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Strong convergence in the weighted setting of operator-valued Fourier series defined by the Marcinkiewicz multipliers

Fonctions de la classe de Marcinkiewicz et la convergence forte des séries d'opérateurs de Fourier associées

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ABSTRACT

Suppose that $1 < p < \infty$ and let w be a bilateral weight sequence satisfying the discrete Muckenhoupt A_p weight condition. We show that every Marcinkiewicz multiplier $\psi : \mathbb{T} \to \mathbb{C}$ has an associated operator-valued Fourier series which serves as an analogue in $\mathfrak{B}\left(\ell^p(w)\right)$ of the usual Fourier series of ψ, and this operator-valued Fourier series is everywhere convergent in the strong operator topology. In particular, we deduce that the partial sums of the usual Fourier series of ψ are uniformly bounded in the Banach algebra of Fourier multipliers for $\ell^p(w)$. These results transfer to the framework of invertible, modulus mean-bounded operators acting on L^p spaces of sigma-finite measures.

RÉSUMÉ

Soient $1 < p < \infty$ et w un poids dans la classe $A_p(\mathbb{Z})$. Cette note établit (dans la topologie forte des opérateurs) la convergence des séries de Fourier (à valeurs dans $\mathfrak{B}\left(\ell^p(w)\right)$) pour les « convolutions de Stieltjes », où ces convolutions sont déterminées par les fonctions ψ appartenant à la classe de Marcinkiewicz $\mathfrak{M}_1(\mathbb{T})$. Les propriétés de convergence pour ces séries de Fourier ayant valeurs dans $\mathfrak{B}\left(\ell^p(w)\right)$ révèlent des propriétés de convergence des séries de Fourier traditionnelles pour les fonctions $\psi \in \mathfrak{M}_1(\mathbb{T})$. En particulier, les sommes partielles de la série de Fourier traditionnelle pour un $\psi \in \mathfrak{M}_1(\mathbb{T})$ quelconque sont uniformément bornées dans la norme des p-multiplicateurs pour $\ell^p(w)$. Ces résultats se transfèrent immédiatement au cadre d'une bijection linéaire arbitraire T telle que T soit un opérateur préservant la disjonction dont le module linéaire est à moyennes bornées.

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1. Introduction

The symbol K with (a possibly empty) set of subscripts denotes a constant which depends only on those subscripts, and which may change in value from one occurrence to another. The characteristic function of an arc $\mathfrak{A} \subseteq \mathbb{T}$ will be symbolized by $\chi_{\mathfrak{A}}$. For our treatment of Marcinkiewicz multipliers we shall make free use of the standard notation for the sequence $\{t_k\}_{k=-\infty}^{\infty}$ of dyadic points of the interval (0.2π), which are defined as $2^{k-1}\pi$ if $k \le 0$, and $2\pi - 2^{-k}\pi$ if $k > 0$. For $1 < p < \infty$, a weight sequence $w \equiv \{w_k\}_{k=-\infty}^{\infty}$ belongs to the class $A_p(\mathbb{Z})$ provided that there is a real constant C (called an $A_p(\mathbb{Z})$ *weight constant for* w) such that

$$\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k\right)\left(\frac{1}{M-L+1}\sum_{k=L}^{M} w_k^{-1/(p-1)}\right)^{p-1} \le C,$$

whenever $L \in \mathbb{Z}$, $M \in \mathbb{Z}$, and $L \le M$. We denote the corresponding sequence space by $\ell^p(w)$. We say that $\psi \in L^\infty(\mathbb{T})$ is a *multiplier for* $\ell^p(w)$ (in symbols, $\psi \in M_{p,w}(\mathbb{T})$) provided that convolution by the inverse Fourier transform of ψ defines a bounded operator on $\ell^p(w)$. Specifically, we require:

Definition 1.1.

