**ESAIM: Control, Optimisation and Calculus of Variations** July 2001, Vol. 6, 499–516
URL: http://www.emath.fr/cocv/

# LOCAL SMALL TIME CONTROLLABILITY AND ATTAINABILITY OF A SET FOR NONLINEAR CONTROL SYSTEM *

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**Abstract**. In the present paper, we study the problem of small-time local attainability (STLA) of a closed set. For doing this, we introduce a new concept of variations of the reachable set well adapted to a given closed set and prove a new attainability result for a general dynamical system. This provide our main result for nonlinear control systems. Some applications to linear and polynomial systems are discussed and STLA necessary and sufficient conditions are obtained when the considered set is a hyperplane.

**Mathematics Subject Classification.** 93B05, 93B03, 93C05, 93C10, 49J53.

Received September 7, 2000. Revised March, 2001.

## 1. Introduction

Small time local controllability is a central property for studying the regularity of Time Minimal problem for control systems [3,24]. In general it is well known that the minimal time function is only lower semicontinuous [6].

There exist different approaches to study the local small time controllability and attainability at a point, leading to different results and requiring different assumptions (see for instance [11, 18, 20, 21]). The problem of attainability of a closed set has been only partially studied using only zero order and one order approaches (*cf.* [2, 8, 23, 24]).

It is worth pointing out that local controllability of a given set is not reduced to the question of local controllability at every point of the set, so it needs a specific study (this fact will appear clearly for instance in Prop. 4.1).

Our main aim consists in studying the problem of small-time local attainability – STLA in short – of a closed set. In the present paper, we propose and study the properties of a new class of local variations of zero and higher order which are *well adapted* to the considered problem. Our approach allows us to obtain a unified treatment of the STLA problem.

Because we wish to obtain attainability for various classes of control systems, we define local variations in the context of a general dynamical system. Such a system is given by a set valued map $R : [0, +\infty) \times \mathbb{R}^n \mapsto \mathbb{R}^n$

*Keywords and phrases:* Attainability, controlability, local variations, polynomial control, linear controls.

* *This work has been done during the visit of the first author in the "Laboratoire de Mathématiques" de l'Université de Bretagne Occidentale and has been also partially supported by the Bulgarian Ministry of Sciences and Higher Education, National Fund for Scientific Research under contracts MM-612/96 and MM-807/98.*

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with closed nonempty values, which is continuous with respect to the first variable and satisfy the following semi-group property: for any point $x \in \mathbb{R}^n$ and for any nonnegative reals $s$ and $t$ the following inclusion holds true:

$$R(R(x,t),s)) \subset R(x,t+s). \tag{1}$$

Throughout the paper we shall say that a closed set $S \subset \mathbb{R}^n$ is small-time locally attainable if and only if for any time $T > 0$, a neighborhood $\mathcal{O}$ of $S$ exists such that

$$\forall x \in \mathcal{O},\ \exists \tau \in\ [0,T],\ R(x,\tau) \cap S \neq \emptyset. \tag{2}$$

This means that $S$ is attained in time not greater than $T$.

Later on, we shall use this notion in the context of nonlinear control systems. In this context, $R$ will be related to the reachable map, so we cover the classical STLA for a point.

The continuity of minimal time for reaching a set is one of the main applications of local attainability. Here, we also derive the Hölder continuity of the minimal time function. Note that Lipschitz continuity of this function has been already obtained in [23,24] using zero order condition. In [17], property of $\frac{1}{2}$-Hölder continuity is derived from first order analysis. We shall provide more precise continuity properties using our higher order approach.

It is well known (since Kalman's work) that for linear systems the STLA property at an equilibrium point can be characterized through a necessary and sufficient condition. When the set $S$ is a hyperplane, we provide also STLA necessary and sufficient conditions for linear and polynomial control systems.

All results presented in the paper are on the question of attainability. But in our setting we consider autonomous systems, so attainability of a set in finite time is equivalent to controllability in finite time for the backward dynamics, so one can translate our results in the framework of small time local controllability of sets.

Let us explain how the paper is organized.

The second section contains preliminaries and different ways for construction of local variations of the reachable set with respect to a closed set.

The third section is devoted to statement of sufficient conditions for small-time local attainability of a closed set for a general dynamical system and then for control systems. From the above STLA conditions we derive some continuity properties of the minimal time to reach a set.

In the fourth section, it is proved that the sufficient conditions for small-time local attainability derived in Section 3 are also necessary for the case of linear and polynomial systems when the set is a hyperplane.

## 2. Local variations with respect to a closed set

### 2.1. Preliminaries

Throughout the paper, we shall use the following notations and definitions we introduce now.

Let us denote by $\mathcal{V}$ the linear space of all analytic vector fields on $\mathbb{R}^n$, considered as a Lie algebra with the Lie product

$$[X,Y] := \frac{\partial Y}{\partial x}X - \frac{\partial X}{\partial x}Y. \tag{3}$$

Given an analytic vector field $Z$ and a positive real $s$, we denote by $\mathrm{Exp}(sZ)\ (x)$ the value of the solution of the equation

$$x'(t)\ =\ sZ(x(t)),\ x(0) = x,\ t \in\ [0,1], \tag{4}$$

at time $t = 1$.

Let $S$ be an arbitrary closed subset of $\mathbb{R}^n$, and $d_S(.)$ denotes the distance function with respect to the set $S$. Let $x_0$ belongs to the boundary $\partial S$ of $S$ and let $B(x_0, r)$ denote the open ball with center $x_0$ and radius $r$.

By $\mathcal{P}$ we shall denote the set of all function $p(t), t \in \mathbb{R}$, of the following type:

$$p(t) = \sum_{i=1}^{k} p_i t^{q_i}, \text{ where } 1 \leq q_1 < q_2 < ... < q_k, \text{ and } 0 \leq p_i, i = 1, ...k.$$

By $o(t)$, we denote a family - parametrized by $t$ – of analytic vector fields $x \mapsto o(t,x)$ on $\mathbb{R}^n$, which is continuous in $(t,x)$ and such that for some $\omega > 1$ the ratio $o(t,x)/t^{\omega}$ is bounded uniformly with respect to $x \in B(x_0, r)$.

With the above notations, we may define a family of analytic vector fields related to the set $S$.

**Definition 2.1.** Let $\mathcal{V}^0_{S,x_0}\left(\mathcal{V}^+_{S,x_0}\right)$ be the set of all families of analytic vector fields $a(t) = a(t,.)$ on $\mathbb{R}^n$ (parametrized on $t \geq 0$), continuous in $(t,x)$ and such that for every element $a(t)$ from $\mathcal{V}^0_{S,x_0}\left(\mathcal{V}^+_{S,x_0}\right)$ there exist some positive reals $r$, $\theta$, $d$, and $c$ such that $a(t,x) \leq c.d_S(x)^d \left(a(t,x) \leq c.t^{\theta}.d_S(x)^d\right)$ for all $x \in B(x_0, r)$.

**Remark 2.1.** The assumption on analyticity of the vector fields is made for simplicity of the exposition. In fact, this definition, Definition 2.2 and all assertions after that, hold true assuming only that the corresponding vector fields are sufficiently smooth. To prove this, one can apply the formalism of the noncommutative vector fields and nilpotent approximations (see for example [1, 7, 9] and [19]).

**Remark 2.2.** The smoothness of the considered vector fields and the definition of Lie bracket (3), imply that the set $\mathcal{V}^0_{S,x_0}\left(\mathcal{V}^+_{S,x_0}\right)$ is a Lie subalgebra of $\mathcal{V}$.

