Available online at www.sciencedirect.com

SCIENCE DIRECT®

C. R. Acad. Sci. Paris, Ser. I 340 (2005) 301–304

COMPTES RENDUS MATHEMATIQUE

http://france.elsevier.com/direct/CRASS1/

Probability Theory/Mathematical Physics

A (one-dimensional) free Brunn–Minkowski inequality

Michel Ledoux

Institut de mathématiques, université Paul-Sabatier, 31062 Toulouse, France

Received 28 October 2004; accepted after revision 17 December 2004

Available online 11 January 2005

Presented by Gilles Pisier

Abstract

We present a one-dimensional version of the functional form of the geometric Brunn–Minkowski inequality in free (non-commutative) probability theory. The proof relies on matrix approximation as used recently by Biane and Hiai et al. to establish free analogues of the logarithmic Sobolev and transportation cost inequalities for strictly convex potentials, that are recovered here from the Brunn–Minkowski inequality as in the classical case. The method is used to extend to the free setting the Otto–Villani theorem stating that the logarithmic Sobolev inequality implies the transportation cost inequality. ***To cite this article: M. Ledoux, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Une inégalité (uni-dimensionnelle) de Brunn–Minkowski libre. Nous présentons une version uni-dimensionnelle de la forme fonctionnelle de l'inégalité géométrique de Brunn–Minkowski en théorie des probabilités libres. L'argument s'appuie sur l'approximation matricielle déjà mise en œuvre récemment par Biane et Hiai et al. pour établir les analogues libres des inégalités de Sobolev logarithmique et de coût du transport pour des potentiels strictement convexes, qui sont ici déduits de l'inégalité de Brunn–Minkowski comme dans le cas classique. La méthode permet, de la même façon, d'étendre au cadre libre le théorème d'Otto–Villani assurant que l'inégalité de Sobolev logarithmique entraîne l'inégalité de transport. ***Pour citer cet article : M. Ledoux, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Brunn–Minkowski inequality and random matrix approximation

In its functional form (known as the Prékopa–Leindler theorem), the Brunn–Minkowski inequality indicates that whenever $\theta \in (0,1)$ and u_1, u_2, u_3 are non-negative measurable functions on $\mathbb{R}^n$ such that

$$u_3\big(\theta x + (1-\theta)y\big) \geqslant u_1(x)^\theta u_2(y)^{1-\theta} \quad \text{for all } x, y \in \mathbb{R}^n,$$

E-mail address: ledoux@math.ups-tlse.fr (M. Ledoux).

1631-073X/$ – see front matter © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.
doi:10.1016/j.crma.2004.12.017

then

$$\int u_3\,\mathrm{d}x \geqslant \left(\int u_1\,\mathrm{d}x\right)^{\theta}\left(\int u_2\,\mathrm{d}x\right)^{1-\theta}.$$

The Brunn–Minkowski inequality has been used recently in the investigation of functional inequalities for strictly log-concave densities such as logarithmic Sobolev or transportation cost inequalities (cf. e.g. [10,15]). The pertinence of Hamilton–Jacobi equations in this investigation has been particularly emphasized in [5,11]. The aim of this Note is to proceed to a similar scheme in the context of one-dimensional free probability theory, using random matrix approximation following the recent investigations by Biane [2] and Hiai, Petz and Ueda [7,8]. We rely specifically on the large deviation asymptotics of spectral measures of unitary invariant Hermitian random matrices put forward by Voiculescu [16] and Ben Arous and Guionnet [1] (cf. [6]). Given a continuous function $Q:\mathbb{R}\to\mathbb{R}$ such that $\lim_{|x|\to\infty}|x|\,\mathrm{e}^{-\varepsilon Q(x)}=0$ for every $\varepsilon>0$, set

$$\widetilde{Z}_N(Q)=\int\limits_A \Delta_N(x)^2\,\mathrm{e}^{-N\sum_{k=1}^N Q(x_k)}\,\mathrm{d}x$$

where $A=\{x_1<x_2<\cdots<x_N\}\subset\mathbb{R}^N$ and $\Delta_N(x)=\prod_{1\leqslant k<\ell\leqslant N}(x_\ell-x_k)$ is the Vandermonde determinant. The large deviation theorem of [16] and [1] (see also [9]) indicates that

$$\lim_{N\to\infty}\frac{1}{N^2}\log\widetilde{Z}_N(Q)=\mathcal{E}_Q(\nu_Q) \tag{1}$$

where, for every probability measure ν on $\mathbb{R}$,

$$\mathcal{E}_Q(\nu)=\iint\log|x-y|\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)-\int Q(x)\,\mathrm{d}\nu(x)$$

is the weighted energy integral with extremal (compactly supported) measure ν_Q maximizing $\mathcal{E}_Q$ (cf. [13,6]). (For the choice of $Q(x)=\frac{x^2}{2}$, ν_Q is the semicircle law.)