(i) For each $x \equiv \{x_k\}_{k=-\infty}^{\infty} \in \ell^p(w)$ and each $j \in \mathbb{Z}$, the series

$$\left(\psi^\vee * x\right)(j) \equiv \sum_{k=-\infty}^{\infty} \psi^\vee(j-k)\, x_k \text{ converges absolutely, and}$$

(ii) the mapping $\mathcal{T}_\psi^{(p,w)} : x \in \ell^p(w) \to \psi^\vee * x$ is a bounded linear mapping of $\ell^p(w)$ into $\ell^p(w)$.

We then call $\mathcal{T}_\psi^{(p,w)}$ the multiplier transform corresponding to ψ, and define the multiplier norm by setting $\|\psi\|_{M_{p,w}(\mathbb{T})} \equiv \left\|\mathcal{T}_\psi^{(p,w)}\right\|_{\mathfrak{B}(\ell^p(w))}$. In particular, it is well-known that $\mathfrak{M}_1(\mathbb{T}) \subseteq M_{p,w}(\mathbb{T})$, where $\mathfrak{M}_1(\mathbb{T})$ is the Banach algebra of periodic Marcinkiewicz multipliers, consisting of all functions $\psi : \mathbb{T} \to \mathbb{C}$ such that

$$\|\psi\|_{\mathfrak{M}_1(\mathbb{T})} \equiv \sup_{z\in\mathbb{T}} |\psi(z)| + \sup_{k\in\mathbb{Z}} \operatorname{var}(\psi, \Delta_k) < \infty$$

(here Δ_k is the dyadic arc of $\mathbb{T}$ specified by $\Delta_k = \left\{e^{i\theta} : \theta \in [t_k, t_{k+1}]\right\}$). Moreover, $\|\psi\|_{M_{p,w}(\mathbb{T})} \le K_{p,C}\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}$. A key structural example of an element of $M_{p,w}(\mathbb{T})$ is furnished, for each $k \in \mathbb{Z}$, by the function $\mathfrak{e}_k(z) \equiv z^k$, whose multiplier transform is $\mathcal{L}^k$, where $\mathcal{L}$ designates the left bilateral shift on $\ell^p(w)$. In particular, for each $\phi \in L^1(\mathbb{T})$, the $n^{\underline{th}}$ partial sum of its Fourier series $s_n\left(\phi, e^{i\theta}\right) \equiv \sum_{k=-n}^{n} \widehat{\phi}(k)\, e^{ik\theta}$ belongs to $M_{p,w}(\mathbb{T})$, with multiplier transform expressed by

$$\mathcal{T}_{s_n(\phi,\cdot)}^{(p,w)} = \sum_{k=-n}^{n} \widehat{\phi}(k)\,\mathcal{L}^k.$$

For further background items concerning our framework, the reader is referred to [1–3]. Our main result can now be stated as follows.

Theorem 1.2. *Suppose that $\psi \in \mathfrak{M}_1(\mathbb{T})$. Then whenever $1 < p < \infty$, and $w \in A_p(\mathbb{Z})$ with an $A_p(\mathbb{Z})$ weight constant C, we have:*

$$\sup\left\{\|s_n(\psi_z, (\cdot))\|_{M_{p,w}(\mathbb{T})} : n \ge 0, z \in \mathbb{T}\right\} \le K_{p,C}\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}, \tag{1.1}$$

where $\psi_z(\cdot) \equiv \psi((\cdot)z)$. Consequently, $\sum_{k=-\infty}^{\infty} z^k \widehat{\psi}(k)\,\mathcal{L}^k$, the Fourier series of $\mathcal{T}_{\psi_z}^{(p,w)}$ relative to the strong operator topology of $\mathfrak{B}\left(\ell^p(w)\right)$, converges in the strong operator topology to $\mathcal{T}_{\psi_z}^{(p,w)}$ at each $z \in \mathbb{T}$.

Thanks to the Dominated Ergodic Estimate Theorem of F.J. Martín-Reyes and A. de la Torre (in the form and notation of Theorem 2.5 in [3]), one can transfer Theorem 1.2 to a broader framework, where the following outcome ensues.