### 2.2. A class of high order variations

Let us consider a generalized dynamical system given by a set valued map $R : [0, +\infty) \times \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfying the semi-group property (1). We do not want to fix the form of the dynamical system too early, because the important tool of our approach is the notion of local variation. Later on we shall consider dynamical systems governed by differential inclusions, affine control systems and, as particular cases, as linear and polynomial control systems.

Using some of the ideas from [3, 10, 11, 14, 21, 22], etc., we define a family of variations which, in our opinion, are useful for studying the problem of local attainability of a set with respect to the considered generalized dynamical system.

**Definition 2.2.** Let $S$ be an arbitrary closed subset of $\mathbb{R}^n$ and let $x_0$ belong to the boundary $\partial S$ of $S$. It is said that the analytic vector field $Z$ belongs to the set $\mathcal{S}^{\alpha}_{x_0}$ if and only if there exist an element $p \in \mathcal{P}$, positive real numbers $r$ and $T$, a family of vector field $a(t) \in \mathcal{V}^+_{S,x_0}$ such that for every point $x$ in $B(x_0, r) \setminus S$ and each $t \in [0, T]$

$$\operatorname{Exp}(t^{\alpha} Z + a(t) + o(t^{\alpha}))(x) \;\in\; R(x, p(t)), \tag{5}$$

where $o(t)$ is defined as in Section 2.1.

**Remark 2.3.** By setting $t := t^{\beta/\alpha}$ one can prove that that the relation $A \in \mathcal{S}^{\alpha}_{x_0}$ implies that $A \in \mathcal{S}^{\beta}_{x_0}$ for every $\beta > \alpha$.

An interesting and difficult problem is to characterize the set $\mathcal{S}^{\alpha}_{x_0}$. In the present paper we prove that this set have the same properties as the corresponding sets of high order variations used for the case when the set

$S$ is a single point. Our proofs are based on the classical formulae of Campbell–Baker–Hausdorff: if $X$ and $Y$ are analytic vector fields on $\mathbb{R}^n$, then

$$\mathrm{Exp}(t_1X)\mathrm{Exp}(t_2Y)(x) = \mathrm{Exp}\big(t_1X + t_2Y + \frac{t_1t_2}{2}[X,Y] + \frac{t_1t_2^2}{12}[Y,[Y,X]] + \frac{t_1^2t_2}{12}[X,[X,Y]] + ...\big)(x), \quad (6)$$

$$\mathrm{Exp}(-t_1X)\mathrm{Exp}(t_2Y)\mathrm{Exp}(t_1X)(x) = \mathrm{Exp}\left(t_2\sum_{k=0}^{\infty}\frac{t_1^k}{k!}(ad^k\ X,Y)\right)(x), \quad (7)$$

where the right-hand sides (with the infinite sums) are convergent for sufficiently small $|t_1|$ and $|t_2|$. Here we have used the following notation: $(ad^0\ X,Y) := Y$, and $(ad^{k+1}\ X,Y) := [X,(ad^k\ X,Y)]$.

Next, we introduce a subset $\mathcal{S}$ of the set $\mathcal{S}^{\alpha}_{x_0}$, which can be used for constructing "new" elements of the set $\mathcal{S}^{\alpha}_{x_0}$, provided that some elements of $\mathcal{S}^{\alpha}_{x_0}$ are already known (*cf.* [12] and [15] where similar sets are also defined).

**Definition 2.3.** It is said that the analytic vector field $Z$ belongs to the set $\mathcal{S}$ if and only if there exist positive real numbers $K$ and $T$, such that for every point $x$ and each $t \in [0,T]$

$$\mathrm{Exp}(tZ)(x) \ \in \ R(x,Kt). \quad (8)$$

**Remark 2.4.** As in [15], it can be proved that the set $\mathcal{S}$ is a convex cone. Moreover, modifying the original proofs, one can prove the following assertions:

**Proposition 2.1. (Sussmann [18]).** *Let $A_1, A_2, ..., A_k$, belong to $\mathcal{S}$ and $A_1 + A_2 + ... + A_k$ belong to $\mathcal{V}^0_{S,x_0}$. Then $[A_i, A_j], i,j = 1,...,k$, belong to $\mathcal{S}^2_{x_0}$.*

**Proposition 2.2. (Hermes [11]).** *Let $A_1$ and $A_2$ belong to $\mathcal{S}$ and $A_1+A_2$ belong to $\mathcal{V}^0_{S,x_0}$. Then $[A_1,[A_1,A_2]]$ $+[A_2,[A_2,A_1]]$ belongs to $\mathcal{S}^3_{x_0}$.*

Here we prove that:

**Proposition 2.3.** *The set $\mathcal{S}^{\alpha}_{x_0}$ is a convex cone.*

*Proof.* Let $A_1$ and $A_2$ belong to $\mathcal{S}^{\alpha}_{x_0}$. According to Definition 2.2, there exist elements $p_i \in \mathcal{P}$, positive real numbers $r_i$ and $T_i$, two families of vector fields $o_i(t)$ and $a_i(t) \in \mathcal{V}^{+}_{S,x_0}$, $i = 1,2$, such that for every point $x$ from $B(x_0,r_i) \setminus S$ and each $t \in [0,T_i]$

$$\mathrm{Exp}(t^{\alpha}A_i + a_i(t) + o_i(t^{\alpha}))(x) \ \in \ R(x,p_i(t)). \quad (9)$$

Let $c > 0$ be an arbitrary real number. By setting $t := \tau.c^{1/\alpha}$ and substituting in (9) we obtain that for every point $x$ from $B(x_0,r_1) \setminus S$ and each $\tau \in [0, T_1/c^{1/\alpha}]$

$$\mathrm{Exp}(\tau^{\alpha}cA_1 + a_1(\tau.c^{1/\alpha}) + o_1(c.\tau^{\alpha}))(x) \ \in \ R(x,p_1(c^{1/\alpha}\tau)). \quad (10)$$

*i.e.* $cA_1$ belongs to $\mathcal{S}^{\alpha}_{x_0}$.

Let $T > 0$ and $r > 0$ be so small that $T < \min(T_1,T_2)$ and for every point $x$ from $B(x_0,r) \setminus S$ and each $t \in [0,T]$

$$\mathrm{Exp}(tA_2 + a_2(t) + o_2(t))(x) \ \in \ R(x,p_2(t)) \cap B(x_0,r_1). \quad (11)$$

Then according to (9), we have that

$$\mathrm{Exp}(t^{\alpha}A_1 + a_1(t) + o_1(t^{\alpha}))\ \mathrm{Exp}(t^{\alpha}A_2 + a_2(t) + o_2(t^{\alpha}))(x) \ \in \ R(x,p_1(t)+p_2(t)).$$

Applying the Campbell–Baker–Hausdorff formula, we obtain that

$$\mathrm{Exp}(t^{\alpha}(A_1 + A_2) + a(t) + o(t^{\alpha}))(x) \;\in\; R(x, p_1(t) + p_2(t)),$$

for some $o(t)$ and $a(t) \in \mathcal{V}^{+}_{S,x_0}$, *i.e.* $A_1 + A_2$ belongs to $\mathcal{S}^{\alpha}_{x_0}$. □

Proposition 2.3 does not give any information on constructing new elements of the set $\mathcal{S}^{\alpha}_{x_0}$. This can be done using the following

**Proposition 2.4.** *Let $A_1$ and $A_2$ belong to $\mathcal{S}^{\alpha}_{x_0}$, $A_1 + A_2$ belongs to $\mathcal{V}^{0}_{S,x_0}$, and $B$ belongs to $\mathcal{S} \cap \mathcal{V}^{0}_{S,x_0}$. Then there exists a real number $\beta$ such that $\beta > \alpha$, and $[B, A_1]$ and $[B, A_2]$ belong to $\mathcal{S}^{\beta}_{x_0}$.*

*Proof.* According to Definition 2.2, there exist elements $p_i \in \mathcal{P}$, positive real numbers $r_i$ and $T_i$, two families of vector fields $o_i(t)$ and $a_i(t) \in \mathcal{V}^{+}_{S,x_0}$, $i = 1, 2$, such that for every point $x$ from $B(x_0, r_i) \setminus S$ and each $t \in [0, T_i]$

$$\mathrm{Exp}(t^{\alpha}A_i + a_i(t) + o_i(t^{\alpha}))(x) \;\in\; R(x, p_i(t)), \tag{12}$$

where $a_i(t,x) \leq c_i.t^{\theta_i}.d_S(x)^{d_i}$ and $o_i(t,x) \leq C_i t^{\omega_i}$, $\omega_i > \alpha$, $i = 1, 2$.

