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HIGHER ORDER FRAMES AND LINEAR CONNECTIONS

by YUEN Ping Cheng

Introduction.

In the first part of this paper we develop some elementary properties of semi-holonomic k-frames parallel to those of holonomic k-frames. Our definition of a semi-holonomic k-frame is essentially equivalent to the one originally given by Ehresmann [1b]; our formulation, however, leads us easily to define a canonical 1-form θ_k on the principal fibre bundle $\bar{H}^k(V_n)$ of semi-holonomic k-frames on a differentiable manifold V_n. If we restrict θ_k to the principal sub-bundle $H^k(V_n)$ of holonomic k-frames on V_n, we obtain the canonical 1-form given by Kobayashi [3]. Our main result is the «Holonomy Theorem» where we give a geometrical interpretation of the holonomy conditions in terms of the canonical 1-form. This result will be useful for studying the integrability of higher order G-structures. These preliminary results served originally as an introductory part to a forthcoming paper which deals with the structure tensors of higher order regular G-structures and higher order geometric structures.

The second part of this paper deals with the higher order linear connections. Let V_n be a differentiable manifold. A linear connection of order k on V_n is an infinitesimal connection on the principal fibre bundle $\bar{H}^k(V_n)$. Its torsion form is defined to be the exterior covariant derivative of θ_k. There is a one-to-one correspondence between the set of linear connections of order k (resp. quasi-holonomic linear connections of order k without torsion) on V_n and the set of invariant sections of the canonical projection $\bar{H}^{k+1}(V_n) \to H^1(V_n)$. We show further that a linear connection of order k on V_n is locally flat if and only if it can be obtained by successive prolongations of a first order linear connection without torsion and curvature. Some of these results have been summarized in [6] and are prepublished in French, in the first part of the author's thesis [7]. If V_n is a differentiable manifold, $T_x(V_n)$ is the tangent vector space of V_n at x.

Part I

HIGHER ORDER FRAMES

1. Semi-holonomic frames.

Let V_n be an n-dimensional C^∞-differentiable manifold. A *first order frame* (or a 1-frame) of V_n *at the point* x is an invertible 1-jet of $\mathbf{R}^n$ into V_n with source $0 \in \mathbf{R}^n$ and target $x \in V_n$. The manifold of all 1-frames of V_n, denoted by $H^1(V_n)$, forms a principal fibre bundle over V_n with natural projection π^1_0 which assigns to each 1-jet its target, the structure group being $L^1_n = GL(n, \mathbf{R})$. The trivial bundle $H^1(\mathbf{R}^n) \approx \mathbf{R}^n \times L^1_n$ can be identified with the group of all affine transformations on $\mathbf{R}^n$. There is a distinguished element in $H^1(\mathbf{R}^n)$, namely the 1-frame e_1 of $\mathbf{R}^n$ defined by the 1-jet of the identity mapping of $\mathbf{R}^n$ onto $\mathbf{R}^n$ with source 0.

Let $h : H^1(\mathbf{R}^n) \to H^1(V_n)$ be a local isomorphism. It induces a local diffeomorphism f of $\mathbf{R}^n$ into V_n with $f \circ \pi^1_0 = \pi^1_0 \circ h$ (pseudo-products); we will denote all natural projections by the same symbol π with indices. We say that h is *1-admissible* if the domain of h contains e_1 and $h(e_1) = j^1_0 f$. (Here $j^1_0 f$ denotes the 1-jet of f with source 0).

The manifold of 1-jets $j^1_{e_1} h$, where h is a 1-admissible local isomorphism of $H^1(\mathbf{R}^n)$ into $H^1(V_n)$, will be denoted by $\bar{H}^2(V_n)$. There are two natural bundle structures on $\bar{H}^2(V_n)$:

i) $\bar{H}^2(V_n)$ forms a principal fibre bundle over $H^1(V_n)$ with natural projection π^2_1 and structure group $\bar{M}^2_n$ consisting of all 1-jets of 1-admissible local isomorphisms of $H^1(\mathbf{R}^n)$ into $H^1(\mathbf{R}^n)$ with source and target e_1. The structure group $\bar{M}^2_n$ acts on $\bar{H}^2(V_n)$ on the right by the composition of jets. Moreover $\pi^2_1(j^1_{e_1} h) = h(e_1) = j^1_0 f$.

ii) $\bar{H}^2(V_n)$ forms a principal fibre bundle over V_n with projection $\pi^2_0 = \pi^1_0 \circ \pi^2_1$ and structure group $\bar{L}^2_n$. Here $\bar{L}^2_n$ is the fibre of $\bar{H}^2(\mathbf{R}^n)$ over the origin $0 \in \mathbf{R}^n$. The multiplication in $\bar{L}^2_n$ is given by: if $g_1 = j^1_{e_1} h_1 \in \bar{L}^2_n$ and $g_2 = j^1_{e_1} h_2 \in \bar{L}^2_n$, then the pseudo-product $h_1 \circ h_2$ is a 1-admissible local isomorphism and $g_1 \cdot g_2 = j^1_{e_1}(h_1 \circ h_2)$ depends only on $j^1_{e_1} h_1$ and $j^1_{e_1} h_2$. Notice there is again a distinguished element in $\bar{H}^2(\mathbf{R}^n) \approx \mathbf{R}^n \times \bar{L}^2_n$, namely the element e_2 defined by the 1-jet of the identity mapping of $H^1(\mathbf{R}^n)$ onto $H^1(\mathbf{R}^n)$ with source e_1. An element $z \in \bar{H}^2(V_n)$ will be called a *semi-holonomic 2-frame* of V_n at the point $x = \pi^2_0(z)$.

We define by recurrence the principal fibre bundle $\bar{H}^k(V_n)$ of semi-holonomic k-frames of V_n. Let us assume that we have defined the principal fibre bundle $\bar{H}^{k-1}(V_n)$ of semi-holonomic $(k-1)$-frames of V_n, with base space V_n, structure group $\bar{L}^{k-1}_n$ and projection π^{k-1}_{k-2} on $\bar{H}^{k-2}(V_n)$. A local isomorphism $u: \bar{H}^{k-1}(\mathbf{R}^n) \to \bar{H}^{k-1}(V_n)$ is said $(k-1)$-admissible if:

i) v is $(k-2)$-admissible, where v is the local isomorphism of $\bar{H}^{k-2}(\mathbf{R}^n)$ into $\bar{H}^{k-2}(V_n)$ induced by u, such that $v \circ \pi^{k-1}_{k-2} = \pi^{k-1}_{k-2} \circ u$.

ii) $u(e_{k-1}) = j^1_{e_{k-2}} v$, where e_{k-1} (resp. e_{k-2}) is the distinguished element in $\bar{H}^{k-1}(\mathbf{R}^n)$ (resp. $\bar{H}^{k-2}(\mathbf{R}^n)$).

The set $\bar{H}^k(V_n)$ of 1-jets of the form $j^1_{e_{k-1}} u$, where u is a $(k-1)$-admissible local isomorphism of $\bar{H}^{k-1}(\mathbf{R}^n)$ into $\bar{H}^{k-1}(V_n)$, forms a principal fibre bundle over V_n with structure group $\bar{L}^k_n$; the underlying set of $\bar{L}^k_n$ is just the fibre of $\bar{H}^k(\mathbf{R}^n)$ over $0 \in R^n$. The space $\bar{H}^k(V_n)$ can also be regarded as a principal fibre bundle over $\bar{H}^{k-1}(V_n)$ with structure group $\bar{M}^k_n = Ker(\bar{L}^k_n \to \bar{L}^{k-1}_n)$. An element z of $\bar{H}^k(V_n)$ will be called a *semi-holonomic k-frame* of V_n at the point x, where x is the projection of z into V_n.

For $m < k$, the natural projection π^k_m of $\bar{H}^k(V_n)$ onto $\bar{H}^m(V_n)$ is compatible with the surjective homomorphism of $\bar{L}^k_n$ onto $\bar{L}^m_n$. The distinguished element e_k in $\bar{H}^k(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}^k_n$ is defined by the 1-jet of the identity mapping of $\bar{H}^{k-1}(\mathbf{R}^n)$ with source e_{k-1}.

2. Canonical form on $\bar{H}^k(V_n)$.

An element $u \in \bar{H}^k(V_n)$ can be written as $u = j^1_{e_{k-1}} h$, where h is a $(k-1)$-admissible local isomorphism of $\bar{H}^{k-1}(\mathbf{R}^n)$ into $\bar{H}^{k-1}(V_n)$; it determines a linear isomorphism $\tilde{u}$ of $\bar{E}^{k-1} = T_{e_{k-1}}(\bar{H}^{k-1}(\mathbf{R}^n))$ onto $T_{u'}(\bar{H}^{k-1}(V_n))$ with $u' = \pi^k_{k-1}(u) \in \bar{H}^{k-1}(V_n)$. Since $\bar{H}^{k-1}(\mathbf{R}^n) \approx \mathbf{R}^n \times \bar{L}^{k-1}_n$, we have a canonical decomposition $\bar{E}^{k-1} = \mathbf{R}^n \oplus \bar{\mathfrak{L}}^{k-1}_n$, where $\bar{\mathfrak{L}}^{k-1}_n$ is the Lie algebra of $\bar{L}^{k-1}_n$. From now on, we will identify $\mathbf{R}^n$ with a vector subspace of $\bar{E}^{k-1}$ given by the canonical decomposition. Since $\tilde{u}$ is a linear isomorphism, $\tilde{u}(\mathbf{R}^n)$ is an n-dimensional vector subspace of $T_{u'}(\bar{H}^{k-1}(V_n))$

transversal to the fibres, called the *horizontal n-plane* associated to the k-frame u.

Let v be the projection of u under π^k_m. The following diagram

$$\begin{array}{ccc} \bar{E}^{k-1} & \xrightarrow{\tilde{u}} & T_{u'}(\bar{H}^{k-1}(V_n)) \\ \downarrow & & \downarrow \\ \bar{E}^{m-1} & \xrightarrow{\tilde{v}} & T_{v'}(\bar{H}^{m-1}(V_n)) \end{array}$$

is commutative, where v' is the projection of v under π^m_{m-1} and where the vertical arrows are the natural projections.

Consider a vector $Z \in T_u(\bar{H}^k(V_n))$. Its image $Z' = T\pi^k_{k-1}(Z)$ under the tangential map $T\pi^k_{k-1}$ is tangent to $\bar{H}^{k-1}(V_n)$ at the point $u' = \pi^k_{k-1}(u)$.

The $\bar{E}^{k-1}$-valued differential 1-form θ_k defined by

$$\theta_k(Z) = \tilde{u}^{-1}(T\pi^k_{k-1}(Z))$$

will be called *the canonical form* on $\bar{H}^k(V_n)$. For $m < k$, we have the following commutative diagram

$$\begin{array}{ccc} T(\bar{H}^k(V_n)) & \xrightarrow{\theta_k} & \bar{E}^{k-1} \\ \downarrow & & \downarrow \\ T(\bar{H}^m(V_n)) & \xrightarrow{\theta_m} & \bar{E}^{m-1} \end{array}$$

where the vertical arrows are the natural projections.

The Lie group $\bar{L}^k_n$ acts naturally on $\bar{E}^{k-1}$ on the left. Each element g of $\bar{L}^k_n$ defines a linear isomorphism $\tilde{g}$ of $\bar{E}^{k-1}$ onto $T_{g'}(\bar{H}^{k-1}(\mathbf{R}^n))$ with $g' = \pi^k_{k-1}(g)$. The right translation $R^{-1}_{g'} = R_{(g')^{-1}}$ determines a linear isomorphism $TR^{-1}_{g'}$ of $T_{g'}(\bar{H}^{k-1}(\mathbf{R}^n))$ onto $\bar{E}^{k-1}$. If we put $\rho(g) = TR^{-1}_{g'} \circ \tilde{g}$, we obtain a linear representation ρ of $\bar{L}^k_n$ on the vector space $\bar{E}^{k-1}$. For $m < k$,

$$\begin{array}{ccc} \bar{E}^{k-1} & \xrightarrow{\rho(g)} & \bar{E}^{k-1} \\ \downarrow & & \downarrow \\ \bar{E}^{m-1} & \xrightarrow{\rho(\pi^k_m(g))} & \bar{E}^{m-1} \end{array}$$

is a commutative diagram, where the vertical arrows are the natural projections.

PROPOSITION I.1. *The canonical form* θ_k *is a pseudo-tensorial 1-form on* $\bar{H}^k(V_n)$ *of type* $(\rho, \bar{E}^{k-1})$, *i.e.*

$$\theta_k(TR_g(Z)) = \rho(g^{-1})\theta_k(Z)$$

for all $Z \in T(\bar{H}^k(V_n))$ *and* $g \in \bar{L}^k_n$.

3. Holonomic Frames.

A diffeomorphism $f: V_n \to V'_n$ induces a principal fibre bundle isomorphism $f^{(k)}$ of $\bar{H}^k(V_n)$ onto $\bar{H}^k(V'_n)$. This isomorphism $f^{(k)}$ possesses the following properties:

i) $\pi^k_m \circ f^{(k)} = f^{(m)} \circ \pi^k_m$ for all $0 \leqslant m < k$;

ii) $f^{(k)}$ is compatible with the canonical forms, i.e. $f^{(k)*}\theta'_k = \theta_k$, where θ_k (resp. θ'_k) is the canonical form on $\bar{H}^k(V_n)$ (resp. $\bar{H}^k(V'_n)$).

THEOREM I.2. *Let* ϕ *be a local diffeomorphism of* $\bar{H}^k(V_n)$ *into* $\bar{H}^k(V'_n)$. *Then locally* $\phi = f^{(k)}$ *for some local diffeomorphism* f *of* V_n *into* V'_n, *if and only if* ϕ *is compatible with the canonical forms, i.e.* $\phi^*\theta'_k = \theta_k$.

It remains to show that the condition is sufficient. For this we will proceed by induction on k.

LEMMA I.3. *Let* ϕ *be a local diffeomorphism of* $H^1(V_n)$ *into* $H^1(V'_n)$ *with* $\phi^*\theta'_1 = \theta_1$. *Then we can locally write* $\phi = f^{(1)}$ *for some local diffeomorphism* f *of* V_n *into* V'_n.

Consider a tangent vector $Z \in T_u(H^1(V_n))$ with $T\pi^1_0(Z) = 0$. The condition $\phi^*\theta'_1 = \theta_1$ implies that $T\pi^1_0(T\phi(Z)) = 0$. Thus ϕ sends a tangent space to the fibre of $H^1(V_n)$ onto a tangent space to the fibre of $H^1(V'_n)$. This means that locally ϕ is a fibre map and induces a map f of V_n into V'_n satisfying $f \circ \pi^1_0 = \pi^1_0 \circ \phi$. We want to show that $\phi = f^{(1)}$. Thus we want to show that for any u with $\pi^1_0(u) = x$ we have $\phi(u) = j^1_x f \circ u$. Let $\xi \in \mathbf{R}^n$. Choose a vector $Z \in T_u(H^1(V_n))$ with $T\pi^1_0(Z) = \tilde{u}(\xi)$. Then $(j^1_x f \circ \tilde{u})(\xi) = (Tf \circ \tilde{u})(\xi) = (Tf \circ T\pi^1_0)(Z) = (T\pi^1_0 \circ T\phi)(Z)$. On the other hand, $(\widetilde{\phi(u)}^{-1} \circ T\pi^1_0 \circ T\phi)(Z) = (\tilde{u}^{-1} \circ T\pi^1_0)(Z) = \xi$. Thus

$\phi(u)=j^1_x f\circ u$ holds proving the lemma.

