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Bernstein–Frechet inequalities for the parameter of the first order autoregressive process

Abdelnasser Dahmani, Megdouda Tari

Laboratory of Applied Mathematics, Department of Mathematics, University of Bejaia, Algeria

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Abstract

The autoregressive process takes an important part in predicting problems leading to decision making. In practice, we use the least squares method to estimate the parameter of the autoregressive process. In the case of the first order autoregressive process, we know that the least squares estimator converges in probability to the unknown parameter θ. In this Note, we show that the least squares estimator converges almost completely to θ and so we construct the inequalities of type Bernstein–Frechet for the coefficient of the first order autoregressive process. Using these inequalities a confidence interval is then obtained. ***To cite this article: A. Dahmani, M. Tari, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Inégalités de Bernstein–Fréchet pour le paramètre du processus autoregressif d'ordre 1. Les processus linéaires, en général et les processus autoregressifs, en particulier jouent un rôle important dans la prévision qui est fondamentale dans la mesure où elle à la base de l'action. En pratique on utilise la méthode des moindres carrées pour estimer les paramètres d'un processus autoregressif en minimisant la somme des erreurs au carrées. Dans le cas d'un processus autoregressif d'ordre 1, on sait que l'estimateur des moindres carrées converge en probabilités vers le paramètre inconnu θ. Dans cette Note, on montre que cet estimateur converge presque complétement vers θ et nous construisons des inégalités de type Bernstein–Fréchet pour le coefficient du processus autoregressif d'ordre 1. Ces inégalités nous ont permis de construire un intervalle de confiance pour ce coefficient. ***Pour citer cet article : A. Dahmani, M. Tari, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

E-mail addresses: a_dahmany@yahoo.fr (A. Dahmani), megtari@yahoo.fr (M. Tari).

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Dans ce travail, nous montrons que l'estimateur des moindres carrées converge complètement vers le paramètre θ du processus autoregressif d'ordre 1 et nous lui construisons comme en [1] des inégalités de type Bernstein–Fréchet. Ces inégalités nous permettront de construire un intervalle de confiance pour ce paramètre.

Soit $(\Omega, \mathcal{F}, P)$ un espace probabilisé. Considérons le processus autoregressif strictement stationnaire d'ordre 1 AR(1) défini par $Y_i = \xi_i + \theta Y_{i-1}$ où θ est le paramètre autorégressif. La suite $(\xi_i)_i$ est un bruit blanc fortement Gaussien. En considérant Y_{i-1} comme une variable exogène, la méthode des moindres carrés ordinaires donne comme estimateur de θ la suite définie par (2).

Nous savons que θ_n converge en probabilité vers θ aussi bien dans le cas stable que dans le cas explosif. Par ailleurs, l'expression $\sqrt{n}(\theta_n - \theta)$ a une distribution limite normale dans le cas stable et l'expression $\sqrt{\sum_{i=1}^{n} Y_{i-1}}(\theta_n - \theta)$ a une distribution normale dans le cas stable et dans le cas explosif. Nous énonçons le résultat suivant. Pour tout ε positif, nous avons

$$P\left\{\sqrt{n}|\theta_n - \theta| > R\right\} \leqslant 2\,\mathrm{e}^{-R^2/2} + 3^{-n/2+1}\,\mathrm{e}^{-n\varepsilon}. \tag{1}$$

A partir duquel nous déduisons que la suite $(\theta_n)_{n\in\mathbb{N}}$ définie en (2) converge presque complètement vers le paramètre θ du processus autoregressif d'ordre 1.

Remarque 1. Les inégalités (4) nous permettent de construire un intervalle de confiance pour le paramètre du processus autoregressif d'ordre 1.

Ainsi, pour R assez grand, par exemple $R = \sqrt{n}\varepsilon$, nous avons $\lim_{n\to+\infty} 2\,\mathrm{e}^{-n\varepsilon/2} + 3^{-n/2+1}\,\mathrm{e}^{-n\varepsilon} = 0$ c'est à dire, pour un seuil donné γ, il existe un entier naturel n_γ pour lequel nous avons $n \geqslant n_\gamma \Rightarrow 2\,\mathrm{e}^{-n\varepsilon/2} + 3^{-n/2+1}\,\mathrm{e}^{-n\varepsilon} \leqslant \gamma$. Par conséquent, $P\{|\theta_{n_\gamma} - \theta| \leqslant \varepsilon\} \geqslant 1 - \gamma$ ce qui veut dire que le paramètre du processus autoregressif d'ordre 1 appartient à l'intervalle fermé de centre $\theta_{n_\gamma+1}$ et de rayon ε avec une probabilité supérieure ou égale à $1 - \gamma$.