Next, we choose a positive number $b > 0$ satisfying the inequality:

$$b \;>\; \max\left\{1, \frac{1}{\theta_i}, \frac{1}{\omega_i - \alpha}\right\}. \tag{13}$$

Let us choose $T > 0$ so small that $T^b < \min(T_1, T_2)$ and for each $\tau \in [0, T]$

$$\mathrm{Exp}(\tau B)\mathrm{Exp}(\tau^{b\alpha}A_2 + a_2(\tau^b) + o_2(\tau^{b\alpha+1}))(x) \;\in\; R(x, \tau + p_2(\tau^b)) \cap B(x_0, r_1).$$

By setting $t := \tau^b$ and substituting in (12), we obtain that for every point $x$ from $B(x_0, r) \setminus S$, and each $\tau \in [0, T]$,

$$\begin{aligned}&\mathrm{Exp}(\tau^{b\alpha}A_1 + a_1(\tau^b) + o_1(\tau^{b\alpha+1}))\;\mathrm{Exp}(\tau B)\\ &\mathrm{Exp}(\tau^{b\alpha}A_2 + a_2(\tau^b) + o_2(\tau^{b\alpha+1}))(x) \;\in\; R(x, p_1(\tau^b) + \tau + p_2(\tau^b)).\end{aligned}$$

Applying the Campbell–Baker–Hausdorff formula two times, we obtain as a result that there exist suitable families of vector fields $\hat{o}_1(t)$ and $\hat{o}_2(t)$, and $\hat{a}_1(t)$ and $\hat{a}_2(t)$ from $\mathcal{V}^{+}_{S,x_0}$ such that

$$\begin{aligned}&\mathrm{Exp}\left(\tau B + \tau^{b\alpha}A_1 + \frac{\tau^{b\alpha+1}}{2}[A_1, B] + \hat{a}_1(\tau^b) + \hat{o}_1(\tau^{b\alpha+1})\right)\\ &\mathrm{Exp}(\tau^{b\alpha}A_2 + a_2(\tau^b) + o_2(\tau^{b\alpha+1}))(x) \;\in\; R(x, p_1(\tau^b) + \tau + p_2(\tau^b)),\end{aligned}$$

and after that

$$\mathrm{Exp}\left(\tau B + \tau^{b\alpha}(A_1 + A_2) + \frac{\tau^{b\alpha+1}}{2}([A_1, B] + [B, A_2]) + \hat{a}_2(\tau^b) + \hat{o}_2(\tau^{b\alpha+1})\right)(x) \;\in\; R(x, p_1(\tau^b) + \tau + p_1(\tau^b)). \tag{14}$$

But

$$[A_1, B] + [B, A_2] \;=\; [A_1 + A_2, B] + 2[B, A_2].$$

Moreover, $A_1 + A_2$ and $B$ belong to $\mathcal{V}^{0}_{S,x_0}$, so $[A_1 + A_2, B]$ also belongs to $\mathcal{V}^{0}_{S,x_0}$. Hence, the inclusion (14) can be written as

$$\mathrm{Exp}(\tau^{b\alpha+1}[B, A_2]) + a(\tau) + o(\tau^{b\alpha+1}))(x) \;\in\; R(x, p(\tau)), \tag{15}$$

where $o(t)$ and $a(t) \in \mathcal{V}^+_{S,x_0}$ are suitable families of vector fields and $p(\tau) := p_1(\tau^b) + \tau + p_1(\tau^b)$. So, $[B, A_2]$ belongs to $\mathcal{S}^{b\alpha+1}_{x_0}$. Changing the order of $A_1$ and $A_2$, one can prove that $[B, A_1]$ also belongs to $\mathcal{S}^{b\alpha+1}_{x_0}$. □

Next, following the approach described in [13], we shall define the following – possibly empty – subset $\mathcal{S}^{\text{fast}}$ of the set $\mathcal{S}$:

**Definition 2.4.** It is said that the analytic vector field $Z$ belongs to the set $\mathcal{S}^{\text{fast}}$ if and only if there exists a positive real number $T$, such that for every point $x$, for each $\mu > 0$ and for each $t \in [0, T]$

$$\operatorname{Exp}(\mu Z)(x) \;\in\; R(x,t). \tag{16}$$

**Proposition 2.5.** *The set $\mathcal{S}^{\text{fast}}$ is a convex cone.*

*Proof.* Let $A_i \in \mathcal{S}^{\text{fast}}$, and $c_i > 0, i = 1, 2$, be arbitrary positive reals. According to Definition 2.4,

$$\operatorname{Exp}(\mu A_i)(x) \;\in\; R(x,t), i = 1, 2,$$

for every point $x$, for each $\mu > 0$ and for each $t \in [0, T]$. Hence, for each positive integer $n$ and for each $t \in [0, T]$,

$$\operatorname{Exp}\left(\frac{\mu c_1}{n} A_1\right) \operatorname{Exp}\left(\frac{\mu c_2}{n} A_2\right)(x) \;\in\; R\left(x, \frac{t}{n}\right)\cdot \tag{17}$$

Applying the Campbell–Baker–Hausdorff formula, we obtain that

$$\operatorname{Exp}\left(\frac{\mu}{n}(c_1A_1 + c_2A_2) + o\left(\frac{\mu}{n}\right)\right)(x) \;\in\; R\left(x, \frac{t}{n}\right)\cdot$$

Taking a composition of the last inclusion $n$-times, we have that

$$\operatorname{Exp}\left(\frac{\mu}{n}(c_1A_1 + c_2A_2) + o\left(\frac{\mu}{n}\right)\right) ...\operatorname{Exp}\left(\frac{\mu}{n}(c_1A_1 + c_2A_2) + o\left(\frac{\mu}{n}\right)\right)(x) \;\in\; R(x,t) \textit{ i.e.}$$

$$\operatorname{Exp}\left(\mu(c_1A_1 + c_2A_2) + n.o\left(\frac{\mu}{n}\right)\right)(x) \;\in\; R(x,t).$$

Taking a limit as $n \to \infty$, we complete the proof. □

**Proposition 2.6.** *Let $\pm A$ belong to the set $\mathcal{S}^{\text{fast}}$, $B$ belong to the set $\mathcal{S}$ and let $(ad^i\ A, B) \equiv 0$ for all $i > k$. Then $(\pm 1)^k (ad^k\ A, B) \in \mathcal{S}^{\text{fast}}$.*

*Proof.* According to Definitions 2.3 and 2.4, there exist positive real numbers $K$ and $T$ such that for every point $x$ and for each $\mu > 0$ and each $t \in [0, T]$,

$$\operatorname{Exp}(\pm\mu A)(x) \;\in\; R(x,t) \text{ and } \operatorname{Exp}(tB)(x) \;\in\; R(x, Kt).$$