To prove the theorem for k we may assume that it has been established for $k-1$. Let $Z\in T_u(\bar{H}^k(V_n))$ with $T\pi^k_{k-1}(Z)=0$. The condition $\phi^*\theta'_k=\theta_k$ implies $(T\pi^k_{k-1}\circ T\phi)(Z)=0$. Thus ϕ is a local fibre map with respect to the fibrations $\bar{H}^k(V_n)\to\bar{H}^{k-1}(V_n)$ and $\bar{H}^k(V'_n)\to\bar{H}^{k-1}(V'_n)$. There exists a local diffeomorphism ψ of $\bar{H}^{k-1}(V_n)$ into $\bar{H}^{k-1}(V'_n)$ such that $\psi\circ\pi^k_{k-1}=\pi^k_{k-1}\circ\phi$. Since $\pi^{k*}_{k-1}\theta_{k-1}=T\pi^{k-1}_{k-2}\circ\theta_k$ (resp. $\pi^{k*}_{k-1}\theta'_{k-1}=T\pi^{k-1}_{k-2}\circ\theta'_k$), we have

$$\begin{aligned}(\pi^{k*}_{k-1}(\psi^*\theta'_{k-1}))(Z)&=(\theta'_{k-1}\circ T\psi\circ T\pi^k_{k-1})(Z)\\&=(\theta'_{k-1}\circ T\pi^k_{k-1}\circ T\phi)(Z)\\&=(\pi^{k*}_{k-1}\theta'_{k-1})(T\phi(Z))\\&=(T\pi^{k-1}_{k-2}\circ\theta'_k\circ T\phi)(Z)\\&=(T\pi^{k-1}_{k-2}\circ\theta_k)(Z)\\&=(\pi^{k*}_{k-1}\theta_{k-1})(Z)\end{aligned}$$

for all $Z\in T_u(\bar{H}^k(V_n))$. As π^k_{k-1} is surjective, we deduce that $\psi^*\theta'_{k-1}=\theta_{k-1}$. By the induction hypothesis, there exists a local diffeomorphism f of V_n into V'_n such that locally $\psi=f^{(k-1)}$. We have thus $f^{(k-1)}\circ\pi^k_{k-1}=\pi^k_{k-1}\circ\phi$ locally. Now we are going to show that locally $\phi=f^{(k)}$. An element $u\in\bar{H}^k(V_n)$ determines a linear isomorphism $\tilde{u}:\bar{E}^{k-1}\to T_{u'}(\bar{H}^{k-1}(V_n))$ with $u'=\pi^k_{k-1}(u)$. Two elements u and v of $\bar{H}^k(V_n)$ are identical if and only if $\tilde{u}=\tilde{v}$. It suffices therefore to show that $\widetilde{\phi(u)}=\widetilde{f^{(k)}(u)}$ for all $u\in\bar{H}^k(V_n)$. Let $\xi\in\bar{E}^{k-1}$. Choose a tangent vector $Z\in T_u(\bar{H}^k(V_n))$ with $\theta_k(Z)=\xi$. We have

$$\theta_k(Z)=(\phi^*\theta'_k)(Z)=(\theta'_k\circ T\phi)(Z)=(\widetilde{\phi(u)})^{-1}\circ T\pi^k_{k-1}\circ T\phi)(Z).$$

On the other hand, $\xi=\theta_k(Z)=(\tilde{u}^{-1}\circ T\pi^k_{k-1})(Z)$. It follows that for all $\xi\in\bar{E}^{k-1}$,

$$\begin{aligned}\widetilde{\phi(u)}(\xi)&=(T\pi^k_{k-1}\circ T\phi)(Z)=(Tf^{(k-1)}\circ T\pi^k_{k-1})(Z)\\&=(Tf^{(k-1)}\circ\tilde{u})(\xi)=\widetilde{f^{(k)}(u)}(\xi).\end{aligned}$$

We have therefore $\phi=f^{(k)}$ locally and our theorem is proved.

COROLLARY I.4. *Let ϕ be a principal fibre bundle isomorphism of $\bar{H}^k(V_n)$ onto $\bar{H}^k(V_n')$. Let f be the diffeomorphism of V_n onto V_n' induced by ϕ. Then $\phi = f^{(k)}$ if and only if $\phi^*\theta_k' = \theta_k$.*

Consider a local diffeomorphism f of an open neighbourhood of $0 \in \mathbf{R}^n$ onto an open set of V_n. It induces a $(k-1)$-admissible local isomorphism $f^{(k-1)}: \bar{H}^{k-1}(\mathbf{R}^n) \to \bar{H}^{k-1}(V_n)$. It follows that $u = j^1_{e_{k-1}} f^{(k-1)}$ is an element of $\bar{H}^k(V_n)$. We say that $u \in \bar{H}^k(V_n)$ is a *holonomic k-frame* of V_n if u can be written as $u = j^1_{e_{k-1}} f^{(k-1)}$ for some local diffeomorphism f of $\mathbf{R}^n$ into V_n. A k-frame u of V_n is holonomic if and only if one can find a representative for u compatible with the canonical forms. The set of holonomic k-frames of V_n forms a principal fibre subbundle $H^k(V_n)$ of $\bar{H}^k(V_n)$. Its structure group is the subgroup L^k_n of $\bar{L}^k_n$ consisting of holonomic elements. Notice there is a group isomorphism between L^k_n and the group of all invertible k-jets of $\mathbf{R}^n$ into $\mathbf{R}^n$ with source and target 0. The space $H^k(V_n)$ can also be regarded as a principal fibre bundle over $H^{k-1}(V_n)$ with structure group $M^k_n = \bar{M}^k_n \cap L^k_n$, kernel of the surjective homomorphism $L^k_n \to L^{k-1}_n$.

4. Relations between $\bar{H}^k(V_n)$, $\bar{P}^k(V_n)$ and $\bar{J}^{k-1}(H^1(V_n))$.

Let W and Y be two C^∞-differentiable manifolds. We will denote by $\bar{J}^k(W, Y)$ the differentiable manifold of semi-holonomic k-jets of W into Y. For the definition of semi-holonomic jets, see the works of Ehresmann. For $m < k$, let p^k_m be the canonical projection of $\bar{J}^k(W, Y)$ onto $\bar{J}^m(W, Y)$. A jet $X \in \bar{J}^k(W, Y)$ is invertible if and only if $p^k_1(X)$ is invertible. Let $\bar{\Pi}^k(W, Y)$ denote the set of invertible jets in $\bar{J}^k(W, Y)$. This set is then the inverse image of $\bar{\Pi}^1(W, Y)$ by the submersion p^k_1. Since $\bar{\Pi}^1(W, Y)$ is an open submanifold of $\bar{J}^1(W, Y) \equiv J^1(W, Y)$, it follows that $\bar{\Pi}^k(W, Y)$ is an open submanifold of $\bar{J}^k(W, Y)$. Moreover, $p^k_m : \bar{\Pi}^k(W, Y) \to \bar{\Pi}^m(W, Y)$ is a submersion.

A semi-holonomic k-frame (resp. holonomic k-frame) of V_n in the sense of Ehresmann is an invertible semi-holonomic k-jet (resp. invertible holonomic k-jet) of $\mathbf{R}^n$ into V_n with source $0 \in \mathbf{R}^n$. The set $\bar{P}^k(V_n)$ (resp.

$P^k(V_n)$) of semi-holonomic k-frames (resp. holonomic k-frames) of V_n in the sense of Ehresmann has a principal fibre bundle structure over V_n, the structure group being the group of all invertible semi-holonomic k-jets (resp. holonomic k-jets) of $\mathbf{R}^n$ into $\mathbf{R}^n$ with source and target $0 \in \mathbf{R}^n$. An element $u \in \bar{P}^k(V_n)$ can then be written as $u = j_0^1 f$, where f is a differentiable mapping of $\mathbf{R}^n$ into $\bar{P}^{k-1}(V_n)$ satisfying the condition:

$$j_0^1(p_{k-2}^{k-1} \circ f) = f(0).$$

Here we have also denoted by p_{k-2}^{k-1} the canonical projection of $\bar{P}^{k-1}(V_n)$ onto $\bar{P}^{k-2}(V_n)$.

THEOREM I.5. *There exists a canonical diffeomorphism ν_k of $\bar{H}^k(V_n)$ onto $\bar{P}^k(V_n)$ satisfying the properties:*

(1) ν_k is a fibre map, i.e. $p_0^k \circ \nu_k = \pi_0^k$;

(2) for $m < k$,

$$\begin{array}{ccc} \bar{H}^k(V_n) & \xrightarrow{\nu_k} & \bar{P}^k(V_n) \\ \pi_m^k \downarrow & & \downarrow p_m^k \\ \bar{H}^m(V_n) & \xrightarrow{\nu_m} & \bar{P}^m(V_n) \end{array}$$

is a commutative diagram;

(3) ν_k, restricted to $H^k(V_n)$, is a diffeomorphism of $H^k(V_n)$ onto $P^k(V_n)$.

We prove the theorem by induction on k. For $k=1$, $H^1(V_n)$ is identical with $P^1(V_n)$ and ν_1 is just the identity map. Let $u = j_{e_1}^1 h$ be an arbitrary element in $\bar{H}^2(V_n)$. If η_1 denotes the «zero section»* of $H^1(\mathbf{R}^n)$ the mapping $u \to \nu_2(u) = j_0^1(\nu_1 \circ h \circ \eta_1)$ defines a diffeomorphism of $\bar{H}^2(V_n)$ onto $\bar{P}^2(V_n)$, because the composition of jets is a differentiable map. Let us assume there exists ν_{k-1} such that, for all $z \in \bar{H}^{k-1}(V_n)$, $\nu_{k-1}(z) = (j_{z'}^1 \nu_{k-1}) \circ z \circ (j_0^1 \eta_{k-2})$ where $z' = \pi_{k-2}^{k-1}(z)$ and η_{k-2} is the «zero section» of the trivial bundle $\bar{H}^{k-2}(\mathbf{R}^n) \approx \mathbf{R}^n \times \bar{L}_n^{k-2}$. Consider then an arbitrary element $y = j_{e_{k-1}}^1 g$ in $\bar{H}^k(V_n)$. If η_{k-1} is the «zero section» of $\bar{H}^{k-1}(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}_n^{k-1}$, $g' = \nu_{k-1} \circ g \circ \eta_{k-1}$ defines a local diffeomorphism of $\mathbf{R}^n$ to $\bar{P}^{k-1}(V_n)$. Since $j_0^1(p_{k-2}^{k-1} \circ g') = g'(0)$, the 1-jet $j_0^1 g'$, which

*) corresponding to $\mathbf{R}^n \times \{e\}$, where e is the unit element.

is independent of the choice of g for y, is an element in $\bar{P}^k(V_n)$. The mapping $y \to \nu_k(y) = j_0^1 g'$ defines a diffeomorphism ν_k of $\bar{H}^k(V_n)$ onto $\bar{P}^k(V_n)$. It is easy to check that ν_k has the desired properties.

Consider the case where $V_n = \mathbf{R}^n$. Let us recall that the underlying set of $\bar{L}_n^k$ is just the fibre of $\bar{H}^k(\mathbf{R}^n)$ over the origin 0. Since the multiplication in $\bar{L}_n^k$ is given by the composition of jets, the restriction of ν_k to $\bar{L}_n^k$ defines a group isomorphism of $\bar{L}_n^k$ onto the group of all invertible semi-holonomic k-jets of $\mathbf{R}^n$ into $\mathbf{R}^n$ with source and target 0. It is easy to see that the diffeomorphism ν_k of the above theorem is compatible with this group isomorphism. We have therefore the following corollary:

COROLLARY I.6. *The principal fibre bundle $\bar{H}^k(V_n)$ (resp. $H^k(V_n)$) is canonically isomorphic to $\bar{P}^k(V_n)$ (resp. $P^k(V_n)$).*

Let E be a locally trivial fibre bundle over V_n. We will denote by $J^k E$ the differentiable manifold of k-jets of local sections of E. Let $\tilde{J}^2 E = J^1(J^1 E)$. The k-th non-holonomic prolongation of E is defined by induction:

$$\tilde{J}^k E = J^1(\tilde{J}^{k-1} E).$$

We define also the semi-holonomic prolongation $\bar{J}^k E$ by restricting ourselves to those local sections such that, for all $0 < m < k$, the local section σ of V_n into $\tilde{J}^m E$ satisfies the condition: $j_x^1(\pi_{m-1}^m \circ \sigma) = \sigma(x)$, where π_{m-1}^m is the natural projection of $\tilde{J}^m E$ onto $\tilde{J}^{m-1} E$. We have

$$J^k E \subset \bar{J}^k E \subset \tilde{J}^k E.$$

THEOREM I.7. *There exists a canonical diffeomorphism μ_k of $\bar{H}^k(V_n)$ onto $\bar{J}^{k-1}(H^1(V_n))$ satisfying the following properties:*

(1) for $k = 1$, μ_1 is just the identity map of $H^1(V_n)$;

(2) μ_k is a fibre map; more explicitly

$$\begin{array}{ccc} \bar{H}^k(V_n) & \xrightarrow{\mu_k} & \bar{J}^{k-1}(H^1(V_n)) \\ \downarrow & & \downarrow \\ V_n & \xrightarrow{id} & V_n \end{array}$$

is a commutative diagram;

(3) for $0 < m < k$, the following diagram

$$\begin{array}{ccc} \bar{H}^k(V_n) & \xrightarrow{\mu_k} & \bar{J}^{k-1}(H^1(V_n)) \\ \downarrow & & \downarrow \\ \bar{H}^m(V_n) & \xrightarrow{\mu_m} & \bar{J}^{m-1}(H^1(V_n)) \end{array}$$

commutes.

We prove the theorem by induction on k. For $k=1$, $J^0(H^1(V_n)) = H^1(V_n)$ by definition and μ_1 is just the identity map of $H^1(V_n)$. Let $u = j^1_{e_1} h$ be an arbitrary element of $\bar{H}^2(V_n)$. Consider the local diffeomorphism f of $\mathbf{R}^n$ into V_n defined by the condition: $\pi^1_0 \circ h = f \circ \pi^1_0$. If η_1 is the «zero section» of $H^1(\mathbf{R}^n) \approx \mathbf{R}^n \times L^1_n$, the mapping

$$x \to \sigma(x) = h \circ \eta_1 \circ f^{-1}(x)$$

defines a local section σ of V_n into $H^1(V_n)$. If we put $\mu_2(u) = j^1_x \sigma$ with $x = \pi^1_0(u)$, the mapping $u \to \mu_2(u)$ defines an injection of $\bar{H}^2(V_n)$ into $\bar{J}^1(H^1(V_n))$. This differentiable mapping μ_2 is surjective. In fact let σ be a local section of V_n into $H^1(V_n)$ with $j^1_x \sigma \in \bar{J}^1(H^1(V_n))$. The target $\sigma(x)$ can be written as $\sigma(x) = j^1_0 f$ for some local diffeomorphism f of $\mathbf{R}^n$ into V_n. Let h be the local isomorphism of $H^1(\mathbf{R}^n)$ into $H^1(V_n)$ defined by the conditions:

i) $\pi^1_0 \circ h = f \circ \pi^1_0$;

ii) $h \circ \eta_1 = \sigma \circ f$.