Let U_1, U_2, U_3 be real-valued continuous functions on $\mathbb{R}$ such that, for every $\varepsilon>0$, $\lim_{|x|\to\infty}|x|\,\mathrm{e}^{-\varepsilon U_i(x)}=0$, $i=1,2,3$. Set

$$u_i(x)=\Delta_N(x)^2\,\mathrm{e}^{-N\sum_{k=1}^N U_i(x_k)}\mathbf{1}_A(x),\quad x\in\mathbb{R}^N,\ i=1,2,3.$$

Since $-\log\Delta_N$ is convex on the convex set A, assuming that, for some $\theta\in(0,1)$ and all $x,y\in\mathbb{R}$, $U_3(\theta x+(1-\theta)y)\leqslant\theta U_1(x)+(1-\theta)U_2(y)$, the Brunn–Minkowski theorem applies to u_1, u_2, u_3 on $\mathbb{R}^N$ to yield

$$\widetilde{Z}_N(U_3)\geqslant\widetilde{Z}_N(U_1)^{\theta}\widetilde{Z}_N(U_2)^{1-\theta}.$$

Taking the limit (1) immediately yields the following free analogue of the functional Brunn–Minkowski inequality on $\mathbb{R}$.

Theorem. *Let U_1, U_2, U_3 be real-valued continuous functions on $\mathbb{R}$ such that, for every $\varepsilon>0$, $\lim_{|x|\to\infty}|x|\,\mathrm{e}^{-\varepsilon U_i(x)}=0$, $i=1,2,3$. Assume that for some $\theta\in(0,1)$ and all $x,y\in\mathbb{R}$,*

$$U_3\big(\theta x+(1-\theta)y\big)\leqslant\theta U_1(x)+(1-\theta)U_2(y).$$

Then

$$\mathcal{E}_{U_3}(\nu_{U_3})\geqslant\theta\mathcal{E}_{U_1}(\nu_{U_1})+(1-\theta)\mathcal{E}_{U_2}(\nu_{U_2}).$$

The free analogue of Shannon's entropy power inequality due to Szarek and Voiculescu [14] may be recovered along the same lines.

2. Free logarithmic Sobolev and transportation cost inequalities

We next show how the preceding free Brunn–Minkowski inequality may be used, following the classical case, to recapture both the free logarithmic Sobolev inequality of Voiculescu [17] (in the form put forward in [3] and extended in [2]) and the free quadratic transportation cost inequality of [4,8] for quadratic and more general strictly convex potentials Q.

Let Q be a real-valued continuous function on $\mathbb{R}$ such that $\lim_{|x|\to\infty}|x|\,\mathrm{e}^{-\varepsilon Q(x)}=0$ for every $\varepsilon>0$. For ν, probability measure on $\mathbb{R}$, define the free entropy of ν (with respect to ν_Q) [17,3,2] as

$$\widetilde{\Sigma}(\nu\mid\nu_Q)=\mathcal{E}_Q(\nu_Q)-\mathcal{E}_Q(\nu)\quad(\geqslant 0).$$

If $\varphi:\mathbb{R}\to\mathbb{R}$ is bounded and continuous, it is convenient to set below $j_Q(\varphi)=\mathcal{E}_{Q-\varphi}(\nu_{Q-\varphi})-\mathcal{E}_Q(\nu_Q)$. For every probability measure ν on $\mathbb{R}$,

$$j_Q(\varphi)\geqslant\int\varphi\,\mathrm{d}\nu+\mathcal{E}_Q(\nu)-\mathcal{E}_Q(\nu_Q)=\int\varphi\,\mathrm{d}\nu-\widetilde{\Sigma}(\nu\mid\nu_Q)$$

with equality for $\nu=\nu_{Q-\varphi}$. In particular $j_Q(\varphi)\geqslant\int\varphi\,\mathrm{d}\nu_Q$.