Then for every positive integer $n$, for every $\sigma \in \{\pm 1\}$, for every point $x$, for each $\mu > 0$, for each $t \in [0, T]$

$$\operatorname{Exp}\left(-\frac{\sigma n.\mu^{1/k}}{t^{1/k}} A\right) \operatorname{Exp}\left(\frac{t.k!}{n^k} B\right) \operatorname{Exp}\left(\frac{\sigma.n.\mu^{1/k}}{t^{1/k}} A\right)(x) \in R\left(x, t\left(1 + \frac{K.k!}{n^k}\right)\right)\cdot$$

Applying the Campbell–Baker–Hausdorff formula, we obtain that

$$\operatorname{Exp}\left(\frac{tk!}{n^k}\sum_{i=0}^{\infty}\frac{(\sigma.n.\mu^{1/k})^i}{i!t^{i/k}}(ad^i\ A,B)\right)(x)\ \in\ R\left(x,t\left(1+\frac{K.k!}{n^k}\right)\right)\cdot$$

According to the assumptions of this proposition,

$$(ad^i\ A,B)\equiv 0\ \text{ for all } i>k.\text{ So,}$$

$$\operatorname{Exp}\left(\frac{tk!}{n^k}\sum_{i=0}^{k}\frac{(\sigma.n.\mu^{1/k})^i}{i!t^{i/k}}(ad^i\ A,B)\right)(x)\ \in\ R\left(x,t\left(1+\frac{K.k!}{n^k}\right)\right)\cdot$$

Taking a limit as $n\to\infty$, we obtain

$$\operatorname{Exp}\left(\mu\sigma^k(ad^k\ A,B)\right)(x)\ \in\ R\left(x,t\right).$$

□

## 3. A sufficient condition for small-time local attainability of a closed set

With the concept of local variations studied in the previous section, we are ready to study the property of local attainability of a set with respect to some dynamical system.

### 3.1. Set-attainability for general dynamical system

Throughout the section we define the supremum of the empty set of $R$ as $-\infty$. We denote by $cl\ A$ the closure of the set $A$.

Associated with an initial condition $x_0$, we call an $R$-trajectory of the generalized dynamical system $R$ any continuous function $x(\cdot):[0,+\infty)\mapsto\mathbb{R}^n$ such that

$$x(0)=x_0 \text{ and } x(t)\in\ R(x,t),\ \forall t\geq 0.$$

For a set $S$ and a point $x\in\mathbb{R}^n$, we define the following set of projections of $x$ on $S$:

$$P_S(x):=\{\pi_x\in S,\ \|\pi_x-x\|=d_S(x)\}\cdot$$

**Theorem 3.1.** *Let $S$ be a closed subset of $\mathbb{R}^n$. Let $\alpha>0$, $s>0$, $r>0$ and $T_0>0$ be given. Let $x_0\in\partial S$. We assume the following conditions:*

**A1** *starting from any $x$ from $cl\ (B(x_0,r)\setminus S)$, there exists a $R$-trajectory $x(\cdot)$ such that for every $t\in[0,T_0]$,*

$$x(t)=x\ +\ a(t;x)\ +\ t^\alpha A(x)\ +\ o(t^\alpha;x)\ \in\ R(t,x);$$

**A2** *there exist positive constants $N$ and $\beta$ such that*

$$\max_{x\in cl\ \ (B(x_0,r)\setminus S)}\ \|o(t^\alpha;x)\|\ \leq\ N.t^{\alpha+\beta};$$

**A3** *there exists some Lipschitz continuous negative function $b(\cdot)$ with a Lipschitz constant $L_b$ on $cl\ (B(x_0,r)\setminus S)$ such that*

$$\max_{x\in cl\ \ (B(x_0,r)\setminus S),\ \pi_x\in P_S(x)}\left\langle\frac{x-\pi_x}{\|x-\pi_x\|},A(x)\right\rangle\leq b(x)<0;$$

**A4** *there exists $L_0 > 0$ such that for all $(x,y)$ in $cl(B(x_0,r) \setminus S)$,*

$$\|A(x) - A(y)\| \leq L_0\|x-y\|, \ i = 1,2\dots\nu;$$

**A5** *there exists a Lipschitz continuous nonnegative function $c(\cdot)$ such that*

$$\max_{x\in cl\ (B(x_0,r)\setminus S)} \|a(t;x)\| \ \leq \ t^s c(x) \quad and \quad \lim_{d_S(x)\to 0,\, x\in cl(B(x_0,r)\setminus S)} c(x) \ = \ 0.$$

*Then for every sufficiently small $T > 0$ there exists a neighborhood $B(x_0,\theta)$ of $x_0$ such that for every point $x \in B(x_0,\theta) \setminus S$ there exists $t \in [0,T]$ such that*

$$R(x,t) \cap S \neq \emptyset.$$

*Proof.* Let $\delta := r/2$. Note $S_\delta := S + \delta B$. We set $L := \max(1, L_0, L_b)$. Without loss of generality, we may assume that $T > 0$ is so small that $T < \min(1, T_0)$ and for every $t$, $0 < t < T$, and for any $x$ from $cl\ (B(x_0,r) \setminus S)$ the following inequality holds true

$$t^s c(x) + t^\alpha \max_{x\in cl\ (B(x_0,r)\setminus S),} \|A(x)\| + Nt^\beta < \min\left(\delta, \frac{|b(x)|}{4L}\right)\cdot \tag{18}$$

Then A1 and (18) imply that $x(t) \in B(x_0, 2\delta)$ for all $x \in B(x_0,\delta)$.

According to Lebourg's Mean Value theorem [16], for any $x \in B(x_0,\theta) \setminus S$, $0 < \theta \leq \delta$ ($\theta$ will be determined later on) we obtain

$$d_S(x(t)) = d_S(x) + \langle \xi, x(t) - x\rangle, \tag{19}$$

where

$$\xi \in co\left\{\frac{u-\pi_u}{\|u-\pi_u\|}, u \in x + [0,1](x(t)-x)\right\}\cdot \tag{20}$$

Since

$$d_S(u) \leq d_S(x) + \|x-u\| \leq \theta + \|x - x(t)\| \leq \theta + \delta \leq 2\delta,$$

we have that $u \in S_{2\delta}$. Let $\pi_u$ be an arbitrary element of $P_S(u)$. Then

$$\left\langle \frac{u-\pi_u}{\|u-\pi_u\|}, x(t) - x\right\rangle = \left\langle \frac{u-\pi_u}{\|u-\pi_u\|}, a(t;x) \ + \ t^\alpha A(x) \ + \ o(t^\alpha;x)\right\rangle\cdot \tag{21}$$

Assumptions A3 and A4 yield

$$\begin{aligned}\left\langle \frac{u-\pi_u}{\|u-\pi_u\|}, A(x)\right\rangle \ &= \ \left\langle \frac{u-\pi_u}{\|u-\pi_u\|}, A(u)\right\rangle + \left\langle \frac{u-\pi_u}{\|u-\pi_u\|}, A(x)-A(u)\right\rangle \\ &\leq \ b(u) + \|A(x)-A(u)\| \leq b(x) + 2L\|x-x(t)\|.\end{aligned} \tag{22}$$

Analogously, Assumption A2 implies that

$$\left\langle \frac{u-\pi_u}{\|u-\pi_u\|}, a(t;x)\right\rangle \ \leq \ t^s.c(x). \tag{23}$$

According to (20), there exist nonnegative reals $p_j$, $j = 1 \dots q$, and points $\pi_u^j \in P_S(u)$ such that

$$\xi = \sum_{j=1}^{q} p_j \frac{u - \pi_u^j}{\|u - \pi_u^j\|}, \; \sum_{j=1}^{q} p_j = 1.$$