It is easy to check that h is 1-admissible and $j^1_x \sigma = \mu_2(j^1_{e_1} h)$. The mapping μ_2 gives then a diffeomorphism of $\bar{H}^2(V_n)$ onto $\bar{J}^1(H^1(V_n))$ with the desired properties. Now, let us assume there exists μ_{k-1} and μ_{k-2} such that, for all $u \in \bar{H}^{k-1}(V_n)$, we have

$$\mu_{k-1}(u) = (j^1_{u'} \mu_{k-2}) \circ u \circ (j^1_0 \eta_{k-2}) \circ \omega^{-1}$$

with $u' = \pi^{k-1}_{k-2}(u)$, $\omega = \pi^k_1(u)$ and where η_{k-2} is the «zero section» of $\bar{H}^{k-2}(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}^{k-2}_n$. Let $z = j^1_{e_{k-1}} h$ be an arbitrary element of $\bar{H}^k(V_n)$. Let f be the local diffeomorphism of $\mathbf{R}^n$ into V_n induced by h. If we denote by η_{k-1} the «zero section» of $\bar{H}^{k-1}(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}^{k-1}_n$, then

$$h' = \mu_{k-1} \circ h \circ \eta_{k-1} \circ f^{-1}$$

defines a local section of V_n into $\bar{J}^{k-2}(H^1(V_n))$ and $j^1_x h'$ determines an element $\mu_k(z)$ of $\bar{J}^{k-1}(H^1(V_n))$ independent of the choice of the representative h for z. It is easy to verify that $z \to \mu_k(z)$ defines a diffeomorphism μ_k of $\bar{H}^k(V_n)$ onto $\bar{J}^{k-1}(H^1(V_n))$ satisfying the required conditions of the theorem.

COROLLARY I.8 [4c] *$\bar{P}^k(V_n)$ and $\bar{J}^{k-1}(H^1(V_n))$ are canonically diffeomorphic.*

5. Local coordinate systems in $\bar{H}^k(V_n)$.

Let $\{x^1, x^2, \ldots, x^n\}$ be the natural coordinate system in $\mathbf{R}^n$. Let U be a coordinate neighbourhood in V_n with a local coordinate system $\{y^1, y^2, \ldots, y^n\}$. Consider an element $u \in H^1(V_n)$ with projection

$$\pi^1_0(u) = y = (y^1, y^2, \ldots, y^n) \in U.$$

The 1-frame u is completely determined by the linear isomorphism

$$\tilde{u} : T_0(\mathbf{R}^n) \longrightarrow T_y(V_n).$$

In terms of local coordinates, $\tilde{u}$ can be expressed by

$$\tilde{u} : v_i \longrightarrow \sum_m y^m_i \bar{v}_m \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant m \leqslant n),$$

where $v_i = (\frac{\partial}{\partial x^i})_0$, $\bar{v}_m = (\frac{\partial}{\partial y^m})_y$ and $det(y^m_i) \neq 0$.

The 1-frame u is therefore completely determined by the set of local coordinates (y^i, y^j_k) with $det(y^j_k) \neq 0$. Thus we can take $\{y^i, y^j_k\}$ as a local coordinate system in $(\pi^1_0)^{-1}(U) \subset H^1(V_n)$. Similarly, we have a global coordinate system $\{x^i, x^j_k\}$ in $H^1(\mathbf{R}^n)$, with respect to which the distinguished element is given by $e_1 = (0, \delta^j_k)$.

The $n+n^2$ vectors $\{s_i = (\frac{\partial}{\partial x^i})_{e_1}, s^j_k = (\frac{\partial}{\partial x^k_j})_{e_1}\}$ form a basis for $E^1 = T_{e_1}(H^1(\mathbf{R}^n))$, and the $n+n^2$ local vector fields $\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^k_j}\}$ are linearly independent. Once again, any 2-frame v is completely determined by the linear isomorphism $\tilde{v}$ associated to v. In terms of local coordinates, we have

$$\tilde{v} : E^1 \longrightarrow T_u(H^1(V_n)) \text{ with } \pi^2_1(v) = u = (y^i, y^j_k)$$

$$\begin{cases} s_i \longrightarrow \sum\limits_m \left(y_i^m \bar{s}_m + \frac{1}{2!} y_{pi}^m \bar{s}_m^p \right) \\ s_k^j \longrightarrow Tu(s_k^j) \end{cases}$$

where $\bar{s}_m = \left(\frac{\partial}{\partial y^m}\right)_u$, $\bar{s}_m^p = \left(\frac{\partial}{\partial y_p^m}\right)_u$ and Tu is the tangential map of u, u being considered as a differentiable map. Thus v is completely determined by the set of local coordinates (y^i, y_j^i, y_{jk}^i) with $det(y_j^i) \neq 0$.

By iteration we have a coordinate neighbourhood $(\pi_0^k)^{-1}(U)$ in $\bar{H}^k(V_n)$ with a local coordinate system $\{y^i, y_{j_1}^i, \dots, y_{j_1 j_2 \cdots j_k}^i\}$ with $det(y_{j_1}^i) \neq 0$. The natural projection of $\bar{H}^k(V_n)$ onto $\bar{H}^m(V_n)$ $(m < k)$ is given by

$$(y^i, y_{j_1}^i, \dots, y_{j_1 \dots j_k}^i) \longrightarrow (y^i, y_{j_1}^i, \dots, y_{j_1 \cdots j_m}^i).$$

If $u = (a^i, a_{j_1}^i, \dots, a_{j_1 \cdots j_k}^i) \in \bar{H}^k(V_n)$, the associated linear isomorphism $\tilde{u}$ can be expressed by

$$\begin{cases} t_j \rightarrow \sum \left(a_j^i \bar{t}_i + \frac{1}{2!} a_{j_1 j}^i \bar{t}_i^{j_1} + \dots + \frac{1}{k!} a_{j_1 \cdots j_{k-1} j}^i \bar{t}_i^{j_1 \cdots j_{k-1}} \right) \\ t_i^{j_1} \rightarrow Tu'(t_i^{j_1}) \\ \dots\dots\dots\dots\dots\dots\dots \\ t_i^{j_1 \cdots j_{k-1}} \rightarrow Tu'(t_i^{j_1 \cdots j_{k-1}}) \end{cases}$$

where

$$t_i^{j_1 \cdots j_m} = \left(\frac{\partial}{\partial x_{j_1 \cdots j_m}^i}\right)_{e_{k-1}}$$

$$\bar{t}_i^{j_1 \cdots j_m} = \left(\frac{\partial}{\partial y_{j_1 \cdots j_m}^i}\right)_{u'}$$

and $u' = \pi_{k-1}^k(u)$. The local coordinates $a_{j_1, \dots, j_k}^i$ are symmetrical with respect to the lower indices if and only if u is a holonomic k-frame of V_n [1c].

6. Holonomy Theorem.

Consider an arbitrary element u in $\bar{H}^k(V_n)$. In this paragraph we give a necessary and sufficient condition for u to be a holonomic k-frame. Let us recall that the horizontal n-plane defined by u is just the image of the $\mathbf{R}^n$-component of $\bar{E}^{k-1} = \mathbf{R}^n \oplus \bar{\mathfrak{L}}_n^{k-1}$ under the linear isomorphism $\tilde{u}$. It is tangent to $H^{k-1}(V_n)$ at the point $u' = \pi^k_{k-1}(u)$, if u is holonomic.

For $k = 1$, there is no distinction between semi-holonomic frames and holonomic frames. For $k \geqslant 2$, $H^k(V_n) \subset \bar{H}^k(V_n)$.

PROPOSITION I.9. *An element u of $\bar{H}^2(V_n)$ is a holonomic 2-frame if and only if the 2-form $d\theta_1$ vanishes on the horizontal n-plane associated to u.*

Let $r_1, r_2, \ldots, r_n$ be a basis for $\mathbf{R}^n$. The canonical form θ_1 on $H^1(V_n)$ can be expressed as follows:

$$\theta_1 = \Sigma\, \theta^i r_i .$$

In terms of a local coordinate system $\{y^i, y^i_j\}$ in $H^1(V_n)$, the components θ^i of θ_1 are given by

$$\theta^i = \sum_j z^i_j \, d y^j ,$$

where (z^i_j) is the inverse matrix of (y^i_j). By exterior differentiation, we get

$$d\theta^i = \Sigma \frac{\partial z^i_p}{\partial y^m_q} \, d y^m_q \wedge d y^p .$$

Let $u = (a^i, a^i_j, a^i_{jk}) \in \bar{H}^2(V_n)$. The horizontal n-plane Q_u associated to u is generated by the n vectors

$$X_i = \Sigma \left(a^j_i \left(\frac{\partial}{\partial y^j}\right)_{u'} + \frac{1}{2!} a^j_{ki} \left(\frac{\partial}{\partial y^j_k}\right)_{u'} \right) \qquad (1 \leqslant i \leqslant n),$$

with $u' = \pi^2_1(u) = (a^i, a^i_j)$. The 2-form $d\theta_1$ vanishes on Q_u if and only if

$$d\theta^i(X_j, X_k) = \Sigma \left(\frac{\partial z^i_m}{\partial y^p_q}\right)_{u'} \begin{vmatrix} \frac{1}{2!} a^p_{qj} & a^m_j \\ \frac{1}{2!} a^p_{qk} & a^m_k \end{vmatrix}$$

is zero for all $1 \leqslant i, j, k \leqslant n$. Since $(z^i_j) = (y^i_j)^{-1}$, we have the relation $z^i_p y^p_k = \delta^i_k$. By differentiation, we get

$$\left(\frac{\partial z^i_m}{\partial y^p_q}\right)_{u'} = - b^q_m b^i_p$$

where $(b^i_j) = (a^i_j)^{-1}$. It follows that

$$d\theta^i(X_j, X_k) = -\frac{1}{2!} \Sigma \left(b^q_m b^i_p (a^m_k a^p_{qj} - a^m_j a^p_{qk}) \right)$$

$$= -\frac{1}{2!} \Sigma \left(b^i_p (a^p_{kj} - a^p_{jk}) \right)$$

Since $det(b^i_j) \neq 0$, we conclude that $d\theta^i(X_j, X_k) = 0$ for all $1 \leqslant i, j, k \leqslant n$ if and only if the a^i_{jk} are symmetrical with respect to their lower indices. Thus our proposition is proved.

For the general case where $k > 2$, we have the following «Holonomy Theorem»:

THEOREM I.10. *An element $u \in \bar{H}^k(V_n)$ is a holonomic k-frame if and only if the following conditions are satisfied:*

i) the horizontal n-plane Q_u associated to u is tangent to the submanifold $H^{k-1}(V_n)$ of $\bar{H}^{k-1}(V_n)$;

ii) the 2-form $d\theta_{k-1}$ vanishes on Q_u.

Let us assume that u is a holonomic k-frame. We can then write $u = j^1_{e_{k-1}} f^{(k-1)}$ for some local diffeomorphism f of $\mathbf{R}^n$ into V_n. If θ_{k-1} and $\hat{\theta}_{k-1}$ are respectively the canonical form on $\bar{H}^{k-1}(V_n)$ and $\bar{H}^{k-1}(\mathbf{R}^n)$, we have $f^{(k-1)*}\theta_{k-1} = \hat{\theta}_{k-1}$. It follows that $f^{(k-1)*} d\theta_{k-1} = d\hat{\theta}_{k-1}$. Now, the 2-form $d\hat{\theta}_{k-1}$ vanishes on the $\mathbf{R}^n$-component of $\bar{E}^{k-1} = \mathbf{R}^n \oplus \bar{\mathfrak{L}}^{k-1}_n$. As a consequence, $d\theta_{k-1}$ vanishes on Q_u. The first condition is obviously necessary.

It remains to show that the conditions are sufficient. The first condition implies that $u' = \pi^k_{k-1}(u)$ is a holonomic $(k-1)$-frame, and that we can find a local coordinate system $\{ y^i, y^i_{j_1}, \dots, y^i_{j_1 \dots j_k} \}$ in $\bar{H}^k(V_n)$ such that $u = (0, a^i_{j_1}, \dots, a^i_{j_1 \dots j_k})$ where $a^i_{j_1 \dots j_m}$ are symmetrical with respect to their lower indices for $2 \leqslant m \leqslant k-1$ and $a^i_{j_1 \dots j_k}$ is symmetrical with respect to the first $k-1$ lower indices. By a change of local coordinate systems, we can even suppose that $a^i_j = \delta^i_j$ and $a^i_{j_1 \dots j_m} = 0$ for

$2 \leqslant m \leqslant k-1$.