Remarque 2. Dans le cas stable, nous savons que l'estimateur θ_n est asymptotiquement normal de moyenne θ et de variance $(1 - \theta^2)$, on peut donc déduire la loi de $\sqrt{n}(\theta_n - \theta)$.

1. Introduction

The study of the autoregressive models constitutes one of the fundamental problems posed by the analysis of the time series in econometrics and statistics [2,4,7,8]. In a general way, the analysis of the autoregressive models makes it possible to establish controls to facilitate development of the forecasts and to lead to the reduction of the undesirable fluctuations. The estimation of the unknown parameters in an autoregressive model is generally done by solving a nonlinear system of Eq. [5]. This method is often heavy and not very precise. We prefer more general methods of a criteria minimization such as the method of least squares or that by sweeping in the space of the parameters. Dufour in 1990 built a confidence interval for processes AR(1) by solving two polynomials of the second degree [3].

Bondarev has obtained in [1] exponential inequalities which have been used to construct a confidence interval for the unknown parameter θ_0 in the equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \theta_0 f\big(t, x(t)\big) + \xi'(t), \qquad x(0) = \xi(0) = 0$$

where ξ' is a Gaussian noise with zero mean and known correlation function.

In practice, we use the least squares method to estimate the parameter of the autoregressive process. In the case of the first order autoregressive process we know that the least squares estimator converges in probability to the unknown parameter θ.

In this Note, we show that this estimator converges almost completely to θ and so we construct, as in [1], the inequalities of type Bernstein–Frechet for the parameter of first order autoregressive process. Theses inequalities will allow us to construct a confidence interval for this parameter.

2. Model and estimator

Let $(\Omega, \mathcal{F}, P)$ be the probability space and let us consider the strictly stationary autoregressive process of first order AR(1) defined by $Y_i = \xi_i + \theta Y_{i-1}$ where θ is the autoregressive parameter. The sequence $(\xi_i)_i$ is a Gaussian strong noise. Considering Y_{i-1} as an output variable, the ordinary least squares method gives an estimator of θ which is the sequence defined by

$$\theta_n = \frac{\sum_{k=1}^n Y_k Y_{k-1}}{\sum_{k=1}^n Y_{k-1}^2}. \tag{2}$$

We know that θ_n converges in probability to θ even in the stable case i.e. $|\theta| < 1$ and in the explosive case ($|\theta| \geqslant 1$). On the other hand, the expression $\sqrt{n}(\theta_n - \theta)$ has a normal limit distribution in the stable case and the expression $\sqrt{\sum_{i=1}^n Y_{i-1}^2}(\theta_n - \theta)$ has a normal distribution in the stable case and in the explosive case.

3. Theorems

We know that if we denote ρ as a density of $Y = (Y_1, Y_2, \ldots, Y_n)$ and p_ξ as a density of $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. The map $Y \to \xi = T(Y)$ is linear and is described by a lower triangular matrix with unit diagonal element, hence of unit determinant, given that T is a bijective application changing the vector ξ on the vector Y. Then $\rho(y) = p_\xi(T(y))$.

Using the changing variables theorem, [6], we obtain

$$\rho(y) = \exp\left(\theta \sum_{k=1}^{n-1} y_k y_{k+1} - \frac{\theta^2}{2} \sum_{k=1}^{n-1} y_k^2\right). \tag{3}$$

Theorem 3.1. *For any $\varepsilon < \frac{1}{2}\log 3$ positive, we have*

$$P\left\{\sqrt{n}|\theta_n - \theta| > R\right\} \leqslant 2\,\mathrm{e}^{-R^2/2\varepsilon} + 3^{-(n-5)/4}\,\mathrm{e}^{n\varepsilon}. \tag{4}$$

Proof. Firstly, we notice that:

$$\theta_n - \theta = \frac{\sum_{k=1}^n Y_{k-1}\xi_k}{\sum_{k=1}^n Y_{k-1}^2}.$$

It follows that,

$$P\left\{\sqrt{n}|\theta_n - \theta| > R\right\} = P\left\{\left|\frac{\frac{1}{\sqrt{n}}\sum_{k=1}^n Y_{k-1}\xi_k}{\frac{1}{n}\sum_{k=1}^n Y_{k-1}^2}\right| > R\right\}.$$

As ξ_k is symmetrically distributed then, for any z positive, we have

$$P\left\{\sqrt{n}|\theta_n - \theta| > R\right\} \leqslant 2P\left\{\frac{z}{\sqrt{n}}\sum_{k=1}^n Y_{k-1}\xi_k - \frac{z^2}{2n}\sum_{k=1}^n Y_{k-1}^2 > \frac{1}{n}\left(Rz - \frac{z^2}{2}\right)\sum_{k=1}^n Y_{k-1}^2\right\}.$$