Then (19) implies that

$$\begin{aligned} d_S(x(t)) \;\; = \;\; & d_S(x) + \sum_{j=1}^{q} p_j \left\langle \frac{u - \pi_u^j}{\|u - \pi_u^j\|}, x(t) - x \right\rangle = d_S(x) + \sum_{j=1}^{q} p_j \left[ \left\langle \frac{u - \pi_u^j}{\|u - \pi_u^j\|}, a(t;x) \right\rangle \right. \\ & \left. + \left\langle \frac{u - \pi_u^j}{\|u - \pi_u^j\|}, t^\alpha A(x) + o(t^\alpha; x) \right\rangle \right] \end{aligned}$$

(accordingly to (22) and (23))

$$\begin{aligned} &\leq d_S(x) + t^s c(x) + t^\alpha \left[ b(x) + 2L\|x - x(t)\| + Nt^\beta \right] \leq d_S(x) + t^s c(x) + t^\alpha \\ &\quad \times \left[ b(x) + 2L \left( t^s c(x) + t^\alpha \max_{x \in \, cl \; (B(x_0, r) \setminus S)} \|A(x)\| + Nt^{\alpha+\beta} \right) + Nt^\beta \right] \end{aligned}$$

(accordingly to (18))

$$\leq d_S(x) + t^s c(x) + t^\alpha \left( b(x) + \frac{1}{2}|b(x)| + \frac{1}{4}|b(x)| \right) \leq d_S(x) + t^s c(x) + t^\alpha \left( \frac{1}{4} b(x) \right) \leq 0$$

for

$$t \geq \left( \frac{4 d_S(x) + 4c(x)}{|b(x)|} \right)^{\frac{1}{\alpha}}. \tag{24}$$

These values of $t$ are available when the following equality holds true

$$T > \left( \frac{4 d_S(x) + 4c(x)}{|b|} \right)^{\frac{1}{\alpha}}, \tag{25}$$

where

$$|b| \; := \min_{x \in \, cl \; (B(x_0, r) \setminus S)} |b(x)|.$$

Choosing $\theta$ small enough (this can be done according A5), we ensure the validity of (25). □

A more refined regularity of $S$ enables us to obtain more precise attainability condition. As an example of this fact we give the following result with a first order Taylor Expansion.

**Proposition 3.2.** *Let $S := \{x \in \mathbb{R}^n, \; \phi(x) \leq 0\}$ be a closed subset of $\mathbb{R}^n$, where $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ is a $C^1$ function. Let $r > 0$ be given. For any $x_0 \in \partial S$, we assume conditions A1, A2, A4, A5 and*

**A3′** *There exists some Lipschitz continuous negative function $b(\cdot)$ on $cl(B(x_0, r) \setminus S)$ such that*

$$\max_{x \in B(x_0, r) \setminus S} \langle \nabla \phi(x), A(x) \rangle \leq b(x).$$

*Then $S$ is small-time locally attainable for the dynamical system $R$.*

*Proof.* The proof follows along the same lines as the proof of Theorem 3.1 using first order Taylor expansion of $\phi$ instead of Lebourg's Mean value theorem, so we omit this proof. $\square$

**Remark 3.1.** When $\phi$ is of class $C^2$, one can combine Taylor expansion of order 2 of $S$ with the condition A1, to obtain a similar result (see also Rem. 3.5 and its example). This idea can be generalized to sets given by a function $\phi$ of class $C^k$.

### 3.2. Set-attainability for nonlinear control system

The properties of trajectories of dynamical systems and the concept of variations developed in Section 2 enable us to state our main results for control. First, we express zero order attainability condition in the context of the following differential inclusion:

$$x'(t) \in F(x(t)). \tag{26}$$

We shall remind that $S$ is STLA for (26) if and only if for any $T > 0$, a neighborhood $\mathcal{O}$ of $S$ exists such that starting from any $x \in \mathcal{O}$ exists a trajectory to (26) reaching $S$ in a time not greater than $T$.

Define the dynamical system $R$ as the reachable map associated with (26):

$$R(x,t) := \{x(t) \,|\, \text{where } x(\cdot) \text{ is a solution to (26) with } x(0) = x\}\cdot$$

Clearly STLA properties of $R$ and (26) are equivalent.

Let us define the set of unit proximal normals [4] at $x_0$ to $S$:

$$NP_S(x_0) := \left\{ \frac{p}{\|p\|} \ ,\ p \neq 0,\ \exists \alpha > 0,\ d_S(x_0 + \alpha p) = \alpha \|p\| \right\}\cdot$$

From Theorem 3.1, one can deduce the following result which is also proved in [3] for smooth manifolds and in [23, 24] for the general case (see also [8]).

**Proposition 3.3. (Zero order sufficient attainability condition).** *Assume that $S \subset \mathbb{R}^n$ is compact and that $F$ is a Lipschitz continuous set-valued map with compact convex values. Suppose that there exists some $\delta > 0$ such that for any $x_0 \in \partial S$*

$$\min_{v \in F(x_0)} \quad \max_{p \in NP_S(x_0)} \langle p, v \rangle \leq -\delta < 0. \tag{27}$$

*Then $S$ is small-time locally attainable for system (26).*

Note that if $NP_S(x_0) = \emptyset$ then (27) is automatically satisfied because $\max \emptyset = -\infty$.

**Remark 3.2.** When $S := \{x \in \mathbb{R}^n,\ \phi(x) \leq 0\}$ with $\phi$ of class $C^1$ and $\nabla\phi(x) \neq 0$ on $\partial S$, one can prove a similar result replacing proximal normals by $\nabla\phi(x)/\|\nabla\phi(x)\|$.

For obtaining high order sufficient conditions, it is required that $F$ contains some regular selection. We now state our main result when $F(x) = f(x) + g(x)U$. But of course, one can prove a similar result when there are some regular vector fields in $F$ (for instance $F(x) \supset f(x) + g(x)U$).

Let us consider the following control system

$$x'(t) = f(x(t)) + g(x(t)).u(t), \tag{28}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, $g := (g_1, g_2, \ldots g_l) : \mathbb{R}^n \mapsto (\mathbb{R}^n)^l$ and $u(t) \in U \subset \mathbb{R}^l$. We assume that the functions $f$ and $g_i$ are smooth enough, *e.g.*, analytic for sake of simplicity. The properties of trajectories of dynamical systems and the concept of variations developped in Section 2 enable us to state our main result for control. For doing this, let us define the following set of vector fields:

$$W := \{[g_i, g_j],\ (ad^k f, g_i),\ \text{ with } i, j = 1 \dots l,\ k \in \mathbb{N}\} \cdot \tag{29}$$

**Theorem 3.4. (High Order Sufficient Attainability condition).** *Let $S$ be a compact subset of $\mathbb{R}^n$. Let $f$ and $g$ be analytic and $U$ be a closed subset of $\mathbb{R}^l$ such that*

$$0 \in \mathrm{Int}(U). \tag{30}$$

*We impose that the zero order sufficient attainability condition is violated:*

**B0′** *let for every point $x_0 \in \partial S$ for which $NP_S(x_0) \neq \emptyset$ the following equality holds true*

$$\min_{u \in U} \langle p, f(x_0) + g(x_0)u \rangle = 0.$$

*Moreover, suppose that for every point $x_0 \in \partial S$, there exists some neighborhood $B(x_0, r)$ of $x_0$ such that:*

**B1** *there exist some constants $C > 0$, $d > 0$ such that*

$$\|f(x)\| \leq C d_S(x)^d,\ \forall x \in\ B(x_0, r) \backslash S; \tag{31}$$

**B2** *there exists two Lipschitz continuous functions $b(\cdot) : cl(B(x_0, r) \backslash S) \mapsto \mathbb{R}_-$ and $w(\cdot) : cl(B(x_0, r) \backslash S) \mapsto \mathbb{R}^n$ such that*[3]

$$w \in\ co\ W,$$

*and for any $x \in \partial S \cap B(x_0, r)$*

$$\max_{p \in\ NP_S(x)} \langle p, w(x) \rangle \leq b(x) < 0. \tag{32}$$

*Then the set $S$ is small-time locally attainable.*

*Proof.* Let $x_0 \in \partial S$. The assumption B1 means that $f \in \mathcal{V}^0_{S,x_0}$ ($B(x_0, r)$ is the neighbourhood of $x_0$ from the Def. 2.1). Because $0 \in Int(U)$, $f \pm \epsilon g_i$ are admissible velocities on $B(x_0, r)$ for $\epsilon > 0$ small enough. This means that $\pm\epsilon g_i$ belong to the set $\mathcal{S}^1_{x_0}$. According to Propositions 2.3, 2.4 and Remark 2.3, the set $co\ W$ is a subset of the set of variations of high order. So, the assumptions A1, A2, A4 and A5 of Theorem 3.1 holds true.