Theorem 1.3. *Suppose that $1 < p < \infty$, (Ω, μ) is a sigma-finite measure space, and $\mathfrak{U} \in \mathfrak{B}\left(L^p(\mu)\right)$ is an invertible, disjoint, modulus mean-bounded operator. Let $\mathcal{E}(\cdot) : \mathbb{R} \to \mathfrak{B}\left(L^p(\mu)\right)$ be the (idempotent-valued) spectral decomposition of $\mathfrak{U}$, and let $\psi \in \mathfrak{M}_1(\mathbb{T})$ be a continuous function. Then*

$$\sup\left\{\left\|\sum_{k=-n}^{n} z^k \widehat{\psi}(k)\,\mathfrak{U}^k\right\|_{\mathfrak{B}(L^p(\mu))} : n \geq 0, z \in \mathbb{T}\right\} \leq K_{p,\mathfrak{C}} \|\psi\|_{\mathfrak{M}_1(\mathbb{T})}, \tag{1.2}$$

where $\mathfrak{C}$ is the common $A_p(\mathbb{Z})$ weight constant of the weights $w^{(x)}$, $x \in \Omega$. Moreover, $\sum_{k=-\infty}^{\infty} z^k \widehat{\psi}(k)\,\mathfrak{U}^k$ the Fourier series (in the strong operator topology of $\mathfrak{B}(L^p(\mu))$) for the Stieltjes convolution $\int_{[0,2\pi]}^{\oplus} \psi_z(e^{it})\, d\mathcal{E}(t)$ converges to $\int_{[0,2\pi]}^{\oplus} \psi_z(e^{it})\, d\mathcal{E}(t)$ in the strong operator topology at each $z \in \mathbb{T}$.

2. Proof of Theorem 1.2

The key to demonstration of Theorem 1.2 resides in the following seminal forerunner.

Theorem 2.1. *Suppose that $1 < p < \infty$, $w \in A_p(\mathbb{Z})$ with an $A_p(\mathbb{Z})$ weight constant C, and $\psi \in \mathfrak{M}_1(\mathbb{T})$. Then we have:*

$$\sup\left\{\left\|\mathcal{T}^{(p,w)}_{s_n(\psi,\cdot)}\mathcal{T}^{(p,w)}_{\chi_{\Delta_m}}\right\|_{\mathfrak{B}(\ell^p(w))} : n \geq 0, m \in \mathbb{Z}\right\} \leq K_{p,C} \|\psi\|_{\mathfrak{M}_1(\mathbb{T})}. \tag{2.1}$$

Proof (Sketch). For each non-negative integer $\mathfrak{m}$, define $\mathcal{I}_{\mathfrak{m}}$ to be the arc $\{e^{i\theta} : t_{-\mathfrak{m}} \leq \theta \leq t_{\mathfrak{m}}\}$, and let $\chi_{\mathfrak{m}}$ symbolize the characteristic function, defined on $\mathbb{T}$, of $\mathcal{I}_{\mathfrak{m}}$. Define $\psi_{\mathfrak{m}} \in BV(\mathbb{T})$ by putting $\psi_{\mathfrak{m}} \equiv \psi\chi_{\mathfrak{m}}$. Temporarily fix an arbitrary non-negative integer n, and observe that there is a non-negative integer ν (in general, depending on n) such that, for arbitrary $z \in \mathbb{T}$,

$$\left\|s_n(\psi_z,\cdot) - s_n\left(((\psi_\nu)_z,\cdot)\right)\right\|_{M_{p,w}(\mathbb{T})} < \|\psi\|_{\mathfrak{M}_1(\mathbb{T})}. \tag{2.2}$$

Keeping this value of ν fixed, we now shift our attention to ψ_ν, in order to circumvent the dependence of ψ_ν on n by reducing matters to the pleasant operator-valued Fourier series phenomena associated with multiplier transforms defined by $BV(\mathbb{T})$ functions (as evinced in Theorem 4.4 of [1] and Theorem 4.1 of [2], which apply to spectral decomposability set in a broader framework that specializes to ours). Temporarily fix $z \in \mathbb{T}$, $m \in \mathbb{Z}$, m non-negative. For all $\varsigma \in \mathbb{T}$, let us consider the following expression.