Let $\{x^1, x^2, \ldots, x^n\}$ be the natural coordinate system in $\mathbf{R}^n$. By iteration, we define a global coordinate system $\{x^i, x^i_{j_1}, \ldots, x^i_{j_1 \cdots j_m}\}$ in $\bar{H}^m(\mathbf{R}^n)$. Let $z^\alpha = x^i_{j_1 \cdots j_p}$ with $\alpha = i n^p + j_p n^{p-1} + \ldots + j_1$. The vectors

$$t_\alpha = \left(\frac{\partial}{\partial z^\alpha}\right)_{e_{k-2}} \qquad (1 \leqslant \alpha \leqslant n^{k-1} + n^{k-2} + \ldots + n)$$

form a basis for $\bar{E}^{k-2}$ and we can write

$$\theta_{k-1} = \sum_\alpha \theta^\alpha t_\alpha.$$

An element $v = (y^i, y^i_{j_1}, \ldots, y^i_{j_1 \cdots j_{k-1}}) \in \bar{H}^{k-1}(V_n)$ defines a linear isomorphism $\tilde{v}$ of $\bar{E}^{k-2}$ onto $T_{v'}(\bar{H}^{k-2}(V_n))$ with $v' = \pi^{k-1}_{k-2}(v)$. In terms of local coordinate systems, $\tilde{v}$ is given by

$$\tilde{v} : t_\alpha \longrightarrow \sum_\beta A^\beta_\alpha \bar{t}_\beta$$

where $1 \leqslant \alpha, \beta \leqslant n^{k-1} + n^{k-2} + \ldots + n$, $\bar{t}_\alpha = \left(\frac{\partial}{\partial \bar{z}^\alpha}\right)_{v'}$ with $\bar{z}^\alpha = y^i_{j_1 \cdots j_p}$. The matrix $A = (A^\beta_\alpha)$ is of the form

$$A = \begin{pmatrix} A^i_j & A^\omega_j \\ 0 & J \end{pmatrix} \qquad \begin{array}{l} 1 \leqslant i, j \leqslant n \\ \\ n < \omega \leqslant n^{k-1} + n^{k-2} + \ldots + n \end{array}$$

where J is the matrix corresponding to the linear isomorphism Tv'. We have therefore $y^i_{j_1 \cdots j_m} = A^\beta_{j_m}$ with $\beta = i n^{m-1} + j_{m-1} n^{m-2} + \ldots + j_1$. Let $B = (B^\beta_\alpha)$ be the inverse matrix of $A = (A^\beta_\alpha)$. The components θ^α of θ_{k-1} can be expressed by

$$\theta^\alpha = \sum_\beta B^\alpha_\beta \, d\bar{z}^\beta.$$

By exterior differentiation, we get

$$d\theta^\alpha = \sum \left(\frac{\partial B^\alpha_\beta}{\partial \bar{z}^\gamma}\right) d\bar{z}^\gamma \wedge d\bar{z}^\beta + \sum \left(\frac{\partial B^\alpha_\beta}{\partial y^i_{j_1 \cdots j_{k-1}}}\right) dy^i_{j_1 \cdots j_{k-1}} \wedge d\bar{z}^\beta.$$

Since $\sum B^\alpha_\mu A^\mu_\nu = \delta^\alpha_\nu$, we obtain by differentiation

$$\frac{\partial B^\alpha_\beta}{\partial \bar{z}^\gamma} = -\sum B^\alpha_\mu B^\nu_\beta \left(\frac{\partial A^\mu_\nu}{\partial \bar{z}^\gamma}\right),$$

$$\frac{\partial B^{\alpha}_{\beta}}{\partial y^{i}_{j_1\cdots j_{k-1}}} = -\Sigma B^{\alpha}_{\mu} B^{\nu}_{\beta} \Big(\frac{\partial A^{\mu}_{\nu}}{\partial y^{i}_{j_1\cdots j_{k-1}}} \Big),$$

hence

$$d\theta^{\alpha} = -\Sigma B^{\alpha}_{\mu} B^{\nu}_{\beta} \Big(\frac{\partial A^{\mu}_{\nu}}{\partial \bar{z}^{\gamma}} \Big) d\bar{z}^{\gamma} \wedge d\bar{z}^{\beta} - \Sigma B^{\alpha}_{\mu} B^{\nu}_{\beta} \Big(\frac{\partial A^{\mu}_{\nu}}{\partial y^{i}_{j_1\cdots j_{k-1}}} \Big) dy^{i}_{j_1\cdots j_{k-1}} \wedge d\bar{z}^{\beta}.$$

Let $u = (0, \delta^{i}_{j}, 0, \dots, a_{j_1\cdots j_k})$ and let Q_u be the horizontal n-plane of $\bar{H}^{k-1}(V_n)$ associated to u. Q_u is generated by the n vectors

$$X_p = \Big(\frac{\partial}{\partial y^{p}}\Big)_{u'} + \frac{1}{k!} \Sigma a^{i}_{j_1\cdots j_{k-1}p} \Big(\frac{\partial}{\partial y^{i}_{j_1\cdots j_{k-1}}} \Big)_{u'}$$

where $u' = \pi^{k}_{k-1}(u)$ and $1 \leqslant p \leqslant n$.

The nullity of $d\theta_{k-1}$ on Q_u implies that $d\theta^{\alpha}(X_p, X_q) = 0$ for all $1 \leqslant p, q \leqslant n$ and $1 \leqslant \alpha \leqslant n^{k-1} + n^{k-2} + \dots + n$. We have then

$$\begin{aligned} 0 &= d\theta^{\alpha}(X_p, X_q) \\ &= \Sigma B^{\alpha}_{\mu}(u') B^{\nu}_{\beta}(u') \Big(\frac{\partial A^{\mu}_{\nu}}{\partial y^{i}_{j_1\cdots j_{k-1}}} \Big)_{u'} \begin{vmatrix} \frac{1}{k!} a^{i}_{j_1\cdots j_{k-1}p} & A^{\beta}_{p}(u') \\ \frac{1}{k!} a^{i}_{j_1\cdots j_{k-1}q} & A^{\beta}_{q}(u') \end{vmatrix} \\ &= \frac{1}{k!} \Sigma B^{\alpha}_{\mu}(u') \, \delta^{\mu}_{\beta} (a^{i}_{j_1\cdots j_{k-2}qp} - a^{i}_{j_1\cdots j_{k-2}pq}) \end{aligned}$$

with $\beta = i n^{k-2} + \dots + j_1$. Since $det(B^{\alpha}_{\beta}(u')) \neq 0$, we obtain

$$a^{i}_{j_1\cdots j_{k-2}pq} = a^{i}_{j_1\cdots j_{k-2}qp}.$$

It follows that the $a^{i}_{j_1\cdots j_k}$ are symmetrical with respect to their lower indices and thus u is a holonomic k-frame.

Let us call $u \in \bar{H}^{k}(V_n)$ a *quasi-holonomic k-frame* if the horizontal n-plane Q_u of $\bar{H}^{k-1}(V_n)$ associated to u is tangent to the submanifold $H^{k-1}(V_n)$. We will denote by $\check{H}^{k}(V_n)$ the set of quasi-holonomic k-frames. We have obviously $H^{k}(V_n) \subset \check{H}^{k}(V_n) \subset \bar{H}^{k}(V_n)$. From the above theorem a quasi-holonomic k-frame u is a holonomic one if and only if $d\theta_{k-1}$ vanishes on the horizontal n-plane Q_u associated to u.

7. Some remarks on $\bar{H}^k(\mathbf{R}^n)$.

In the preceding paragraphs, $\mathbf{R}^n \times \bar{L}_n^k$ has been identified with $\bar{H}^k(\mathbf{R}^n)$. In this identification, a couple $(x, g) \in \mathbf{R}^n \times \bar{L}_n^k$ is identified with the element $t_x^{(k)}(g) \in \bar{H}^k(\mathbf{R}^n)$, where t_x denotes the translation in $\mathbf{R}^n$ sending the origin 0 to the point x. The tangent space $\bar{E}^k$ to $\bar{H}^k(\mathbf{R}^n)$ at the distinguished element e_k has a canonical Lie algebra structure. Let us say a few words on this Lie algebra structure. Let $u = (x, g) \in \bar{H}^k(\mathbf{R}^n)$. The translation t_x in $\mathbf{R}^n$ induces an automorphism $t_x^{(k)}$ of $\bar{H}^k(\mathbf{R}^n)$ which commutes with the right translations of $\bar{L}_n^k$ on $\bar{H}^k(\mathbf{R}^n)$, i.e.

$$t_x^{(k)} \circ R_b = R_b \circ t_x^{(k)}$$

for all $b \in \bar{L}_n^k$. In particular, $t_x^{(k)} \circ R_g = R_g \circ t_x^{(k)}$ gives a diffeomorphism of $\bar{H}^k(\mathbf{R}^n)$ onto itself that we will denote by t_u. We call a vector field on $\bar{H}^k(\mathbf{R}^n)$ *invariant* if it is invariant with respect to all diffeomorphisms of the form t_u, where u is an arbitrary element of $\bar{H}^k(\mathbf{R}^n)$. There is a one-to-one correspondence between $\bar{E}^k$ and the set of invariant vector fields on $\bar{H}^k(\mathbf{R}^n)$. If X, Y are two invariant vector fields on $\bar{H}^k(\mathbf{R}^n)$, so is the bracket $[X, Y]$. The vector space $\bar{E}^k$, endowed with this multiplication, becomes a Lie algebra over the field of real numbers. The Lie algebra $\bar{\mathfrak{L}}_n^k$ of $\bar{L}_n^k$ is a Lie subalgebra of $\bar{E}^k = \mathbf{R}^n \oplus \bar{\mathfrak{L}}_n^k$.

To every differentiable map f of a differentiable manifold W into $\bar{H}^k(\mathbf{R}^n)$, we can associate a differential 1-form $\omega_f = f^{-1} df$ with values in the Lie algebra $\bar{E}^k$ defined by $\omega_f(X) = (T t_{f(x)}^{-1} \circ Tf)(X)$ for all X in $T_x(W)$. In particular, if $W = \bar{H}^k(\mathbf{R}^n)$ and if f is the identity map of $\bar{H}^k(\mathbf{R}^n)$, we get a differential 1-form ω on $\bar{H}^k(\mathbf{R}^n)$ with values in $\bar{E}^k$, called the *invariant form* on $\bar{H}^k(\mathbf{R}^n)$.

PROPOSITION I.11. *The invariant form ω on $\bar{H}^k(\mathbf{R}^n)$ satisfies the equation*

$$d\omega + [\omega, \omega] = 0.$$

We recall that the form $[\omega, \omega]$ is defined by $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$ for all vector fields X, Y on $\bar{H}^k(\mathbf{R}^n)$. Since the module of vector fields on $\bar{H}^k(\mathbf{R}^n)$ is generated by the invariant vector fields, it suffices to prove the equation for two invariant vector fields X and Y. We have

$$d\omega(X,Y)=X\omega(Y)-Y\omega(X)-\omega([X,Y])$$
$$=-\omega([X,Y])=-[\omega(X),\omega(Y)]$$

proving the proposition.

REMARK: We have adopted the following convention for the exterior product:

$$(\alpha\wedge\beta)(X_1,X_2,\dots,X_{p+q})=\Sigma(-1)^{\varepsilon}\alpha(X_{i_1},\dots,X_{i_p})\beta(X_{i_{p+1}},\dots,X_{i_{p+q}}),$$

where the summation runs over all permutations $i_1,\dots,i_p,i_{p+1},\dots,i_{p+q}$ of $\{1,2,\dots,p+q\}$ and where ε denotes the signature of the corresponding permutation. With this convention, we have the following formula: if α is a p-form, then

$$d\alpha(X_1,\dots,X_{p+1})=\sum_{i=1}^{p+q}(-1)^{i+1}X_i\alpha(X_1,\dots,\hat{X}_i,\dots,X_{p+1})$$
$$+\sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1}).$$

Kumpera pointed out to me that the above Lie algebra structure on $\bar{E}^k$ comes from a canonical Lie group structure on $\bar{H}^k(\mathbf{R}^n)$. Since $(x,g)\in\mathbf{R}^n\times\bar{L}^k_n$ is identified with $t_x^{(k)}(g)=t_x^{(k)}\circ R_g(e_k)=R_g\circ t_x^{(k)}(e_k)$, we have $(t_x^{(k)}\circ R_g)\circ(t_{x'}^{(k)}\circ R_{g'})=t_x^{(k)}\circ t_{x'}^{(k)}\circ R_g\circ R_{g'}=t_{x+x'}^{(k)}\circ R_{g'g}$. Let ${}^t\bar{L}^k_n$ denote the underlying set of $\bar{L}^k_n$ endowed with the following multiplication: $g*h=hg$ where $g*h$ denotes the product in ${}^t\bar{L}^k_n$ and hg denotes the product in $\bar{L}^k_n$. With the identification $\bar{H}^k(\mathbf{R}^n)=\mathbf{R}^n\times\bar{L}^k_n$, $\bar{H}^k(\mathbf{R}^n)$ becomes a Lie group isomorphic to $\mathbf{R}^n\times{}^t\bar{L}^k_n$. Moreover, if $u=(x,g)$, $u'=(x',g')$, then

$$uu'=(x+x',g'g)=t_{x+x'}^{(k)}\circ R_{g'g}(e_k)$$
$$=(t_x^{(k)}\circ R_g)\circ(t_{x'}^{(k)}\circ R_{g'})(e_k)$$
$$=(t_x^{(k)}\circ R_g)(u')=t_u(u'),$$

where t_u is the diffeomorphism defined in the opening paragraph of this section. In fact, t_u is no other than the left translation defined by u in the Lie group $\overline{H}^k(\mathbf{R}^n)$. The Lie algebra structure on $\overline{E}^k$ defined above is precisely the Lie algebra of the Lie group $\overline{H}^k(\mathbf{R}^n)$. The invariant form ω is simply the Maurer-Cartan form of the Lie group $\overline{H}^k(\mathbf{R}^n)$.

Part II

HIGHER ORDER CONNECTIONS

1. Linear connections of order k.

An infinitesimal connection Γ^k in the principal fibre bundle of semi-holonomic k-frames $\overline{H}^k(V_n)$ over V_n will be called a *linear connection of order* k of V_n. Let ω_k be its connection form. We will sometimes say that ω_k is a linear connection of order k of V_n. If D *is* the exterior covariant differentiation relative to ω_k, the tensorial 2-form $\Theta_k = D\,\theta_k$ (resp. $\Omega_k = D\,\omega_k$) will be called the *torsion form* (reps. *curvature form*) of Γ^k or ω_k. For $Y, Z \in T(\overline{H}^k(V_n))$, $g \in \overline{L}^k_n$, we have

$$\Theta_k(TR_g(Y), TR_g(Z)) = \rho(g^{-1})\Theta_k(Y, Z).$$

where ρ is the linear representation of $\overline{L}^k_n$ on $\overline{E}^{k-1}$ defined in Part I. If Y or Z is a vertical vector, then $\Theta_k(Y, Z) = 0$.

The linear representation ρ induces a representation of $\overline{\mathfrak{L}}^k_n$ on $\overline{E}^{k-1}$: if $A \in \overline{\mathfrak{L}}^k_n$, $\xi \in \overline{E}^{k-1}$, we put

$$A\ \xi = \lim_{t \to 0} \frac{1}{t}(\rho(a_t)\xi - \xi)$$

where $a_t = exp\ tA$ is the 1-parameter group of transformations of $\overline{L}^k_n$ generated by A. In particular, if ξ is vertical, i.e. $\xi \in \overline{\mathfrak{L}}^{k-1}_n$, we have

$$A\xi = -[T\pi^k_{k-1}(A), \xi].$$

THEOREM II.1 (structure equations) *Let ω_k be a linear connection of order k. Then*

$$\Omega_k = d\omega_k + \omega_k \wedge \omega_k$$

$$\Theta_k = d\theta_k + \omega_k \wedge \theta_k + 3\,[T\pi^k_{k-1} \circ \omega_k, T\pi^k_{k-1} \circ \omega_k].$$

The first structure equation is well known. Let us show the second structure equation:

$$\Theta_k(X, Y) = d\theta_k(X, Y) + \omega_k(X)\theta_k(Y) - \omega_k(Y)\theta_k(X)$$
$$+ 3\,[T\pi^k_{k-1} \circ \omega_k(X),\, T\pi^k_{k-1} \circ \omega_k(Y)]$$

for all vectors $X \in T_u(\overline{H}^k(V_n))$ and $Y \in T_u(\overline{H}^k(V_n))$. It is sufficient to verify the equality in the following three special cases:

i) X and Y are horizontal. In this case, $\omega_k(X) = 0$, $\omega_k(Y) = 0$ and the equation reduces to the definition of Θ_k.

ii) X and Y are vertical. Let $X = A^*_u$ and $Y = B^*_u$, where A^* and B^* are the fundamental vector fields on $\overline{H}^k(V_n)$ corresponding to $A = \omega_k(X)$ and $B = \omega_k(Y)$ respectively. We have

$$\Theta_k(X, Y) = 0;$$
$$d\theta_k(X, Y) = X\theta_k(B^*) - Y\theta_k(A^*) - \theta_k([A^*, B^*]_u)$$
$$= -[T\pi^k_{k-1}(A), T\pi^k_{k-1}(B)];$$
$$\omega_k(X)\theta_k(Y) = A\,\theta_k(B^*_u)$$
$$= -[T\pi^k_{k-1}(A), T\pi^k_{k-1}(B)];$$
$$\omega_k(Y)\theta_k(X) = -[T\pi^k_{k-1}(B), T\pi^k_{k-1}(A)];$$

and

$$[T\pi^k_{k-1} \circ \omega_k(X), T\pi^k_{k-1} \circ \omega_k(Y)] = [T\pi^k_{k-1}(A), T\pi^k_{k-1}(B)].$$

The equality holds.

iii) X is vertical and Y is horizontal. Let $X = A^*_u$ with $A = \omega_k(X) \in \overline{\mathfrak{L}}^k_n$. We can extend Y to an invariant horizontal vector field $\tilde{Y}$ on $\overline{H}^k(V_n)$. We have then

$$d\theta_k(X, Y) = X\theta_k(\tilde{Y}) - Y\theta_k(A^*) - \theta_k([A^*, \tilde{Y}]_u).$$

Since $\theta_k(A^*)$ is constant, $Y\theta_k(A^*) = 0$. As $\tilde{Y}$ is an invariant horizontal vector field, $[A^*, \tilde{Y}] = 0$. Let $a_t = \exp tA$ be the 1-parameter group of transformations of $\overline{L}^k_n$ generated by $A \in \overline{\mathfrak{L}}^k_n$.

$$d\theta_k(X, Y) = A^*_u\theta_k(\tilde{Y})$$
$$= \lim_{t\to 0}\frac{1}{t}(\rho(a_t^{-1})\theta_k(\tilde{Y}) - \theta_k(\tilde{Y}))$$

$$= -A\,\theta_k(Y).$$

Now, $\omega_k(Y)=0$, $\Theta_k(X,Y)=0$ and $\omega_k(X)\theta_k(Y)=A\,\theta_k(Y)$. The equality therefore holds.