By virtue of the probability properties, we have for any ε positive

$$P\left\{\sqrt{n}|\theta_n-\theta|>R\right\}\leqslant 2P\left\{\frac{z}{\sqrt{n}}\sum_{k=1}^{n}Y_{k-1}\xi_k-\frac{z^2}{2n}\sum_{k=1}^{n}Y_{k-1}^2>\left(Rz-\frac{z^2}{2}\right)\varepsilon\right\}+2P\left\{\frac{1}{n}\sum_{k=1}^{n}Y_{k-1}^2\leqslant\varepsilon\right\}. \quad (5)$$

Let us now bound the first probability of the right-hand side of the latter inequality.

According to the Chernoff inequality, we obtain the following result:

$$\begin{aligned}&P\left\{\frac{z}{\sqrt{n}}\sum_{k=1}^{n}Y_{k-1}\xi_k-\frac{z^2}{2n}\sum_{k=1}^{n}Y_{k-1}^2>\left(Rz-\frac{z^2}{2}\right)\varepsilon\right\}\\&\quad\leqslant\exp\left(-\left(Rz-\frac{z^2}{2}\right)\varepsilon\right)E\left[\exp\left(\frac{z}{\sqrt{n}}\sum_{k=1}^{n}Y_{k-1}\xi_k-\frac{z^2}{2n}\sum_{k=1}^{n}Y_{k-1}^2\right)\right]\end{aligned} \quad (6)$$

where E design the mathematical expectation.

Using the properties of the conditionals expectations (ξ_{k+1} does not depend on the variables $Y_1, Y_2, \ldots, Y_k$), we have

$$E\left[\exp\left(\frac{z}{\sqrt{n}}\sum_{k=1}^{n}Y_{k-1}\xi_k-\frac{z^2}{2n}\sum_{k=1}^{n}Y_{k-1}^2\right)\right]=1 \quad (7)$$

indeed,

$$\begin{aligned}&E\left[\exp\left(\frac{z}{\sqrt{n}}\sum_{k=1}^{n}Y_{k-1}\xi_k-\frac{z^2}{2n}\sum_{k=1}^{n}Y_{k-1}^2\right)\right]\\&\quad=E\left[\exp\left(\frac{z}{\sqrt{n}}\sum_{k=1}^{n-1}Y_{k-1}\xi_k-\frac{z^2}{2n}\sum_{k=1}^{n-1}Y_{k-1}^2\right)\right]E\left[\exp\left(\frac{z}{\sqrt{n}}Y_{n-1}\xi_n-\frac{z^2}{2n}Y_{n-1}^2\right)|\mathcal{F}_{n-1}\right]\end{aligned}$$

where $\mathcal{F}_n$ is the σ-algebra generated by $\xi_1, \xi_2, \ldots, \xi_n$ and, since

$$E\left[\exp\left(\frac{z}{\sqrt{n}}Y_{n-1}\xi_n-\frac{z^2}{2n}Y_{n-1}^2\right)\Big|\mathcal{F}_{n-1}\right]=1$$

the assertion results from n recurrence.

By virtue of the relations (6) and (7) and taking $z=R$, it follows

$$P\left\{\frac{z}{\sqrt{n}}\sum_{k=1}^{n}Y_{k-1}\xi_k-\frac{z^2}{2n}\sum_{k=1}^{n}Y_{k-1}^2>\left(Rz-\frac{z^2}{2}\right)\varepsilon\right\}\leqslant\exp\left(-\frac{R^2}{2}\varepsilon\right). \quad (8)$$

We will bound now, the second probability of the right-hand side of the expression (5). According to the Chernoff inequality, it follows for any u positive

$$P\left\{\frac{1}{n}\sum_{k=1}^{n}Y_{k-1}^2\leqslant\varepsilon\right\}\leqslant\exp(u\varepsilon)E\exp\left(-\frac{u}{n}\sum_{k=1}^{n}Y_{k-1}^2\right).$$

In accordance with the properties of the absolute continuity of the measure generated by $\xi_1, \xi_2, \ldots, \xi_n$ in comparison with the measure generated by $Y_1, Y_2, \ldots, Y_n$ we have

$$E\exp\left(-\frac{u}{n}\sum_{k=1}^{n}Y_{k-1}^2\right)=E\left[\rho(\xi_1,\ldots,\xi_n)\exp\left(-\frac{u}{n}\sum_{k=1}^{n}\xi_k^2\right)\right].$$