$$F(\varsigma) \equiv \mathcal{T}^{(p,w)}_{((\psi_\nu)_z\chi_{\Delta_m})_\varsigma}. \tag{2.3}$$

On the right-hand side of (2.3) we can apply in succession the following items from [2]: Theorem 4.1; Theorem 4.5; and (3.2). Along with careful simplifications, this procedure shows that for arbitrary fixed $z \in \mathbb{T}$,

$$\sup\left\{\left\|s_N\left(\left((\psi_\nu)_z\chi_{\Delta_m}\right)_\varsigma,\cdot\right)\right\|_{M_{p,w}(\mathbb{T})} : N \geq 0, \varsigma \in \mathbb{T}\right\} \leq K_{p,C}\left\|\left\{(\psi_\nu)_z\chi_{\Delta_m}\right\}(\cdot)\right\|_{BV(\mathbb{T})}. \tag{2.4}$$

In order to profit from this estimate, notice that the $BV(\mathbb{T})$ function involved in the majorant of (2.4) – specifically, $\varsigma \in \mathbb{T} \mapsto \{(\psi_\nu)_z\chi_{\Delta_m}\}(\varsigma)$ – vanishes outside at most two disjoint closed subarcs of the fixed dyadic arc Δ_m, and coincides with ψ_z on each of these subarcs. Hence if we confine z to the arc $\mathcal{A}_m \equiv \{e^{i\theta} : 0 \leq \theta \leq t_{m+2} - t_{m+1}\}$, straightforward reasoning yields

$$\left\|\left\{(\psi_\nu)_z\chi_{\Delta_m}\right\}(\cdot)\right\|_{BV(\mathbb{T})} \leq K\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}.$$

Applying this to (2.4) we infer that

$$\sup\left\{\left\|s_N\left(\left((\psi_\nu)_z\chi_{\Delta_m}\right)_\varsigma,\cdot\right)\right\|_{M_{p,w}(\mathbb{T})} : N \geq 0, \varsigma \in \mathbb{T}, z \in \mathcal{A}_m\right\} \leq K_{p,C}\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}. \tag{2.5}$$

By specializing the result in (2.5) to the case where the parameters $\varsigma \in \mathbb{T}$ and $z \in \mathcal{A}_m$ are both taken to be 1, we arrive at the following central estimate.

$$\sup\left\{\left\|s_N(\psi_\nu\chi_{\Delta_m},\cdot)\right\|_{M_{p,w}(\mathbb{T})} : N \geq 0\right\} \leq K_{p,C}\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}. \tag{2.6}$$

Extensive calculations proceeding from (2.6) can be carried out to show that

$$\sup\left\{\left\|\chi_{\Delta_m}s_N(\psi_\nu,\cdot)\right\|_{M_{p,w}(\mathbb{T})} : N \geq 0\right\} \leq K_{p,C}\|\psi\|_{\mathfrak{M}_1(\mathbb{T})}.$$

We omit the details here for expository reasons. Applying this last estimate to the fixed but arbitrary non-negative integer n in (2.2), we readily deduce (2.1) with the aid of standard features of A_p weighted spaces. □

Proof of Theorem 1.2. When Theorem 2.1 is specialized to the setting $p = 2$ and applied in conjunction with the Littlewood–Paley inequalities for weighted spaces, we easily see that (1.1) holds for all $A_2(\mathbb{Z})$ weights. By invoking a suitable version of the recent "streamlined" rendition of Rubio de Francia's Extrapolation Theorem (see Theorem 3.1 of [4]), we readily obtain (1.1) in the full range $1 < p < \infty$. The remaining conclusion of Theorem 1.2 can now be seen from this general case of (1.1) by calculations based on the norm density in $\ell^p(w)$ of the finitely supported vectors. □

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