Let us fix some real $e$ from the interval $(0, 1)$ and let $\delta := \min(|b(x)|\ :\ x \in S \cap clB(x_0, r))$. The compactness of $S$ and (32) imply that $b(x) + e\delta < 0$ for every point $x$ from $\partial S \cap B(x_0, r))$.

By $L > 0$ be greater than the Lipschitz constants of $w$ and $b$ on $B(x_0, r)$. We set $\xi = \min(r/2, e\delta/(2L))$. Let $x$ be an arbitrary point from the set $B(x_0, \xi)$ and $\pi_x$ be an arbitrary point from the set $P_S(x)$. Clearly,

$$\frac{x - \pi_x}{\|x - \pi_x\|} \in NP_S(\pi_x) \ \text{ and}$$

$$\|x_0 - \pi_x\| \leq \|x - x_0\| + \|x - \pi_x\| \leq 2\|x - x_0\| \leq 2\xi \leq r.$$

Then our choice of $L$ and (32) imply that

$$\left\langle \frac{x - \pi_x}{\|x - \pi_x\|}, w(x) \right\rangle \leq \left\langle \frac{x - \pi_x}{\|x - \pi_x\|}, w(\pi_x) \right\rangle + \|w(x) - w(\pi_x)\|$$

[3]Notation $co$ means convex hull.

$$\leq b(\pi_x) + L\|x - \pi_x\| \leq b(x) + 2.L\|x - \pi_x\| \leq b(x) + 2L\xi \leq b(x) + e.\delta < 0.$$

Hence, the Assumption A3 of Theorem 3.1 is also fulfilled. Applying Theorem 3.1 with $A = w$, we obtain that for every sufficiently small $T > 0$ there exists a neighborhood $B(x_0, \theta)$ of $x_0$ such that for every point $x \in B(x_0, \theta) \setminus S$ there exists $t \in [0, T]$ such that

$$R(x, t) \cap S \neq \emptyset.$$

Using an easy compactness argument one can complete the proof. □

**Remark 3.3.** Using the same approach, one can easily obtain results for systems of the form

$$x'(t) = f(x(t), u(t))$$

as soon as it is possible to construct a set of local variations.

Also using ideas of [3] and [20], it is possible to construct more general class of local variations for sets under suitable assumptions.

The assumption of analyticity of the dynamics can be weakened in the spirit of Remark 2.2.

**Remark 3.4.** If we consider only variations of the form

$$W^1 := \{[g_i, g_j],\ i, j = 1 \ldots l\},$$

we obtain the result of [17] for regular submanifolds without the additional assumption of continuous balanced vector fields which is not needed in our setting.

In a similar fashion to that in Proposition 3.2, it is possible to obtain the following:

**Corollary 3.5.** *Let $S := \{x \in \mathbb{R}^n,\ \phi(x) \leq 0\}$ be a compact subset where $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ is a $C^1$ function. Let $f$ and $g$ be analytic and (30) holds true.*

**Part I.** *Assume that there exists a Lipschitz continuous negative function $b(\cdot)$ such that*

**CO**

$$\forall x_0 \in \partial S,\ \min_{u \in U} \langle \nabla\phi(x_0), f(x_0) + g(x_0)u \rangle \leq b(x_0) < 0$$

*Then $S$ is small-time locally attainable.*

**Part II.** *Assume that*

**CO′**

$$\forall x_0 \in \partial S,\ \min_{u \in U} \langle \nabla\phi(x_0), f(x_0) + g(x_0)u \rangle = 0$$

*and that for any $x_0 \in \partial S$, there exists some neighborhood $B(x_0, r)$ of $x_0$ on which the following conditions hold true:*

**C1** *There exist some constant $C > 0, d > 0$ such that*

$$\|f(x)\| \leq C(\phi(x))^d,\ \forall x \in\ cl(B(x_0, r) \backslash S) \tag{33}$$

**C2** *There exists a Lipschitz continuous function $b(\cdot) : cl(B(x_0, r) \backslash S) \mapsto \mathbb{R}_-$ and $w(\cdot) : cl(B(x_0, r) \backslash S) \mapsto \mathbb{R}^n$ such that $w \in co\ W$ and for every point $x \in cl(B(x_0, r) \backslash S)$*

$$\langle \nabla\phi(x), w(x) \rangle \leq b(x) < 0.$$

*Then the set $S$ is small-time locally attainable.*

This corollary is a direct consequence of Proposition 3.3 and Theorem 3.4, when our approach of local variations is used in a first order Taylor expansion of $\phi$.

When $S$ is a regular set of the form

$$S = \{x \in \mathbb{R}^n,\ \phi(x) \leq 0\}$$

with $\phi$ of class $C^1$ or an intersection of such sets one can obtain more explicit condition:

**Corollary 3.6.** *Let $S = \bigcap_{j\in J} \{x \in \mathbb{R}^n,\ \phi_j(x) \leq 0\}$ be a compact subset where $\phi_j : \mathbb{R}^n \mapsto \mathbb{R}$ are $C^1$ functions with nonvanishing gradients on $\partial S$. Let $f$ and $g$ be analytic and (30) holds true. For any $x_0 \in \partial S$, denote*

$$J(x_0) := \{j \in J,\ \phi_j(x_0) = 0\}\cdot$$

*Assume that conditions C1 and conditions CO, CO′, C2 with $\phi$ replaced with $\phi_j$ for any $j \in J(x_0)$ hold true.*

*Then the same conclusions as in Corollary 3.5 are valid.*

**Remark 3.5.** As in Remark 3.1, one can easily obtain a sufficient condition using second order Taylor expansion of $\phi$ when it is regular enough. As an illustration of this fact, we give the following example in $\mathbb{R}^2$. The set

$$S := \{(x, y) \,|\, y \leq x^2\}$$

is STLA for the control system

$$(x'(t), y'(t)) = (u(t), 0), \quad u(t) \in [-1, 1].$$

From previous results, we can derive regularity of the minimal time for reaching $S$.

### 3.3. Minimal time to reach a set

Note that the minimal time function is in general only lower semicontinuous [6], see also [24] for conditions insuring the Lipschitz continuity.

Let $\Theta(x_0)$ be the minimal time $\tau$ for which there exists a solution $y(\cdot)$ to (28) starting from $x_0$ and reaching $S$ in time $\tau$, namely $x(\tau) \in S$.

**Corollary 3.7.** *Suppose that the Assumptions A1–A5 of Theorem 3.1 hold. Then*

$$\Theta(x) \leq \text{const.}\ d_S(x)^{\frac{1}{\alpha}} \tag{34}$$

*for every $x$ from some neighborhood of $x_0$. Moreover $\Theta$ is $\frac{1}{\alpha}$- Hölder continuous in this neighborhood of $x_0$.*

*Proof.* Fix $x$ in $B(x_0, r) \setminus S$. In the same manner to that in Theorem 3.1, starting from $x$ there exists some trajectory such that $d_S(x(t)) = 0$ for $t$ satisfying (24). In the present corollary, the fact that the functions $c$, $d_S$ are Lipschitz continuous and equal to 0 on $\partial S$ so (25) implies that for any $\pi_x \in NP_S(x)$,

$$\Theta(x) \leq C\|x - \pi_x\|^{\frac{1}{\alpha}},$$

which gives (34) (where $C > 0$ is a constant).