Assume now that (Q is C^1 and such that) $Q(x)-\frac{c}{2}x^2$ is convex for some $c>0$. For bounded continuous functions $f,g:\mathbb{R}\to\mathbb{R}$ such that $g(x)\leqslant f(y)+\frac{c}{2}|x-y|^2$, we may apply the free Brunn–Minkowski theorem, as in the classical case (cf. [11]), to $U_1=Q-(1-\theta)g$, $U_2=Q+\theta f$ and $U_3=Q$. Thus, by the theorem, $j_Q((1-\theta)g)+\frac{1-\theta}{\theta}j_Q(-\theta f)\leqslant 0$. As $\theta\to 0$, it follows that for every probability measure ν,

$$\int g\,\mathrm{d}\nu-\int f\,\mathrm{d}\nu_Q\leqslant\widetilde{\Sigma}(\nu\mid\nu_Q)$$

(in other words $j_Q(g)\leqslant\int f\,\mathrm{d}\nu_Q$). By the Monge–Kantorovitch–Rubinstein theorem (cf. e.g. [15]), this is the dual form of the free quadratic transportation cost inequality

$$\mathrm{W}_2(\nu,\nu_Q)^2\leqslant\frac{1}{c}\widetilde{\Sigma}(\nu\mid\nu_Q)\tag{2}$$

recently put forward in [4] for the semicircle law associated to the quadratic potential, and in [8] for strictly convex potentials (where $W_2(\nu,\nu_Q)$ is the Wasserstein distance between ν and ν_Q).

The free logarithmic Sobolev inequality of [17], extended to strictly convex potentials in [2], follows in the same way from the free Brunn–Minkowski theorem. We follow [2] where the matrix approximation is used similarly to this task. Fix a probability measure ν with compact support and smooth density p on $\mathbb{R}$. Define a C^1 function R on $\mathbb{R}$ such that $R(x)=2\int\log|x-y|\,\mathrm{d}\nu(y)$ on $\mathrm{supp}(\nu)$, $R(x)=Q(x)$ for $|x|$ large, and such that $R(x)\geqslant 2\int\log|x-y|\,\mathrm{d}\nu(y)$ everywhere. By the uniqueness theorem of extremal measures of weighted potentials (cf. [13]), it is easily seen that the energy functional $\mathcal{E}_R$ is maximized at the unique point $\nu_R=\nu$. Define then f, with compact support, by $f=Q-R+C$ where the constant C $(=\mathcal{E}_Q(\nu_Q)-\mathcal{E}_R(\nu_R))$ is chosen so that $j_Q(f)=0$. Let $g_t(x)=\inf_{y\in\mathbb{R}}[f(y)+\frac{1}{2t}(x-y)^2]$, $t>0$, $x\in\mathbb{R}$, be the infimum-convolution of f with the quadratic cost, solution of the Hamilton–Jacobi equation $\partial_t g_t+\frac{1}{2}{g_t'}^2=0$ with initial condition f. As in the classical case (cf. [11]), apply the Brunn–Minkowski theorem to $U_1=Q-\frac{1}{\theta}g_t$, $t=\frac{1-\theta}{c\theta}$, $U_2=Q$, $U_3=Q-f$, to get that $j_Q((1+ct)g_t)\leqslant j_Q(f)=0$ for every $t>0$. In particular therefore, $\int(1+ct)g_t\,\mathrm{d}\nu\leqslant\widetilde{\Sigma}(\nu\mid\nu_Q)$, and, since $\nu=\nu_R=\nu_{Q-f}$, as $t\to 0$,

$$\widetilde{\Sigma}(\nu\mid\nu_Q)=\int f\,\mathrm{d}\nu\leqslant\frac{1}{2c}\int f'^2\,\mathrm{d}\nu.$$

Now, $f'=Q'-Hp$ where $Hp(x)=\mathrm{p.v.}\int\frac{2p(y)}{x-y}\,\mathrm{d}y$ is the Hilbert transform (up to a multiplicative factor) of the (smooth) density p of ν. Hence the preceding amounts to the free logarithmic Sobolev inequality

$$\widetilde{\Sigma}(\nu\mid\nu_Q)\leqslant\frac{1}{2c}\int[Hp-Q']^2\,\mathrm{d}\nu=\frac{1}{2c}\mathrm{I}(\nu\mid\nu_Q)\tag{3}$$

as established in [2], where $\mathrm{I}(\nu \mid \nu_Q)$ is known as the free Fisher information of ν with respect to ν_Q [17,3]. Careful approximation arguments to reach arbitrary probability measures ν (with density in $\mathrm{L}^3(\mathbb{R})$) are described in [7].