The projection π^k_m of $\bar{H}^k(V_n)$ onto $\bar{H}^m(V_n)$ being compatible with the natural surjection of $\bar{L}^k_n$ onto $\bar{L}^m_n$ $(m<k)$, any linear connection ω_k (of order k) induces a linear connection ω_m of order m, given by

$$\pi^{k*}_m\omega_m = T\pi^k_m \circ \omega_k\,.$$

PROPOSITION II.2 *Any linear connection ω_k of order k induces canonically a linear connection ω_m of order $m<k$ given by*

$$\pi^{k*}_m\omega_m = T\pi^k_m \circ \omega_k\,.$$

We have the relations:

$$\pi^{k*}_m\Omega_m = T\pi^k_m \circ \Omega_k\,,$$

$$\pi^{k*}_m\Theta_m = T\pi^{k-1}_{m-1} \circ \Theta_k\,.$$

Let us verify only the last formula. We know that

$$\pi^{k*}_m\theta_m = \theta_m \circ T\pi^k_m = T\pi^{k-1}_{m-1}\circ\theta_k\,.$$

As a consequence, $\pi^{k*}_m d\theta_m = T\pi^{k-1}_{m-1}\circ d\theta_k$. From the second structure equation, we obtain

$$\begin{aligned}
\pi^{k*}_m\Theta_m &= \pi^{k*}_m\, d\theta_m + \pi^{k*}_m\,\omega_m \wedge \pi^{k*}_m\,\theta_m \\
&\qquad + 3\,[T\pi^m_{m-1}\circ\pi^{k*}_m\,\omega_m,\ T\pi^m_{m-1}\circ\pi^{k*}_m\,\omega_m] \\
&= T\pi^{k-1}_{m-1}\circ d\theta_k + T\pi^k_m\circ\omega_k\wedge T\pi^{k-1}_{m-1}\circ\theta_k \\
&\qquad + 3\,[T\pi^k_{m-1}\circ\omega_k,\ T\pi^k_{m-1}\circ\omega_k] \\
&= T\pi^{k-1}_{m-1}\circ(d\theta_k+\omega_k\wedge\theta_k+3\,[T\pi^k_{k-1}\circ\omega_k,\ T\pi^k_{k-1}\circ\omega_k]) \\
&= T\pi^{k-1}_{m-1}\circ\Theta_k\,.
\end{aligned}$$

COROLLARY II.3 *If the torsion form (resp. the curvature form) of ω_k vanishes identically on $T(\bar{H}^k(V_n))$, the induced connection ω_m $(m<k)$ is without torsion (resp. without curvature).*

Let ω_k be a linear connection of V_n. We say that ω_k is *quasi-ho-*

lonomic if the connection form ω_k, restricted to $T(H^k(V_n))$, defines a connection in the principal fibre bundle $H^k(V_n)$ over V_n. If ω_k is quasi-holonomic, all induced connections ω_m $(m < k)$ are quasi-holonomic. The canonical connection in $\bar{H}^k(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}^k_n$ is quasi-holonomic.

2. Second order linear connections.

Let u be an element of $\bar{H}^2(V_n)$. Consider a coordinate neighbourhood U of $a_0 = \pi^2_0(u)$ with a system of local coordinates $\{x^1, x^2, \ldots, x^n\}$. The 2-frame u can be represented by a set of local coordinates (x^i, x^i_j, x^i_{jk}) with $det(x^i_j) \neq 0$. Let U' be another coordinate neighbourhood of a_0 with a system of local coordinates $\{y^1, y^2, \ldots, y^n\}$. The same u is represented by (y^i, y^i_j, y^i_{jk}). The changes of local coordinates are given by

$$y^i = y^i(x)$$

$$y^i_j = \Sigma \left(\frac{\partial y^i}{\partial x^m}\right) x^m_j$$

$$y^i_{jk} = \Sigma \left(\frac{\partial^2 y^m}{\partial x^j \partial x^k}\right) x^i_m + \Sigma \left(\frac{\partial y^p}{\partial x^j}\right)\left(\frac{\partial y^q}{\partial x^k}\right) x^i_{pq}\,.$$

An element $g \in \bar{L}^2_n$ can be represented by $u = (a^i_j, a^i_{jk})$ with $det(a^i_j) \neq 0$. In terms of these coordinates, the multiplication in $\bar{L}^2_n$ is given by

$$(a^i_j, a^i_{jk}).(b^i_j, b^i_{jk}) = (\Sigma\, a^i_m b^m_j,\ \Sigma\, a^i_m b^m_{jk} + \Sigma\, a^i_{pq} b^p_j b^q_k).$$

The action of $\bar{L}^2_n$ on $\bar{H}^2(V_n)$ is given by

$$(x^i, x^i_j, x^i_{jk})(a^i_j, a^i_{jk}) = (x^i,\ \Sigma\, x^i_m a^m_j,\ \Sigma\, x^i_m a^m_{jk} + \Sigma\, x^i_{pq} a^p_j a^q_k).$$

Let α be the automorphism of $\bar{L}^2_n$ defined by $\alpha(a^i_j, a^i_{jk}) = (a^i_j, a^i_{kj})$. It is evident that α leaves fixed every element in L^2_n. Moreover, $\alpha^2 =$ *identity.* We have immediately

PROPOSITION II.4 *There exists an involutive automorphism* α *of* $\bar{L}^2_n$ *such that* L^2_n *is the subgroup of all the fixed points of* α.

THEOREM II.5 *The homogeneous space* $\bar{L}^2_n / L^2_n$ *is weakly reductive: there exists a vector subspace* $\mathfrak{M}$ *of* $\bar{\mathfrak{L}}^2_n$ *such that*

$$\bar{\mathfrak{L}}^2_n = \mathfrak{L}^2_n \oplus \mathfrak{M} \ (\,direct\ sum\,),$$

$$ad(L_n^2)\mathfrak{M} \subset \mathfrak{M},$$

where $\mathfrak{L}_n^2$ *(resp.* $\overline{\mathfrak{L}}_n^2$*) is the Lie algebra of* L_n^2 *(resp.* $\overline{L}_n^2$*).*

This result is an immediate consequence of the following lemma proved in [2].

LEMMA II.6 *Let* α *be an involutive automorphism of a Lie group* $\overline{G}$. *The set of fixed points of* α *forms a Lie subgroup* G *of* $\overline{G}$. *Moreover, the homogeneous space* $\overline{G}/G$ *is weakly reductive: there exists a vector subspace* $\mathfrak{M}$ *of the Lie algebra* $\overline{\mathfrak{G}}$ *of* $\overline{G}$ *such that*

$$\overline{\mathfrak{G}} = \mathfrak{G} \oplus \mathfrak{M} \ (\textit{direct sum})$$

$$ad(G)\mathfrak{M} \subset \mathfrak{M}$$

where $\mathfrak{G}$ *is the Lie algebra of* G. *The vector space* $\mathfrak{M}$ *can be given by*

$$\mathfrak{M} = \{X \in \overline{\mathfrak{G}} : T\alpha(X) = -X\}.$$

Let $\mathfrak{M}$ be the vector subspace of $\overline{\mathfrak{L}}_n^2$ defined by the above lemma. If $X \in \mathfrak{M}$, $Y \in \mathfrak{M}$, $T\alpha([X, Y]) = [T\alpha(X), T\alpha(Y)] = [-X, -Y] = [X, Y]$ showing that $[X, Y] \in \mathfrak{L}_n^2$, i.e. $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{L}_n^2$. We have therefore the following result.

COROLLARY II.7 *The homogeneous space* $\overline{L}_n^2/L_n^2$ *is a symmetric space.*

For the rest of this section, we fix once for all a decomposition $\overline{\mathfrak{L}}_n^2 = \mathfrak{L}_n^2 \oplus \mathfrak{M}$, where $\mathfrak{M}$ is the vector subspace defined in the theorem II.5. We denote by i the canonical injection of $H^2(V_n)$ into $\overline{H}^2(V_n)$.

Let $\bar{\omega}_2$ be a connection form in $\overline{H}^2(V_n)$. We can write $i^*\bar{\omega}_2 = \omega_2 + t$, where ω_2 (resp. t) is the $\mathfrak{L}_n^2$-component (resp. $\mathfrak{M}$-component) of $i^*\bar{\omega}_2$. Since $ad(L_n^2)\mathfrak{M} \subset \mathfrak{M}$, ω_2 defines a connection in the principal fibre bundle $H^2(V_n)$ over V_n and t is a $\mathfrak{M}$-valued tensorial *1*-form on $H^2(V_n)$, called the *quasi-holonomic form* of $\bar{\omega}_2$. Inversely, the couple (ω_2, t) determines a connection $\bar{\omega}_2$ in $\overline{H}^2(V_n)$. In fact, if $\xi \in T_u(\overline{H}^2(V_n))$ with $u \in H^2(V_n)$, we can write $\xi = \xi' + \xi''$, where ξ' is a horizontal vector with respect to the connection ω_2 and ξ'' is a vertical vector. Let us put $\bar{\omega}_2(\xi) = t(\xi') + u^{-1}(\xi'')$. Now, if $\bar{\xi} \in T_v(\overline{H}^2(V_n))$ where $v \notin H^2(V_n)$, there exist $u \in H^2(V_n)$ and $g \in \overline{L}_n^2$ such that $v = ug$ and $\bar{\xi} = TR_g(\xi)$ for some $\xi \in T_u(\bar{H}^2(V_n))$. It is easy to check that $\bar{\omega}_2(\bar{\xi}) = ad(g^{-1})\bar{\omega}_2(\xi)$ does not depend on the choice of u and g. The mapping $\xi \to \bar{\omega}_2(\xi)$ gives the required connection form on $\bar{H}^2(V_n)$. Besides, $i^*\bar{\omega}_2 = \omega_2 + t$. We have thus established the following result.

PROPOSITION II.8 *There is a one-to-one correspondence between the set of all second order connections $\bar{\omega}_2$ of V_n and the set of all couples (ω_2, t), where ω_2 is a connection form in $H^2(V_n)$ and t is a $\mathfrak{M}$-valued tensorial 1-form on $H^2(V_n)$; the correspondence is given by*

$$i^*\bar{\omega}_2 = \omega_2 + t.$$

COROLLARY II.9 *A linear connection $\bar{\omega}_2$ is quasi-holonomic if and only if its associated quasi-holonomic form t vanishes identically on $H^2(V_n)$.*

Let ϕ be a tensorial form on $\bar{H}^2(V_n)$. From the structure equation

$$\bar{D}\phi = d\phi + \bar{\omega}_2 \wedge \phi$$

where $\bar{D}\phi$ is the exterior covariant derivative of ϕ with respect to $\bar{\omega}_2$, we deduce that

$$\begin{aligned} i^*(\bar{D}\phi) &= i^*d\phi + i^*\bar{\omega}_2 \wedge i^*\phi \\ &= d(i^*\phi) + i^*\bar{\omega}_2 \wedge i^*\phi \\ &= d(i^*\phi) + \omega_2 \wedge i^*\phi + t \wedge i^*\phi. \end{aligned}$$

The induced form $i^*\phi$ is a tensorial form on $H^2(V_n)$. If D is the exterior covariant differentiation with respect to ω_2, we have

$$D(i^*\phi) = d(i^*\phi) + \omega_2 \wedge i^*\phi.$$

Thus

$$i^*(\bar{D}\phi) = D(i^*\phi) + t \wedge i^*\phi.$$

Let $\bar{\Omega}_2$ (resp. Ω_2) be the curvature form of $\bar{\omega}_2$ (resp. ω_2). From the structure equation

$$\bar{\Omega}_2 = d\bar{\omega}_2 + [\bar{\omega}_2, \bar{\omega}_2]$$

we have

$$\begin{aligned} i^*\bar{\Omega}_2 &= i^*(d\bar{\omega}_2) + i^*([\bar{\omega}_2, \bar{\omega}_2]) \\ &= d(i^*\bar{\omega}_2) + [i^*\bar{\omega}_2, i^*\bar{\omega}_2] \end{aligned}$$

$$=\Omega_2+Dt+[t,t]$$

The form $Dt+[t,t]$ is a tensorial 2-form on $H^2(V_n)$. We may call it the *quasi-holonomic curvature* of $\bar{\omega}_2$.

From the structure equation

$$\bar{\Theta}_2=d\theta_2+\bar{\omega}_2\wedge\theta_2+3\,[T\pi^2_1\circ\bar{\omega}_2,\,T\pi^2_1\circ\bar{\omega}_2]$$

we have

$$\begin{aligned} i^*\bar{\Theta}_2&=i^*d\theta_2+i^*\bar{\omega}_2\wedge i^*\theta_2+3\,[T\pi^2_1\circ i^*\bar{\omega}_2,\,T\pi^2_1\circ i^*\bar{\omega}_2]\\ &=\Theta_2+t\wedge i^*\theta_2+3\,[T\pi^2_1\circ(\omega_2+t),\,T\pi^2_1\circ(\omega_2+t)]\,. \end{aligned}$$

The form

$$\begin{aligned} T=\;&t\wedge i^*\theta_2+3\,[T\pi^2_1\circ(\omega_2+t),\,T\pi^2_1\circ(\omega_2+t)]\\ &-3\,[T\pi^2_1\circ\omega_2,\,T\pi^2_1\circ\omega_2] \end{aligned}$$

is a tensorial 2-form on $H^2(V_n)$, which may be called the *quasi-holonomic torsion* of $\bar{\omega}_2$.