Substituting the expression (3) in the last relation, we write

$$E\exp\left(-\frac{u}{n}\sum_{k=1}^{n}Y_{k-1}^2\right)\leqslant E\left[\exp\left(-\frac{u}{2n}\sum_{k=1}^{n-1}\xi_k^2\right)\exp\left(\theta\sum_{k=1}^{n-1}\xi_k\xi_{k+1}-\frac{1}{2}\left(\theta^2+\frac{u}{n}\right)\sum_{k=1}^{n-1}\xi_k^2\right)\right]. \tag{9}$$

For a suitably chosen λ such as

$$E\exp\left(\lambda\theta\sum_{k=1}^{n-1}\xi_k\xi_{k+1}-\frac{\lambda}{2}\left(\theta^2+\frac{u}{n}\right)\sum_{k=1}^{n-1}\xi_k^2\right)=1. \tag{10}$$

This choice is always possible, for example, taking $(\lambda\theta)^2=\lambda(\theta^2+\frac{u}{n})$ or even $\lambda=1+\frac{u}{n\theta^2}$. Let $p=1+\frac{n\theta^2}{u}$ and $q=1+\frac{u}{n\theta^2}$. Applying the Hölder inequality to the right-hand side of (9), we have

$$E\exp\left(-\frac{u}{n}\sum_{k=1}^{n}Y_{k-1}^2\right)\leqslant\left[E\exp\left(-\frac{1}{2}\left(\theta^2+\frac{u}{n}\right)\sum_{k=1}^{n-1}\xi_k^2\right)\right]^{u/(u+n\theta^2)}.$$

For $u=n$, we obtain

$$E\exp\left(-\frac{u}{n}\sum_{k=1}^{n}Y_{k-1}^2\right)\leqslant\left[E\exp\left(-\frac{1}{2}(\theta^2+1)\sum_{k=1}^{n-1}\xi_k^2\right)\right]^{1/(1+\theta^2)}.$$

Taking into account the Lyapunov inequality and the stationary of the process Y_i, we have the inequality

$$E\exp\left(-\frac{u}{n}\sum_{k=1}^{n}Y_{k-1}^2\right)\leqslant\left[E\exp\left(-\sum_{k=1}^{n-1}\xi_k^2\right)\right]^{1/2}.$$

The following result also holds, since ξ_i and ξ_i^2 are independently and identically distributed random variables.

$$E\exp\left(-\sum_{k=1}^{n-1}\xi_{k-1}^2\right)=\prod_{k=1}^{n-1}E\exp(-\xi_{k-1}^2)=\prod_{k=1}^{n-1}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2-x^2/2}\,\mathrm{d}x=3^{-(n-1)/2}$$

we have then

$$P\left\{\frac{1}{n}\sum_{k=1}^{n}Y_{k-1}^2\leqslant\varepsilon\right\}\leqslant 3^{-(n-1)/4}\exp(n\varepsilon). \tag{11}$$

Considering the relations (8) and (11) together and taking in to account the expression (5) we obtain the result. □

Corollary 3.2. *The sequence* $(\theta_n)_{n\in\mathbb{N}}$ *defined in* (2) *converges almost completely to the parameter* θ *of the first order autoregressive process.*

Proof. The convergence almost complete follows from the inequalities (4).

Indeed, applying the Cauchy rule on the positive real term sequences u_n where the general term is defined by $u_n=2\,\mathrm{e}^{-\frac{n}{2}\varepsilon^3}+3^{-(n-5)/4}\,\mathrm{e}^{n\varepsilon}$ It follows that $\sum_{n=1}^{\infty}P\{\sqrt{n}|\theta_n-\theta|>R\}<+\infty$. which yields the result. □

Remark 1. The inequalities (4) give us the possibility to construct a confidence interval for the parameter of the first order autoregressive process.

For large R, such as $R=\varepsilon\sqrt{n}$, it follows $\lim_{n\to+\infty}2\,\mathrm{e}^{-\frac{n}{2}\varepsilon^3}+3^{-(n-5)/4}\,\mathrm{e}^{n\varepsilon}=0$ which means, for a given level γ, we can found a natural integer n_γ such that, $\forall n\geqslant n_\gamma$ we have $2\,\mathrm{e}^{-\frac{n}{2}\varepsilon^3}+3^{-(n-5)/4}\,\mathrm{e}^{n\varepsilon}\leqslant\gamma$ Consequently,

$P\{|\theta_{n_\gamma} - \theta| \leqslant \varepsilon\} \geqslant 1 - \gamma$, which means that the parameter of the first order autoregressive process belongs to the inclusive interval of centre θ_{n_γ} and radius ε with a probability greater or equal to $1 - \gamma$.

Remark 2. In the stable case, it is well known that the estimator θ_n is asymptotically normal with mean θ and variance $(1 - \theta^2)$, this statement allows us to deduce the law of $\sqrt{n}(\theta_n - \theta)$.

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