The Hamilton–Jacobi approach may be used to prove, as in the classical case, the free analogue of the Otto–Villani theorem [12] (cf. [15,5,11]) stating that, for a given probability measure $\mathrm{d}\mu = \mathrm{e}^{-Q}\,\mathrm{d}x$ on $\mathbb{R}$ (with a C^1 potential Q such that $\lim_{|x|\to\infty} |x|\,\mathrm{e}^{-\varepsilon Q(x)} = 0$ for every $\varepsilon > 0$), the free logarithmic Sobolev inequality (3) always implies the free transportation cost inequality (2). To this task, given a compactly supported C^1 function f on $\mathbb{R}$ and $a \in \mathbb{R}$, set $j_t = j_Q((a+ct)g_t)$ and $f_t = (a+ct)g_t - j_t$ so that $j_Q(f_t) = 0$. Denote for simplicity by ν_t the extremal measure for the potential $Q - f_t$. Then the logarithmic Sobolev inequality (3) can be expressed as $\int f_t\,\mathrm{d}\nu_t \leqslant \frac{1}{2c}\int f_t'^2\,\mathrm{d}\nu_t$. In other words,

$$c(a+ct)\int g_t\,\mathrm{d}\nu_t - cj_t \leqslant -(a+ct)^2\int \partial_t g_t\,\mathrm{d}\nu_t.$$

On the support of ν_t (cf. [13]),

$$2\int \log|x-y|\,\mathrm{d}\nu_t(y) = Q - f_t + C_t$$

where $C_t = \iint \log|x-y|\,\mathrm{d}\nu_t\,\mathrm{d}\nu_t + \mathcal{E}_{Q-f_t}(\nu_t)$. Since $j_Q(f_t) = \mathcal{E}_{Q-f_t}(\nu_t) - \mathcal{E}_Q(\nu_Q) = 0$, it follows that $\int \partial_t f_t\,\mathrm{d}\nu_t = 0$. Therefore, $cj_t \geqslant (a+ct)\partial_t j_t$ and hence $(a+ct)^{-1} j_t$ is non-increasing in t. In particular, $\frac{1}{a+1} j_{1/c} \leqslant \frac{1}{a} j_0$ which for $a = 0$ amounts to $j_Q(g) \leqslant \int f\,\mathrm{d}\nu_Q$, that is the dual form of (2). This approach through the Hamilton–Jacobi equations has some similarities with the use of the (complex) Burgers equation in [4].

Acknowledgements

I thank Philippe Biane for useful comments and references.

References

[1] G. Ben Arous, A. Guionnet, Large deviations for Wigner's law and Voiculescu's noncommutative entropy, Probab. Theory Related Fields 108 (1997) 517–542.

[2] P. Biane, Logarithmic Sobolev inequalities, matrix models and free entropy, Acta Math. Sinica 19 (2003) 1–11.

[3] P. Biane, R. Speicher, Free diffusions, free entropy and free Fisher information, Ann. Inst. H. Poincaré 37 (2001) 581–606.

[4] P. Biane, D. Voiculescu, A free probability analogue of the Wasserstein distance on the trace-state space, Geom. Funct. Anal. 11 (2001) 1125–1138.

[5] S. Bobkov, I. Gentil, M. Ledoux, Hypercontractivity of Hamilton–Jacobi equations, J. Math. Pures Appl. 80 (2001) 669–696.

[6] F. Hiai, D. Petz, The Semicircle Law, Free Random Variables and Entropy, Math. Surveys and Monographs, vol. 77, American Mathematical Society, 2000.

[7] F. Hiai, D. Petz, Y. Ueda, Inequalities related to free entropy derived from random matrix approximation (2003).

[8] F. Hiai, D. Petz, Y. Ueda, Free transportation cost inequalities via random matrix approximation, Probab. Theory Related Fields 130 (2004) 199–221.

[9] K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices, Duke Math. J. 91 (1998) 151–204.

[10] M. Ledoux, The Concentration of Measure Phenomenon, Math. Surveys and Monographs, vol. 89, American Mathematical Society, 2001.

[11] M. Ledoux, Measure concentration, transportation cost, and functional inequalities, Summer School on Singular Phenomena and Scaling in Mathematical Models, Bonn, 2003.

[12] F. Otto, C. Villani, Generalization of an inequality by Talagrand, and links with the logarithmic Sobolev inequality, J. Funct. Anal. 173 (2000) 361–400.

[13] E.B. Saff, V. Totik, Logarithmic Potentials with External Fields, Grundlehren Math. Wiss., vol. 316, Springer, 1997.

[14] S. Szarek, D. Voiculescu, Volumes of restricted Minkowski sums and the free analogue of the entropy power inequality, Commun. Math. Phys. 178 (1996) 563–570.

[15] C. Villani, Topics in Optimal Transportation, Grad. Stud. Math., vol. 58, American Mathematical Society, 2003.

[16] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, I, Commun. Math. Phys. 155 (1993) 71–92.

[17] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, V. Noncommutative Hilbert transforms, Invent. Math. 132 (1998) 189–227.