If $\bar{\omega}_2$ is quasi-holonomic, its associated quasi-holonomic form t vanishes identically on $H^2(V_n)$. Therefore, the quasi-holonomic curvature and the quasi-holonomic torsion of $\bar{\omega}_2$ are zero.

3. $\mathfrak{S}$-connections.

Let u be an arbitrary element of L^1_n. There exists a unique automorphism f of the vector space $\mathbf{R}^n$ such that $u=j^1_0 f$. The induced map $f^{(k-1)}:\bar{H}^{k-1}(\mathbf{R}^n)\to\bar{H}^{k-1}(\mathbf{R}^n)$ is a $(k-1)$-admissible isomorphism, and $j^1_{e_{k-1}}f^{(k-1)}\in L^k_n$. The mapping $u\to\iota^k(u)=j^1_{e_{k-1}}f^{(k-1)}$ gives a canonical identification of L^1_n with a subgroup of L^k_n (hence of $\bar{L}^k_n$). For $m<k$, $\iota^m=\pi^k_m\circ\iota^k$.

An invariant section of the fibration $\bar{H}^{k+1}(V_n)\to H^1(V_n)$, i.e. a lift ϕ_{k+1} of $H^1(V_n)$ into $\bar{H}^{k+1}(V_n)$ compatible with the canonical homomorphism $\iota^{k+1}:L^1_n\to\bar{L}^{k+1}_n$, will be called an *$\mathfrak{S}$-connection* of order k of V_n. It is given by a reduction of the structure group of $\bar{H}^{k+1}(V_n)$ from $\bar{L}^{k+1}_n$ to L^1_n. There is a one-to-one correspondence between the set of all $\mathfrak{S}$-connections (of order k) of V_n and the set of all semi-holonomic connections (of order k) defined in the sense of Ehresmann on the principal bundle $H^1(V_n)$. [4c] .

We say that an $\mathcal{E}$-connection ϕ_{k+1} is *symmetrical* or *holonomic* (resp. *quasi-holonomic*) if

$$\phi_{k+1}(H^1(V_n))\subset H^{k+1}(V_n) \text{ (resp. } \phi_{k+1}(H^1(V_n))\subset \check{H}^{k+1}(V_n)).$$

If ϕ_{k+1} is symmetrical (resp. quasi-holonomic), all projections $\phi_{m+1}=\pi^{k+1}_{m+1}\circ\phi_{k+1}$ of ϕ_{k+1} are symmetrical.

Consider an open set U of V_n with a system of local coordinates $\{x^1, x^2, \dots, x^n\}$. In terms of the induced local coordinates, a lift ϕ_{k+1} of $H^1(V_n)$ into $\bar{H}^{k+1}(V_n)$ can be expressed by

$$(x^i, x^i_j) \rightarrow (x^i, x^i_j, \dots, x^i_{j_1 j_2 \dots j_{k+1}}).$$

If ϕ_{k+1} is invariant, the functions $x^i_{j_1 j_2}, \dots, x^i_{j_1\dots j_{k+1}}$ can be written in the form

$$x^i_{j_1 j_2} = -\sum \Gamma^i_{m_1 m_2}\, x^{m_1}_{j_1}\, x^{m_2}_{j_2}$$

$$x^i_{j_1 j_2 j_3} = -\sum \Gamma^i_{m_1 m_2 m_3}\, x^{m_1}_{j_1}\, x^{m_2}_{j_2}\, x^{m_3}_{j_3}$$

$$\dots\dots\dots\dots$$

$$x^i_{j_1 j_2 \dots j_{k+1}} = -\sum \Gamma^i_{m_1 m_2 \dots m_{k+1}}\, x^{m_1}_{j_1}\, x^{m_2}_{j_2} \dots x^{m_{k+1}}_{j_{k+1}}$$

where $\Gamma^i_{m_1 m_2}, \dots, \Gamma^i_{m_1 m_2 \dots m_{k+1}}$ are differentiable functions defined on U. These are the *Christoffel symbols* of the $\mathcal{E}$-connection ϕ_{k+1}. They are not entirely arbitrary; they have to satisfy certain conditions when we change the local coordinates system. It is clear that ϕ_{k+1} is symmetrical if and only if all the Christoffel symbols are symmetrical with respect to their lower indices.

Let us consider some particular cases:

case (i): $k=1$.

Let Γ^i_{rs} (resp. $\bar{\Gamma}^i_{rs}$) be the Christoffel symbols of a first order $\mathcal{E}$-connection ϕ_2 relative to a coordinate neighbourhood U (resp. $\bar{U}$) with a local coordinates system $\{x^1, x^2, \dots, x^n\}$ (resp. $\{\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n\}$). If $U \cap \bar{U} \neq \phi$, we obtain easily the classical formula for the Christoffel symbols of a linear connection

$$\Gamma^i_{jk} = \sum \bar{\Gamma}^\alpha_{\beta\gamma}\left(\frac{\partial \bar{x}^\beta}{\partial x^j}\right)\left(\frac{\partial \bar{x}^\gamma}{\partial x^k}\right)\left(\frac{\partial x^i}{\partial \bar{x}^\alpha}\right) + \sum\left(\frac{\partial^2 \bar{x}^\alpha}{\partial x^j \partial x^k}\right)\left(\frac{\partial x^i}{\partial \bar{x}^\alpha}\right).$$

The quantities Γ^i_{jk} define then a linear connection of V_n. On the other hand, if $u \in H^1(V_n)$, the lift $\phi_2(u)$ of u determines a horizontal n-plane $Q_{\phi_2(u)}$ of $H^1(V_n)$ at u. Since ϕ_2 is compatible with $\iota^2: L^1_n \to \bar{L}^2_n$, it is easy to check that the distribution $u \to Q_{\phi_2(u)}$ defines an infinitesimal connection on $H^1(V_n)$, thus a linear connection ω_1 of V_n. The quantities Γ^i_{jk} are simply the classical Christoffel symbols of the associated linear connection ω_1. In fact, if $X_j = \sum x^i_j\left(\frac{\partial}{\partial x^i}\right)_x$ $(1 \leqslant j \leqslant n)$ is a basis for $T_x(V_n)$, with $x \in U$, the horizontal lift of X_j at $u = (x^i, x^i_j) \in H^1(V_n)$ with respect to ω_1, is given by

$$X^*_j = \sum x^i_j\left(\frac{\partial}{\partial x^i}\right)_u + \sum x^i_{jk}\left(\frac{\partial}{\partial x^i_k}\right)_u$$

where $x^i_{jk} = -\sum \Gamma^i_{rs} x^r_j x^s_k$. Let $r^j_i = \left(\frac{\partial}{\partial x^i_j}\right)_{e_1}$ $(1 \leqslant i, j \leqslant n)$ be a basis for $\mathfrak{L}^1_n$. The components of $\omega_1 = \sum \omega^i_j r^j_i$ can be expressed by

$$\omega^i_j = \sum y^i_k\left(dx^k_j + \sum C^k_{mp} x^p_j dx^m\right)$$

where (y^i_k) is the inverse matrix of (x^i_k) and C^k_{mp} are the classical Christoffel symbols of the linear connection ω_1. Consequently, $\omega^i_j(X^*_k) = 0$ for all indices $1 \leqslant i, j, k \leqslant n$. It follows that

$$x^i_{jk} = -\sum \Gamma^i_{rs} x^r_j x^s_k = -\sum C^i_{rs} x^r_j x^s_k.$$

Since $det(x^i_j) \neq 0$, we have $\Gamma^i_{jk} = C^i_{jk}$.

PROPOSITION II.10 [4a] *(i) There is a one-to-one correspondence between the set of first order linear connections of V_n and the set of invariant sections of $H^1(V_n)$ into $\bar{H}^2(V_n)$.*

(ii) Two linear connections of V_n have the same torsion if and only if the images of $H^1(V_n)$ by the corresponding invariant sections are contained in a principal subbundle of $\bar{H}^2(V_n)$ having the structure group L^2_n.

It remains to prove the second part of the proposition. Let $\phi_2, \bar{\phi}_2$

be two invariant sections of $H^1(V_n)$ into $\bar{H}^2(V_n)$. In terms of local coordinates, these $\mathfrak{S}$-connections are given by

$$(x^i, x^i_j) \longrightarrow \Phi_2(x^i, x^i_j) = (x^i, x^i_j, -\Sigma\, \Gamma^i_{rs} x^r_j x^s_k),$$

$$(x^i, x^i_j) \longrightarrow \bar{\Phi}_2(x^i, x^i_j) = (x^i, x^i_j, -\Sigma\, \bar{\Gamma}^i_{rs} x^r_j x^s_k).$$

where Γ^i_{jk}, $\bar{\Gamma}^i_{jk}$ are the corresponding Christoffel symbols. As $\Phi_2(x^i, x^i_j)$ and $\bar{\Phi}_2(x^i, x^i_j)$ are on the same fibre of $\bar{H}^2(V_n)$, there exists an element $(\delta^i_j, g^i_{jk}) \in \bar{M}^2_n = Ker(\bar{L}^2_n \to \bar{L}^1_n)$ such that

$$(x^i, x^i_j, -\Sigma\, \bar{\Gamma}^i_{rs} x^r_j x^s_k) = (x^i, x^i_j, -\Sigma\, \Gamma^i_{rs} x^r_j x^s_k)(\delta^i_j, g^i_{jk}).$$

It follows that

$$\Sigma\, \bar{\Gamma}^i_{rs} x^r_j x^s_k = \Sigma\, \Gamma^i_{rs} x^r_j x^s_k - \Sigma\, x^i_m g^m_{jk},$$

Consequently, we have

$$(*) \quad \Sigma\, (\bar{\Gamma}^i_{rs} - \bar{\Gamma}^i_{sr}) x^r_j x^s_k = \Sigma\, (\Gamma^i_{rs} - \Gamma^i_{sr}) x^r_j x^s_k - \Sigma\, x^i_m (g^m_{jk} - g^m_{kj}).$$

If the two linear connections have the same torsion, that is if $\Gamma^i_{rs} - \Gamma^i_{sr} = \bar{\Gamma}^i_{rs} - \bar{\Gamma}^i_{sr}$, we have $\Sigma x^i_m (g^m_{jk} - g^m_{kj}) = 0$. Since $det(x^i_m) \neq 0$, we get $g^m_{jk} = g^m_{kj}$, which shows that $(\delta^i_j, g^i_{jk}) \in M^2_n = \bar{M}^2_n \cap L^2_n$. Hence the condition is necessary.

If Φ_2 and $\bar{\Phi}_2$ map $H^1(V_n)$ into the same principal subbundle of $\bar{H}^2(V_n)$ having the structure group L^2_n, we still have the formula $(*)$ with $g^m_{jk} = g^m_{kj}$. Consequently,

$$\Sigma\, (\bar{\Gamma}^i_{rs} - \bar{\Gamma}^i_{sr}) x^r_j x^s_k = \Sigma\, (\Gamma^i_{rs} - \Gamma^i_{sr}) x^r_j x^s_k.$$

Since $det(x^i_j) \neq 0$, we get

$$\bar{\Gamma}^i_{rs} - \bar{\Gamma}^i_{sr} = \Gamma^i_{rs} - \Gamma^i_{sr}.$$

Hence the connections have the same torsion, proving that the condition is sufficient.

Case (ii): $k = 2$

An element of $\bar{L}^3_n$ can be represented by a set of coordinates $(a^i_j, a^i_{jk}, a^i_{jkm})$ with $det(a^i_j) \neq 0$. The multiplication is given by

$$(a^i_j, a^i_{jk}, a^i_{jkm}) . (b^i_j, b^i_{jk}, b^i_{jkm}) = (\Sigma\, a^i_r b^r_j, \Sigma\, (a^i_{rs} b^r_j b^s_k + a^i_r b^r_{jk}),$$

$$\Sigma (a^i_{rst} b^r_j b^s_k b^t_m + a^i_{rs} b^r_{jk} b^s_m + a^i_{rs} b^r_k b^s_{jm} + a^i_{rs} b^r_j b^s_{km} + a^i_r b^r_{jkm})).$$

If $u = (x^i, x^i_j, x^i_{jk}, x^i_{jkm}) \in \bar{H}^3(V_n)$, the action of $\bar{L}^3_n$ on $\bar{H}^3(V_n)$ can be expressed by

$$(x^i, x^i_j, x^i_{jk}, x^i_{jkm})(a^i_j, a^i_{jk}, a^i_{jkm}) = (x^i, \Sigma\, x^i_r a^r_j, \Sigma\, (x^i_{rs} a^r_j a^s_k + x^i_r a^r_{jk}),$$
$$\Sigma\, (x^i_{rst} a^r_j a^s_k a^t_m + x^i_{rs} a^r_{jk} a^s_m + x^i_{rs} a^r_k a^s_{jm} + x^i_{rs} a^r_j a^s_{km} + x^i_r a^r_{jkm})).$$

Consider an $\mathcal{E}$-connection ϕ_3 of order 2. In terms of local coordinates, ϕ_3 is given by

$$(x^i, x^i_j) \longrightarrow (x^i, x^i_j, -\Sigma\, \Gamma^i_{rs} x^r_j x^s_k, -\Sigma\, \Gamma^i_{rst} x^r_j x^s_k x^t_m)$$

where Γ^i_{rs}, Γ^i_{rst} are the Christoffel symbols. If $\bar{\Gamma}^i_{rs}$, $\bar{\Gamma}^i_{rst}$ are the Christoffel symbols of ϕ_3 in an other local coordinates system, we have

$$\Gamma^i_{jk} = \Sigma\, \bar{\Gamma}^\alpha_{\beta\gamma} (\frac{\partial \bar{x}^\beta}{\partial x^j})(\frac{\partial \bar{x}^\gamma}{\partial x^k})(\frac{\partial x^i}{\partial \bar{x}^\alpha}) + \Sigma\, (\frac{\partial^2 \bar{x}^\alpha}{\partial x^j \partial x^k})(\frac{\partial x}{\partial x^\alpha}),$$

$$\bar{\Gamma}^i_{jkm} = \Sigma (\frac{\partial x^r}{\partial \bar{x}^j})(\frac{\partial x^s}{\partial \bar{x}^k})(\frac{\partial x^t}{\partial \bar{x}^m}) \{ \Gamma^\alpha_{rst}(\frac{\partial \bar{x}^i}{\partial x^\alpha}) - (\frac{\partial^3 \bar{x}^i}{\partial x^r \partial x^s \partial x^t}) +$$
$$\Gamma^\alpha_{rs}(\frac{\partial^2 \bar{x}^i}{\partial x^\alpha \partial x^t}) + \Gamma^\alpha_{rt}(\frac{\partial^2 \bar{x}^i}{\partial x^s \partial x^\alpha}) + \Gamma^\alpha_{st}(\frac{\partial^2 \bar{x}^i}{\partial x^r \partial x^\alpha}) \}.$$

By direct computations, we have the following result:

PROPOSITION II.11 *Let Γ^i_{jk}, Γ^i_{jkm} be the Christoffel symbols of a second order $\mathcal{E}$-connection of V_n. If the induced first order $\mathcal{E}$-connection is symmetrical, then the following quantities*

$$A^i_{jkm} = \Gamma^i_{jkm} - \Gamma^i_{kjm},$$

$$B^i_{jkm} = \Gamma^i_{jkm} - \Gamma^i_{mkj},$$

$$C^i_{jkm} = \Gamma^i_{jkm} - \Gamma^i_{jmk}$$

are respectively the components of a $(1,3)$-tensor on V_n. The given $\mathcal{E}$-connection is symmetrical if and only if these three tensors are zero.

