{} & \tau(e'')F(mf).\tau(g)F(f).F'(g)\tau(f)
\end{array}$$

commutes;

- for each pair (g, n), where n is a 2-morphism from f to f', there is a similar coherence condition.

DEFINITION. A *2-tetradesis from* $\mathfrak{t}$ *to* $\mathfrak{t}' = (F', \tau, F)$ is defined as an element $d = (\mathfrak{t}', \delta, \mathfrak{t})$, where:

- $\delta(e)$ is a 2-morphism from $\tau(e)$ to $\tau'(e)$, for each object e of C,
- $\delta(f)$ is a 3-morphism from $\tau'(f).F'(f)\delta(e)$ to $\delta(e')F(f).\tau(f)$

for each 1-morphism f from e to e', satisfying the coherence condition:

$$\begin{array}{ccc}
\tau'(f'f).F'(f'f)\delta(e) & \xRightarrow{\ \tau'(f',f).F'(f'f)\delta(e)\ } & \tau'(f')F(f).F'(f')\tau'(f).F'(f'f)\delta(e) \\
 & & \Big\Downarrow{\scriptstyle \tau'(f')F(f).F'(f')\delta(f)} \\
{\scriptstyle \delta(f'f)}\Big\Downarrow & & \tau'(f')F(f).F'(f')\delta(e')F(f).F'(f')\tau(f) \\
 & & \Big\Downarrow{\scriptstyle \delta(f')F(f).F'(f)\tau(f)} \\
\delta(e'')F(f'f).\tau(f'f) & \xRightarrow[\ \delta(e'')F(f'f).\tau(f',f)\ ]{} & \delta(e'')F(f'f).\tau(f')F(f).F'(f')\tau(f)
\end{array}$$

and other ones with 2-morphisms which we will not use here.

There is no difficulty to define 3-tetradeses.

The 2-cocycles of p_A (i.e. the maps from $\Pi \times \Pi$ to A satisfying:

$$f(x, y) + f(xy, z) = x f(y, z) + f(x, y, z)$$

[10]) are the tetralgebras over $p_2 \cdot p_A$. A morphism between two tetralgebras f and f' is given by a map g from Π to A such that $g(1)=0$ (the unique possible 1-morphism of A) and

$$f(x, y) + g(xy) = g(x) + x g(y) + f'(x, y).$$

Thus the underlying set of the second cohomology group $H^2(p_A)$ is the set of the components (for the first law) of the tetralimit (which is a 2-groupoïd there) of $p_2 \cdot p_A$. Therefore it seems probable that the underlying set of each group $H^n(p_A)$ can be looked as the set of the components (for the first law) of a «n-limit» of $p_n \cdot p_A$ (which would be a n-groupoid). Whence two remarks. Given a category C *there is no obstruction to develop an abelian cohomology of C over p* (from C to $\mathcal{G}_a$), since $H^0(p)$, $H^1(p)$, $H^2(p)$ (seen as the set of the components of the limit of $p_0 \cdot p$, the catalimit of $p_1 \cdot p$, the tetralimit of $p_2 \cdot p$) have an obvious abelian group structure. The difficulty of a *non abelian cohomology of C over p* (from C to Cat) appears to be due not only to the lack of good tools (exact sequences for instance) but also to the necessity of having a higher «dimension» for the objects of the codomain of p.

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