Repeating arguments of [3], we shall prove the Hölder continuity. Let $y_1$ and $y_2$ be elements of $B(x_0, r/m)$ where $m > 0$ is a constant we choose later on.

Suppose $\Theta(y_1) \leq \Theta(y_2)$. Fix $\varepsilon > 0$. There exists a control $u_\varepsilon$ such that the trajectory $x_1(\cdot)$ to (28) starting from $y_1$ reach $S$ in some time $\tau$ with

$$\Theta(y_1) \leq \tau \leq \Theta(y_1) + \varepsilon.$$

Denote by $x_2(\cdot)$ the solution to

$$x'(t) = f(x(t)) + g(x(t))u_\varepsilon(t),\ t \in [0,\tau],\ x(0) = y_2.$$

Choose $m > 0$ large enough such that $x_2(\tau) \in B(x_0, r)$.

Grönwall's Lemma yields the existence of some $K > 0$ such that

$$\|x_1(\tau) - x_2(\tau)\| \le K\|y_1 - y_2\|.$$

By (34),

$$\Theta(x_2(\tau)) \le C.d_S(x_2(\tau))^{\frac{1}{\alpha}},$$

so

$$\Theta(y_2) \le \tau + \Theta(x_2(\tau)) \le \Theta(y_1) + \epsilon + C.d_S(x_2(\tau))^{\frac{1}{\alpha}} \le \Theta(y_1) + C.\|x_1(\tau) - x_2(\tau)\|^{\frac{1}{\alpha}} + \epsilon$$

(because $x_1(\tau) \in S$)

$$\le \Theta(y_1) + C.(K\|y_1 - y_2\|)^{\frac{1}{\alpha}} + \epsilon.$$

Since $\Theta(y_1)$ and $\Theta(y_2)$ do not depend on $\epsilon$, we obtain that

$$\Theta(y_2) \le \Theta(y_1) + C.K^{\frac{1}{\alpha}}\|y_1 - y_2\|^{\frac{1}{\alpha}}.$$

Symmetric arguments when $\Theta(y_2) \le \Theta(y_1)$ complete the proof. □

## 4. Necessary and sufficient small-time local attainability conditions

This section is devoted to linear and polynomial control systems. We show that in this context necessary and sufficient attainability condition can be obtained. In both cases the set $S$ will be the following hyperplane going through the origin and normal to a given vector $n \in \mathbb{R}^n \setminus \{0\}$

$$H \ := \ \{x \in \mathbb{R}^n :\ \langle n, x\rangle \ = \ 0\}\cdot$$

### 4.1. Linear control systems

Let us consider the following linear control system on $\mathbb{R}^n$:

$$x'(t) \ = \ Ax(t) \ + \ \sum_{i=1}^{m} u_i b_i, \quad u := (u_1, ..., u_m) \in \mathbb{R}^m, \tag{35}$$

where $A$ is a constant matrix of dimension $n\times n$ and $b_1 \in \mathbb{R}^n, i = 1, 2, ..., m$. To study the problem of attainability of the set $H$ with respect to the control system (35), one can use the approach described in previous sections. But, exploiting the linearity of the system, we shall obtain a necessary and sufficient condition in a more direct way. In fact, we prove the following:

**Proposition 4.1.** *The hyperplane $H$ is locally attainable with respect to the control system (35) if and only if there exist a positive integer $k$ and a vector $u^0 = (u^0, ..., u^0_m)$ such that*

$$\left\langle n, A^k \left(\sum_{i=1}^{m} u_i^0 b_i\right)\right\rangle \ \neq \ 0. \tag{36}$$

*Proof.* Sufficiency: Let us assume that the condition (36) holds true. Let us denote by $b = \sum_{i=1}^{m} u_i^0 b_i$ and define the matrix $P$ by

$$Px \;:=\; Ax \;-\rho.\left\langle n, A^{k+1}x\right\rangle b, \quad \text{where } \rho \;:=\; \frac{1}{\langle n, A^k b\rangle}\cdot$$

One can directly check that the following relations hold true:

$$\left\langle n, A^k P x\right\rangle \;=\; 0 \;\text{ for every } x \in \mathbb{R}^n, \tag{37}$$

$$\left\langle n, P^i b\right\rangle \;=\; 0 \;\text{ for } i \;\neq\; k, \;\text{ and } \left\langle n, P^k b\right\rangle \;=\; \left\langle n, A^k b\right\rangle \cdot \tag{38}$$

Let $y$ be an arbitrary point which does not belong to $H$. Take an arbitrary $c > 0$ and an integrable function $v(.) : [0,1] \;\to\; \mathbb{R}^+$ for which

$$\int_0^1 \frac{(1-\theta)^k}{k!} v(\theta)\mathrm{d}\theta \;=\; 1, \quad \text{and set } v_t^c\;(s) \;=\; c.v\left(\frac{s}{t}\right), s \in [0,t]. \tag{39}$$

Let us consider the solution $x_t^c(.)$ of

$$x'(s) \;=\; Px(s) \;+\; v_t^c(s)b, \;\; x(0) \;=\; 0, \;\; s \in [0,t], \; t > 0. \tag{40}$$

By the definition of $P$, $x_t^c(s)$ is a trajectory of (35) and

$$x_t^c(s) \;=\; \mathrm{Exp}(sP)y \;+\; \int_0^s \mathrm{Exp}((s-\theta)P)bv_t^c(\theta)\mathrm{d}\theta, \;\; s \in [0,t]. \tag{41}$$

By setting $\tau := \theta/t$ and $\omega := s/t$ , we obtain that

$$\int_0^s \mathrm{Exp}((s-\theta)P)bv_t^c(\theta)\mathrm{d}\theta \;=\; c.t.\int_0^\omega \mathrm{Exp}(t(\omega-\tau)P)bv(\tau)\mathrm{d}\tau.$$

Then

$$x_t^c(t) \;=\; \sum_{i=0}^{\infty} c.\beta_i.t^{i+1}P^i b,$$

where

$$\beta_i \;:=\; \int_0^1 \frac{(1-\theta)^i}{i!} v(\theta)\mathrm{d}\theta.$$

The point $x_t^c(t)$ belongs to $H$ if and only if the equality $\langle n, x_t^c(t)\rangle \;=\; 0$ holds true. According to (37) and (38) we have

$$\langle n, x_t^c(t)\rangle \;=\; \langle n, \;\mathrm{Exp}(tP)y\rangle \;+\; \sum_{i=0}^{\infty} c.t^{i+1}.\beta_i.\left\langle n, P^i b\right\rangle \;=\; \langle n, \;\mathrm{Exp}(tP)y\rangle \;+\; c.t^{k+1}.\left\langle n, P^k b\right\rangle \cdot$$

By setting

$$c \;:=\; -\frac{\langle n, \;\mathrm{Exp}(tP)y\rangle}{c.t^{k+1}.\left\langle n, P^k b\right\rangle},$$

we obtain that $x_t^c(t)$ belongs to $H$.