4. Linear connections and $\mathcal{E}$-connections.

The Lie group $\bar{L}^{k+1}_n$ (resp. L^{k+1}_n) acts linearly on $\bar{E}^k$ (*resp.* $E^k = T_{e_k}(H^k(\mathbf{R}^n)))$ on the left. We denote by $\bar{S}^k T$ (resp. $S^k T$) the associated vector bundle of $\bar{H}^{k+1}(V_n)$ (resp. $H^{k+1}(V_n)$) with standard fibre $\bar{E}^k$ (resp. E^k) and structure group $\bar{L}_n^{k+1}$ (resp. L_n^{k+1}). For $k=0$, $S^0T=T(V_n)$.

PROPOSITION II.12 *The vector bundle* $\bar{S}^kT$ *(resp.* S^kT*) is canonically isomorphic to the vector bundle* $T(\bar{H}^k(V_n))/\bar{L}_n^k$ *(resp.* $T(H^k(V_n))/L_n^k$*).*

An element $u\in\bar{H}^{k+1}(V_n)$ determines a linear isomorphism $\tilde{u}$ of $\bar{E}^k$ onto $T_{u'}(\bar{H}^k(V_n))$ with $u'=\pi_k^{k+1}(u)$. On the other hand, u can be considered as a linear isomorphism of $\bar{E}^k$ onto the fibre $(\bar{S}^kT)_x$ over x, where x is the projection of u on V_n. We have then a linear isomorphism $\tilde{u}\circ u^{-1}$ of $(\bar{S}^kT)_x$ onto $T_{u'}(\bar{H}^k(V_n))$. If v is another element of $\bar{H}^{k+1}(V_n)$ with projection $x=\pi_0^{k+1}(v)$, we can write $v=ug$ for a unique $g\in\bar{L}_n^{k+1}$. Similarly, we have a linear isomorphism $\tilde{v}\circ v^{-1}:(\bar{S}^kT)_x\to T_{v'}(\bar{H}^k(V_n))$, where $v'=\pi_k^{k+1}(v)$. Now, $v=u\circ\rho(g)$ and $\tilde{v}=TR_{g'}\circ\tilde{u}\circ\rho(g)$ with $g'=\pi_k^{k+1}(g)\in\bar{L}_n^k$. Consequently, $\tilde{v}\circ v^{-1}=TR_{g'}\circ\tilde{u}\circ u^{-1}$. Since $\bar{H}^{k+1}(V_n)\to\bar{H}^k(V_n)$ is surjective, we get an isomorphism of $\bar{S}^kT$ onto $T(\bar{H}^k(V_n))/\bar{L}_n^k$. Similarly, one establishes an isomorphism of S^kT onto $T(H^k(V_n))/L_n^k$.

P. Libermann showed that $T(\bar{H}^k(V_n))/\bar{L}_n^k$ (resp. $T(H^k(V_n))/L_n^k$) is canonically isomorphic to $\bar{J}^kT$ (resp. J^kT), the k-th semi-holonomic (resp. holonomic) prolongation of the vector bundle $T(V_n)$. Thus, we have an isomorphism of $\bar{S}^kT$ (resp. S^kT) onto $\bar{J}^kT$ (resp. J^kT).

$H^{k+1}(V_n)$ being a principal fibre subbundle of $\bar{H}^{k+1}(V_n)$ and the action of L_n^{k+1} on E^k being the restriction of that of $\bar{L}_n^{k+1}$ on $\bar{E}^k$, the vector bundle S^kT can be considered as a vector subbundle of $\bar{S}^kT$.

The projection π_{m+1}^{k+1} of $\bar{H}^{k+1}(V_n)$ onto $\bar{H}^{m+1}(V_n)$ induces a surjection p_m^k of $\bar{S}^kT$ onto $\bar{S}^mT$. Moreover, the restriction of p_m^k to each fibre of $\bar{S}^kT$ is linear. Similarly, we have a projection of S^kT onto S^mT for $m<k$.

An $\mathfrak{S}$-connection $\phi_{k+1}:H^1(V_n)\to\bar{H}^{k+1}(V_n)$ induces a splitting of the following exact sequence of vector bundles

$$0\longrightarrow\bar{N}^k\longrightarrow\bar{S}^kT\longrightarrow T(V_n)\longrightarrow 0$$

where $\bar{N}^k$ is the kernel of the projection $\bar{S}^kT\to T(V_n)$. More precisely, we have the following result:

THEOREM II.13 *There exists a one-to-one correspondence between the set of* $\mathfrak{S}$*-connections of order* k *of* V_n *and the set of splittings of the exact sequence of vector bundles over* V_n:

$$0 \longrightarrow \bar{N}^k \longrightarrow \bar{S}^k T \longrightarrow T(V_n) \longrightarrow 0.$$

Let us first prove two lemmas:

LEMMA II.14 *Let* $\bar{E}^k = \mathbf{R}^n \oplus \bar{\mathfrak{L}}^k_n$ *be the canonical decomposition of* $\bar{E}^k$ *defined by the canonical connection in* $\bar{H}^k(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}^k_n$. *For every other decomposition of* $\bar{E}^k$ *of the form* $\bar{E}^k = Q^k \oplus \bar{\mathfrak{L}}^k_n$, *there exists a unique* $g \in \bar{M}^{k+1} = Ker(\bar{L}^{k+1}_n \to L^1_n)$ *such that* $\rho(g)(\mathbf{R}^n) = Q^k$.

We prove the lemma by induction on k. For $k=1$, we have the canonical decomposition $E^1 = \mathbf{R}^n \oplus \mathfrak{L}^1_n$. Let $E^1 = Q^1 \oplus \mathfrak{L}^1_n$ be another decomposition of E^1. Consider a local section σ_1 of $H^1(\mathbf{R}^n) \to \mathbf{R}^n$ such that $\sigma_1(0) = e_1$ and $T\sigma_1(\mathbf{R}^n) = Q^1$. Let f be the admissible local isomorphism of $H^1(\mathbf{R}^n)$ into $H^1(\mathbf{R}^n)$ defined by the condition: $f \circ \eta_1 = \sigma_1$, where η_1 is the «zero section» of $H^1(\mathbf{R}^n) = \mathbf{R}^n \times L^1_n \to \mathbf{R}^n$. The 1-jet $j^1_{e_1} f = g$ defines an element $g \in \bar{M}^2_n = Ker(\bar{L}^2_n \to L^1_n)$ satisfying the property: $\rho(g)(\mathbf{R}^n) = Q^1$. Uniqueness follows from the fact that the neutral element is the only element of $\bar{M}^2_n$ leaving stable the two components of $E^1 = \mathbf{R}^n \oplus \mathfrak{L}^1_n$.

Let us assume that the lemma is proved for $m \leqslant k-1$. If $\bar{E}^k = Q^k \oplus \bar{\mathfrak{L}}^k_n$ is a decomposition of $\bar{E}^k$, we may consider a local section σ_k of $\bar{H}^k(\mathbf{R}^n) \to \mathbf{R}^n$ satisfying the conditions: $\sigma_k(0) = e_k$ and $T\sigma_k(T_0(\mathbf{R}^n)) = Q^k$. Now,

$$\bar{E}^{k-1} = T\pi^k_{k-1}(\bar{E}^k) = T\pi^k_{k-1}(Q^k) \oplus T\pi^k_{k-1}(\bar{\mathfrak{L}}^k_n) = T\pi^k_{k-1}(Q^k) \oplus \bar{\mathfrak{L}}^{k-1}_n.$$

From the induction hypothesis, there is a unique $g' \in \bar{M}^k = Ker(\bar{L}^k_n \to L^1_n)$ such that $\rho(g')(\mathbf{R}^n) = T\pi^k_{k-1}(Q^k)$. Let h be the admissible local isomorphism of $\bar{H}^k(\mathbf{R}^n)$ into $\bar{H}^k(\mathbf{R}^n)$ defined by the condition: $h \circ \eta_k = R_{g'} \circ \sigma_k$ where η_k is the «zero section» of $\bar{H}^k(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}^k_n \to \mathbf{R}^n$. The 1-jet $j^1_{e_k} h$ defines an element g of $\bar{M}^{k+1} = Ker(\bar{L}^{k+1}_n \to L^1_n)$ such that $\rho(g)(\mathbf{R}^n) = Q^k$. Suppose that there is another $\bar{g} \in \bar{M}^{k+1}$ satisfying the condition: $\rho(\bar{g})(\mathbf{R}^n) = Q^k$. We have then $\rho(\pi^{k+1}_k(\bar{g}))(\mathbf{R}^n) = T\pi^k_{k-1}(Q^k)$. Consequently, $g' = \pi^{k+1}_k(\bar{g})$. We can write $\bar{g} = gm_0$ where m_0 is an element of $Ker(\bar{L}_n^{k+1} \to \bar{L}_n^k)$. Since the neutral element is the only element of $Ker(\bar{L}_n^{k+1} \to \bar{L}_n^k)$ leaving stable the two components of $\bar{E}^k = \mathbf{R}^n \oplus \bar{\mathfrak{L}}_n^k$, we conclude that $\bar{g} = g$ proving the uniqueness of g.

LEMMA II.15 *The Lie group* $\iota^{k+1}(L_n^1)$ *is the largest subgroup of* $\bar{L}_n^{k+1}$ *which leaves invariant the two direct summands of* $\bar{E}^k = \mathbf{R}^n \oplus \bar{\mathfrak{L}}_n^k$.

It is easy to check that $\iota^{k+1}(L_n^1)$ leaves invariant the two direct summands of $\bar{E}^k = \mathbf{R}^n \oplus \bar{\mathfrak{L}}_n^k$. Now, consider an element $g \in \bar{L}_n^{k+1}$ such that $\rho(g)(\mathbf{R}^n) = \mathbf{R}^n$. Let $g_0 = \pi_1^{k+1}(g)$. The action of $\iota^{k+1}(g_0).g^{-1}$ on $\mathbf{R}^n \subset \bar{E}^k$ is trivial. Consequently, we have $g = \iota^{k+1}(g_0) \in \iota^{k+1}(L_n^1)$ in virtue of the preceeding lemma.

Let us go back to the proof of the theorem. We have seen that there is a mapping F of the set of $\mathcal{E}$-connections of order k of V_n into the set of splittings of the exact sequence of vector bundles over V_n:

$$0 \longrightarrow \bar{N}^k \longrightarrow \bar{S}^k T \longrightarrow T(V_n) \longrightarrow 0.$$

This mapping F is injective. Let us consider two $\mathcal{E}$-connections ϕ_{k+1} and ψ_{k+1} which induce the same splitting

$$F(\phi_{k+1}) = F(\psi_{k+1}) : T(V_n) \to \bar{S}^k T.$$

If $y \in T(V_n)$, we can write $y = q_1(u, \xi)$, where $u \in H^1(V_n)$, $\xi \in \mathbf{R}^n$ and q_1 is the natural projection of $H^1(V_n) \times \mathbf{R}^n$ onto $T(V_n)$. The condition $F(\phi_{k+1})(y) = F(\psi_{k+1})(y)$ implies that

$$q_{k+1}(\phi_{k+1}(u), \xi) = q_{k+1}(\psi_{k+1}(u), \xi),$$

where we have denoted by q_{k+1} the natural projection of $\bar{H}^{k+1}(V_n) \times \bar{E}^k$ onto $\bar{S}^k T$. From the above lemma, we deduce that $\phi_{k+1}(u) = \psi_{k+1}(u)$ for all $u \in H^1(V_n)$. Let us show that F is surjective. Consider a splitting of the exact sequence

$$0 \longrightarrow \bar{N}^k \longrightarrow \bar{S}^k T \longrightarrow T(V_n) \longrightarrow 0$$

given by the lift $\sigma : T(V_n) \to \bar{S}^k T$. Let x be an arbitrary element of V_n. An element u of the fibre of $\bar{H}^{k+1}(V_n)$ over x determines a linear isomorphism of $\bar{E}^k$ onto $(\bar{S}^k T)_x$. The image $u^{-1}(\sigma(T_x(V_n)))$ is a vector subspace of $\bar{E}^k$. More exactly, we have $\bar{E}^k = u^{-1}(\sigma(T_x(V_n))) \oplus \bar{\mathfrak{L}}_n^k$. From the lemma II.14, there exists a $g \in \bar{M}^{k+1} = Ker(\bar{L}_n^{k+1} \to L_n^1)$ such that $\rho(g)(\mathbf{R}^n) = u^{-1}(\sigma(T_x(V_n)))$. The element $v = ug \in \bar{H}^{k+1}(V_n)$ defines therefore a linear isomorphism of $\bar{E}^k = \mathbf{R}^n \oplus \bar{\mathfrak{L}}_n^k$ onto $(\bar{S}^k T)_x$, mapping $\mathbf{R}^n$ onto $\sigma(T_x(V_n))$. Every element of $\bar{H}^{k+1}(V_n)$ lying on the fibre over x and having the same property is of the form $v g_0$ with $g_0 \in \iota^{k+1}(L_n^1)$. Since x is arbitrary, we obtain in this way a principal subbundle of $\bar{H}^{k+1}(V_n)$ with structure group $\iota^{k+1}(L_n^1)$, hence the $\mathfrak{S}$-connection that we are looking for.

The vector bundle $T(\bar{H}^k(V_n))/\bar{L}_n^k$ is isomorphic to $\bar{S}^k T$. We have therefore a one-to-one correspondence between the set of linear connections of order k of V_n and the set of splittings of the exact sequence of vector bundles

$$0 \longrightarrow \bar{N}^k \longrightarrow \bar{S}^k T \longrightarrow T(V_n) \longrightarrow 0.$$

From the preceeding result, we have

THEOREM II.16 *There is a one-to-one correspondence between the set of linear connections of order k and the set of $\mathfrak{S}$-connections of the same order.*

Consider an $\mathfrak{S}$-connection $\phi_{k+1} : H^1(V_n) \to \bar{H}^{k+1}(V_n)$. Let $\phi_k = \pi_k^{k+1} \circ \phi_{k+1}$. If $u \in H^1(V_n)$, $\phi_{k+1}(u)$ determines a horizontal n-plane of $\bar{H}^k(V_n)$ at $\phi_k(u) \in \bar{H}^k(V_n)$. We obtain thus a field of n-planes of $\bar{H}^k(V_n)$ defined on $\phi_k(H^1(V_n))$. It is easy to check that this local field is invariant with respect to the right translations defined by the elements of $\iota^k(L_n^1)$ on $\bar{H}^k(V_n)$. Consequently, we can extend it to a global field of n-planes of $\bar{H}^k(V_n)$ invariant with respect to the right translations of $\bar{L}_n^k$ on $\bar{H}^k(V_n)$. We obtain thus a linear connection ω_k of order k of V_n. This correspondence $\phi_{k+1} \to \omega_k$ is exactly the one we have established in the above theorem. For $k = 1$, we have a one-to-one correspondence between the set of symmetrical linear connections of V_n and the set of invariant sections of $H^1(V_n)$ into $H^2(V_n)$ (cf. Prop. I.9 and Prop. II.10). Let us assume that there is a one-to-one correspondence between the set of symmetrical $\mathfrak{S}$-connections of order m ($m \leqslant k-1$) and the set of quasi-holonomic linear connections of the same order having zero torsion. If ϕ_{k+1} is a symmetrical $\mathcal{E}$-connection of order k, the corresponding linear connection ω_k is quasi-holonomic and without torsion (cf. Theorem I.10). Inversely let ω_k be a quasi-holonomic linear connection having zero torsion and let ϕ_{k+1} be the corresponding $\mathcal{E}$-connection established in the above theorem. The connection projection ω_{k-1} (of order $k-1$) of ω_k is a quasi-holonomic connection without torsion. From the induction hypothesis, the corresponding $\mathcal{E}$-connection ϕ_k is symmetrical. It is easy to check that $\phi_k = \pi_k^{k+1} \circ \phi_{k+1}$. Hence $\phi_{k+1}(H^1(V_n)) \subset H^{k+1}(V_n)$ from the «Holonomy Theorem». We have thus established the following result:

COROLLARY II.17 *There is a one-to-one correspondence between the set of symmetrical $\mathcal{E}$-connections and the set of quasi-holonomic linear connections without torsion.*

5. Pseudo-connections and multi-connections.

A *pseudo-connection* of order k of V_n is a couple (ψ_{k+1}, Ψ_{k+1}), where Ψ_{k+1} is a homomorphism of $\bar{L}_n^k$ into $\bar{L}_n^{k+1}$ and ψ_{k+1} is a differentiable lift of $\bar{H}^k(V_n)$ into $\bar{H}^{k+1}(V_n)$ such that

$$\psi_{k+1}(ug) = \psi_{k+1}(u)\,\Psi_{k+1}(g)$$

for all $u \in \bar{H}^k(V_n)$ and $g \in \bar{L}_n^k$. It follows that Ψ_{k+1} is a lift of $\bar{L}_n^k$ into $\bar{L}_n^{k+1}$. The condition of compatibility implies that an invariant vector field of $\bar{H}^k(V_n)$ can be lifted to an invariant vector field of $\bar{H}^{k+1}(V_n)$. We obtain thus an infinitesimal connection in the principal fibre bundle $\bar{H}^{k+1} \to \bar{H}^k(V_n)$, or equivalently, a splitting of the exact sequence of vector bundles over V_n

$$0 \longrightarrow \bar{N}_k^{k+1} \longrightarrow \bar{S}^{k+1}T \longrightarrow \bar{S}^k T \longrightarrow 0$$

where $\bar{N}_k^{k+1}$ is the kernel of $\bar{S}^{k+1}T \to \bar{S}^k T$.

Consider a pseudo-connection (ψ_{k+1}, Ψ_{k+1}) of V_n. The lift ψ_{k+1} of $\bar{H}^k(V_n)$ into $\bar{H}^{k+1}(V_n)$ defines an absolute parallelism on $\bar{H}^k(V_n)$. If $Z \in T_u(\bar{H}^k(V_n))$, we put $\alpha(Z) = \widetilde{\psi_{k+1}(u)}^{-1}(Z)$. The mapping $Z \to \alpha(Z)$ defines a differentiable 1-form α on $\bar{H}^k(V_n)$ with values in $\bar{E}^k$. There is an induced linear representation of $\bar{L}_n^k$ on $\bar{E}^k$ given by

$$\sigma = \rho \circ \Psi_{k+1},$$

where we have denoted by ρ the linear representation of $\bar{L}_n^{k+1}$ on $\bar{E}^k$. If $Z \in T(\bar{H}^k(V_n))$, we have $\alpha(TR_g(Z)) = \sigma(g^{-1})\alpha(Z)$, i.e. α is a pseudotensorial 1-form on $\bar{H}^k(V_n)$, called the *pseudo-connection form* of (ψ_{k+1}, Ψ_{k+1}).

A *multi-connection* of order k of V_n is given by a sequence of pseudo-connections (ψ_{m+1}, Ψ_{m+1}), $m = 1, 2, \dots, k$ such that $\Psi_{m+1} \circ \iota^m = \iota^{m+1}$. The composite map $\phi_{k+1} = \psi_{k+1} \circ \psi_k \circ \cdots \circ \psi_2$ defines an $\mathcal{E}$-connection of V_n. Inversely, given a sequence of homomorphisms Ψ_{m+1}: $\bar{L}_n^m \to \bar{L}_n^{m+1}$ such that $\Psi_{m+1} \circ \iota^m = \iota^{m+1}$ $(m = 1, 2, \dots, k)$, an $\mathcal{E}$-connection $\phi_{k+1}: H^1(V_n) \to \bar{H}^{k+1}(V_n)$ determines a multi-connection of order k of V_n.

We are going to define a natural sequence of group homomorphisms

$$L_n^1 \xrightarrow{\Lambda_2} \bar{L}_n^2 \xrightarrow{\Lambda_3} \cdots\cdots\cdots \to \bar{L}_n^k \xrightarrow{\Lambda_{k+1}} \bar{L}_n^{k+1} \to \cdots$$

satisfying the conditions: $\pi_k^{k+1} \circ \Lambda_{k+1} = identity$, $\Lambda_{k+1} \circ \iota^k = \iota^{k+1}$ for $k = 2, 3, \dots$. We put $\Lambda_2 = \iota^2$, the canonical injection of L_n^1 into $\bar{L}_n^2$. It induces a lift of $H^1(\mathbf{R}^n) = \mathbf{R}^n \times L_n^1$ into $\bar{H}^2(\mathbf{R}^n) = \mathbf{R}^n \times \bar{L}_n^2$. We will denote this lift by the same symbol Λ_2. Let $u = j_{e_1} f \in \bar{L}_n^2$, where f is an admissible local isomorphism of $H^1(\mathbf{R}^n)$ into $H^1(\mathbf{R}^n)$. Consider the local isomorphism h of $\bar{H}^2(\mathbf{R}^n)$ into $\bar{H}^2(\mathbf{R}^n)$ defined by the condition:

$$h \circ \eta_2 = R_u \circ \Lambda_2 \circ R_{u'}^{-1} \circ f \circ \eta_1,$$

where $u' = \pi_1^2(u)$ and η_i $(i = 1, 2)$ are the «zero sections». The 1-jet $j_{e_2}^1 h$ depends uniquely on u and the mapping $u \to \Lambda_3(u) = j_{e_2}^1 h$ defines a group homomorphism of $\bar{L}_n^2$ into $\bar{L}_n^3$ satsfying the required conditions. Let us assume that we have defined homomorphisms $\Lambda_2, \Lambda_3, \dots, \Lambda_k$ satsifying the required conditions. Let $v = j_{e_{k-1}}^1 b \in \bar{L}_n^k$, where b is an admissible local isomorphism of $\bar{H}^{k-1}(\mathbf{R}^n)$ into $\bar{H}^{k-1}(\mathbf{R}^n)$. Consider the admissible local isomorphism g of $\bar{H}^k(\mathbf{R}^n)$ into $\bar{H}^k(\mathbf{R}^n)$ defined by the condition:

$$g \circ \eta_k = R_v \circ \Lambda_k \circ R_{v'}^{-1} \circ b \circ \eta_{k-1}$$

with $v' = \pi_{k-1}^k(v)$ and η_i $(i = k-1, k)$ are the «zero sections». It is easy to check that the mapping $v \to \Lambda_{k+1}(v) = j_{e_k}^1 g$ defines a group homomorphism of $\bar{L}_n^k$ into $\bar{L}_n^{k+1}$ with the desired properties. We obtain thus a natural sequence of group homomorphisms

$$L_n^1 \xrightarrow{\Lambda_2} \bar{L}_n^2 \longrightarrow \cdots \longrightarrow \bar{L}_n^k \xrightarrow{\Lambda_{k+1}} \bar{L}_n^{k+1} \longrightarrow \cdots$$

PROPOSITION II.18 *There is a one-to-one correspondence between the set of* $\mathcal{E}$*-connections of order* k *of* V_n *and the set of multi-connections of the form* $\{(\lambda_m, \Lambda_m)\}_{2 \leqslant m \leqslant k}$, *where the* Λ_m *are the homomorphisms of the natural sequence.*

6. Prolongations of linear connections.

We have seen that a linear connection of order 1 of V_n can be given by an invariant section ϕ_2 of $H^1(V_n)$ into $\bar{H}^2(V_n)$. We are going to construct a lift of $\phi_2(H^1(V_n))$ into $\bar{H}^3(V_n)$. Let $u = j^1_{e_1} f \in \phi_2(H^1(V_n))$, where f is an admissible local isomorphism of $H^1(\mathbf{R}^n)$ into $H^1(V_n)$. Let h be the admissible local isomorphism of $\bar{H}^2(\mathbf{R}^n)$ into $\bar{H}^2(V_n)$ defined by: $h \circ \eta_2 = \phi_2 \circ f \circ \eta_1$. The mapping $u \to \phi_2^3(u) = j^1_{e_1} h$ defines a lift of $\phi_2(H^1(V_n))$ into $\bar{H}^3(V_n)$. The composite mapping $\phi_3 = \phi_2^3 \circ \phi_2$ defines an invariant section of $H^1(V_n)$ into $\bar{H}^3(V_n)$. The $\mathcal{E}$-connection ϕ_3 obtained by this way or the corresponding linear connection of order 2 will be called the first prolongation of ϕ_2.The principal subbundle $\phi_3(H^1(V_n))$ of $\bar{H}^3(V_n)$, possesses the following property: for every $v \in \phi_3(H^1(V_n))$, there exists an admissible local isomorphism g of $\bar{H}^2(\mathbf{R}^n)$ into $\bar{H}^2(V_n)$ such that $v = j^1_{e_2} g$ and that g maps the (local) zero section of $\bar{H}^2(\mathbf{R}^n)$ into $\phi_2(H^1(V_n))$. By means of this property, we can construct a lift ϕ_3^4 of $\phi_3(H^1(V_n))$ into $\bar{H}^4(V_n)$ and the composite mapping $\phi_4 = \phi_3^4 \circ \phi_3$ defines an $\mathcal{E}$-connection of order 3, called the second prolongation of ϕ_2. Notice that the projections of ϕ_4 are respectively ϕ_3 and ϕ_2. By iterations, we construct the k-th prolongation of ϕ_2.

If we consider only the prolongations of linear connections of order 1 of V_n, we do not obtain all the linear connections of higher order of V_n, A linear connection of order k is called *simple* if it is the $(k-1)$-th prolongation of a first order linear connection of V_n.

Let ω_k (resp. ω'_k) be a linear connection of order k of V_n (resp. V'_n). We will say that ω_k is *equivalent* to ω'_k if there exists a diffeomorphism f of V_n onto V'_n such that $f^{(k)*}\omega'_k = \omega_k$.

A linear connection ω_k is called *locally flat* if it is locally equivalent to the canonical connection in the trivial bundle $\bar{H}^k(\mathbf{R}^n)=\mathbf{R}^n\times\bar{L}^k_n$.

THEOREM II.19 *A linear connection of order k is locally flat if and only if it is simple, without torsion and without curvature.*

It is well known that a first order connection is locally flat if and only if its torsion and curvature are zero. For $k>1$, the conditions are obviously necessary, because the canonical connection in $\bar{H}^k(\mathbf{R}^n)$ is simple, without torsion and without curvature. Let us show that the conditions are sufficient. Consider such a linear connection ω_k. The connection projection ω_1 of order 1 of ω_k is locally flat, because its torsion and its curvature are both zero. Since ω_k is simple, we can obtain ω_k by taking the successive prolongations of ω_1. Let ϕ_{k+1} be the invariant section of $H^1(V_n)$ into $\bar{H}^{k+1}(V_n)$ corresponding to ω_k. We put $\phi_k=\pi^{k+1}_k\circ\phi_{k+1}$. For all $y\in H^1(V_n)$, the horizontal n-plane of $\bar{H}^k(V_n)$ associated to the $(k+1)$-frame $\phi_{k+1}(y)$ is tangent to $\phi_k(H^1(V_n))$, because ω_k is simple. From the «Holonomy Theorem», we have $\phi_{k+1}(H^1(V_n))\subset H^{k+1}(V_n)$. On the other hand, the nullity of the curvature form of ω_k implies that the distribution of n-planes of $\bar{H}^k(V_n)$ defined by ω_k is involutive. Let W be the maximal integral submanifold passing through $u\in\phi_k(H^1(V_n))$. We have $W\subset\phi_k(H^1(V_n))$. The canonical form θ_k (resp. $\hat{\theta}_k$) of $\bar{H}^k(V_n)$ (resp. $\bar{H}^k(\mathbf{R}^n)$), restricted to W (resp. $Q=\eta_k(\mathbf{R}^n)$), will be denoted by θ_W (resp. $\hat{\theta}_Q$). These forms θ_W and $\hat{\theta}_Q$ have their values in $\mathbf{R}^n\subset\bar{E}^{k-1}$. Consider the 1-form $\beta=p_1^*\theta_W-p_2^*\hat{\theta}_Q$ on the product manifold $W\times Q$, where p_i $(i=1,2)$ are the projections on W and Q respectively. In terms of a basis $\{a^1,a^2,\dots,a^n\}$ for $\mathbf{R}^n$, the components β_i of β are linearly independant. Consider now the module $\mathfrak{M}$ of vector fields X on $W\times Q$ such that $\beta_i(X)=0$ for $i=1,2,\dots,n$. If $X\in\mathfrak{M}$, $Y\in\mathfrak{M}$, we have

$$d\beta(X,Y)=X\beta(Y)-Y\beta(X)-\beta([X,Y])=-\beta([X,Y]).$$

On the other hand, $d\beta(X,Y)=0$. Consequently, $[X,Y]\in\mathfrak{M}$ showing that $\mathfrak{M}$ is involutive. Therefore, there exists a maximal integral submanifold M of dimension n passing through $(u,e_k)\in W\times Q$. For any non-zero vector Z tangent to $p_2^{-1}(e_k)$, $\beta(Z)\neq 0$. We can find an open neighbourhood U of e_k in Q and a differentiable section λ of U into $W\times Q$ such that we have $\lambda(U) \subset M$. Let $b = p_1 \circ \lambda$. The form β vanishes identically on M, we have $\lambda^*\beta = 0$, showing that $\hat{\theta}_Q = b^*\theta_W$. We can now extend b to a local isomorphism $\check{b}$ of $\overline{H}^k(\mathbf{R}^n)$ into $\overline{H}^k(V_n)$ satisfying $\hat{\theta}_k = \tilde{b}^*\theta_k$. In virtue of theorem I.2, we can find an open neighbourhood N (resp. N') of $0 \in \mathbf{R}^n$ (resp. $x = \pi_0^k(u) \in V_n$) and a diffeomorphism f of N onto N' such that locally $\tilde{b} = f^{(k)}$. Consequently, ω_k is locally flat.

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