Necessity: we assume that

$$\left\langle n, A^k b_i\right\rangle \;=\; 0 \;\text{ for all } i = 1,2,..,m, \text{ and } k = 1,2,... \tag{42}$$

First, let us assume that the hyperplane $H$ is invariant under the mapping $A$, *i.e.* $AH \subset H$. This implies that $A^*n = \gamma n$ for some real $\gamma$ (by $A^*$ we have denoted the transposed matrix of the matrix $A$). Let us choose the point $n$ which does not belong to the hyperplane $H$. Let $x(\cdot)$ be an arbitrary trajectory of (35) starting from the point $n$, *i.e.*

$$x(t) = \operatorname{Exp}(tP)n + \sum_{i=1}^{m} \int_0^s \operatorname{Exp}((t-\theta)A)b_i v_i(\theta)\mathrm{d}\theta. \tag{43}$$

Then

$$\langle n, x(t)\rangle = \langle n, \operatorname{Exp}(tA)n\rangle + \sum_{i=1}^{m} \left\langle n, \int_0^s \operatorname{Exp}((t-\theta)A)b_i v_i(\theta)\mathrm{d}\theta \right\rangle =$$

(accordingly to (42))

$$= \sum_{i=0}^{\infty} \left\langle \frac{t^k}{k!}(A^*)^k n, n \right\rangle + 0 = \sum_{i=0}^{\infty} \frac{\gamma^k t^k}{k!}|n|^2 = \mathrm{e}^{t\gamma}|n|^2 > 0,$$

*i.e.* $x(t)$, does not belong to $H$ for every $t \geq 0$ .

Next, we assume that the hyperplane $H$ is not invariant with respect to the matrix $A$. This implies the existence of a point $x_0 \in H$ for which $\langle n, Ax_0\rangle > \varepsilon > 0$. Then there exists a positive number $\delta$ such that $\langle n, Ax\rangle > \varepsilon > 0$ for every $x \in B(x_0,\delta) \setminus H$. Starting from an arbitrary point from $x \in B(x_0,\delta) \setminus H$ for which $\langle n, x\rangle > 0$, the corresponding trajectory can not reach $H$ without leaving the set $B(x_0,\delta)$. But this implies that the set $H$ is not small-time locally attainable with respect to the trajectories of the system (35). $\square$

**Example 4.1.** Let $\varepsilon > 0$ be an arbitrary real number. We set $H := \{(x,y) : x = 0\}$ and consider the following two-dimensional control system

$$\begin{aligned} x' &= +y, \quad x(0) = \sin\left(\frac{\pi}{2} - \varepsilon\right) \\ y' &= -x, \quad y(0) = \cos\left(\frac{\pi}{2} - \varepsilon\right). \end{aligned}$$

One can directly check that if $\varepsilon > 0$ is sufficiently small, then the trajectory starting from a point, which is sufficiently closed to $(x(0), y(0))$, go away from the set $H$, and then, after a finite time, this trajectory reaches $H$.

## 4.2. **Polynomial control systems**

Next, we consider the following polynomial control system on $\mathbb{R}^n$:

$$x'(t) = P(x(t)) + \sum_{i=1}^{m} u_i(t)b_i, \quad u \in \mathbb{R}^m, \tag{44}$$

where $P : \mathbb{R}^n \to \mathbb{R}^n$. As in [13], we shall assume the existence of an odd positive integer $p$ such that if $P = (P_1, ..., P_n)$, then each coordinate $P_i, i = 1, ..., n$, is a homogeneous polynomial with respect to the coordinates $x = (x_1, ..., x_n)$ of degree $p$, *i.e.* $P_i(\lambda x) = \lambda^p P_i(x), i = 1, ..., n$. The system (44) is from the so called class of odd nonlinear systems (*cf.* [5]). To study the problem of attainability of the set $H$ with respect to the control system (44), we use the approach described in the previous sections and we prove a sufficient and necessary condition.

**Proposition 4.2.** *Let $B^P$ be the smallest vector space containing the vectors $b_1, ..., b_m$, and which is invariant under the mapping $P$. Then the hyperplane $H$ is locally attainable with respect to the control system (44) if and only if there exists an element $b$ of $B^P$ such that*

$$\langle n, b\rangle \neq 0. \tag{45}$$

*Proof.* Sufficiency: define $R(x,t)$ to be the closure of the union of reachable sets of the system (44) from the point $x$ at times $\tau \le t$, *i.e.*

$$R(x,t) := cl\Big( \bigcup_{\tau\in[0,t]} \{x(\tau)\,|\,\text{where } x(\cdot) \text{ is a solution to (44) with } x(0)=x\,\}\cdot$$

Let $t>0$ and $s>0$ be arbitrary reals and $i$ be an arbitrary index, $1\le i\le m$. Since

$$\mathrm{Exp}\left(\frac{t}{n}\left(P\pm\frac{sn}{t}b_i\right)\right)(x) \;\in\; R\left(x,\frac{t}{n}\right) \;\subset\; R\,(x,t)\,,$$

we obtain after taking a limit $n\to\infty$ that

$$\mathrm{Exp}\,(\pm s b_i))\,(x) \;\in\; R\,(x,t)\,.$$

This means that $b_i\in\mathcal{S}^{\text{fast}}$, $i=1,2,...,m$. Proposition 2.5 implies that the vector space $L$ spanned by the vectors $b_i, i=1,2,...,m$, is a subset of $\mathcal{S}^{\text{fast}}$. Let $b\in L$ be an arbitrary vector. Considered as a constant vector field, we have according to the homogenety of $P$ that $(ad^k\; b, P)\equiv 0$ for all $k>p$. Hence, applying Proposition 2.6, we obtain that $\pm P(b)=(ad^p b,P)\in\mathcal{S}^{\text{fast}}$. From here we can conclude that the set $B^P \subset \mathcal{S}^{\text{fast}}$. Let $y$ be an arbitrary point which does not belong to $H$. Without loss of generality, we may assume that $\langle n,y\rangle<0$. Since $B^P$ is a linear subspace, the assumption (45) implies the existence of $b\in B^P$ such that $\langle n,b\rangle>0$. Application of Theorem 3.1 with $A(y)\equiv b$ and $a(t;x)\equiv 0$ implies that $R(y,t)\cap H^\epsilon\ne\emptyset$, where $H^\epsilon := \{z\in R^n|\;\langle n,z\rangle>\epsilon\}$ and $\epsilon>0$ is a sufficiently small. But this implies the existence of a trajectory of the system (44) reaching $H$ at some moment of time not greater than $t$.

Necessity: let for all $b\in B^P$ we have that

$$\langle n,b\rangle \;\equiv\; 0. \tag{46}$$

First, let us assume that the hyperplane $H$ is invariant under the mapping $P$, *i.e.* $P(x)\in H$ for every point $x$ from $H$. The assumptions on $P$ and the assumption (46) imply that the set $H$ is invariant with respect to the trajectories of the following control system:

$$x'(t) \;=-P(x(t)) \;+\; \sum_{i=1}^{m} u_i(t)b_i. \tag{47}$$

So, if we assume the existence of a trajectory of (44) starting from a point $x\notin H$ and reaching $H$ at some point $y$, then this implies the existence of a trajectory of the system (47) starting from $y$ and reaching the point $x$. This contradicts the invariance property of the set $H$ with respect to (47) and completes the proof in this case.

Next, we assume that the hyperplane $H$ is not invariant with respect to the mapping $P$. This implies the existence of a point $x_0\in H$ for which $\langle n,P(x_0)\rangle \;>\; \varepsilon>0$. Then there exists a positive number $\delta$ such that $\langle n,P(x)\rangle \;>\; \varepsilon>0$ for every $x\in B(x_0,\delta)\setminus H$. Starting from an arbitrary point from $x\in B(x_0,\delta)\setminus H$ for which $<n,x>\;>\;0$, the corresponding trajectory can not reach $H$ without leaving the set $B(x_0,\delta)$. But this means that the set $H$ is not small-time local attainable with respect to (47). □

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