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A SINGULAR PERTURBATION PROBLEM IN EXACT CONTROLLABILITY OF THE MAXWELL SYSTEM *

John E. Lagnese[1]

Abstract. This paper studies the exact controllability of the Maxwell system in a bounded domain, controlled by a current flowing tangentially in the boundary of the region, as well as the exact controllability the same problem but perturbed by a dissipative term multiplied by a small parameter in the boundary condition. This boundary perturbation improves the regularity of the problem and is therefore a singular perturbation of the original problem. The purpose of the paper is to examine the connection, for small values of the perturbation parameter, between observability estimates for the two systems, and between the optimality systems corresponding to the problem of norm minimum exact control of the solutions of the two systems from the rest state to a specified terminal state.

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1. Introduction

Let Ω be a bounded, open, connected set in $\mathbb{R}^3$ with smooth boundary Γ, and let $T > 0$. We consider the Maxwell system

$$\begin{gathered}\begin{cases}\varepsilon E_t - \operatorname{rot} H = 0\\ \mu H_t + \operatorname{rot} E = 0 \quad \text{in } \mathcal{Q} := \Omega\times(0,T)\end{cases}\\ \nu\wedge E = J \text{ on } \Sigma := \Gamma\times(0,T)\\ E(0)=E_0, \quad H(0)=H_0 \text{ in } \Omega,\end{gathered} \tag{1.1}$$

as well as the perturbed system

$$\begin{gathered}\begin{cases}\varepsilon E^\delta_t - \operatorname{rot} H^\delta = 0\\ \mu H^\delta_t + \operatorname{rot} E^\delta = 0 \quad \text{in } \mathcal{Q}\end{cases}\\ \nu\wedge E^\delta - \delta\,\nu\wedge(H^\delta\wedge\nu) = J \text{ on } \Sigma,\ \delta>0,\\ E^\delta(0)=E_0, \quad H^\delta(0)=H_0 \text{ in } \Omega.\end{gathered} \tag{1.2}$$

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[1] Department of Mathematics, Georgetown University, Washington, DC 20057, U.S.A.; e-mail: `lagnese@math.georgetown.edu`

Here $\wedge$ denotes vector cross product, ν is the exterior pointing unit normal vector to Γ and $\varepsilon = (\varepsilon^{jk}(x))$, $\mu = (\mu^{jk}(x))$ are positive definite 3×3 Hermitian matrices with $C^\infty(\overline{\Omega})$ entries. The function J is taken from the class

$$\mathcal{U} = \mathcal{L}^2_\tau(\Sigma) := \{J|\, J \in L^2(0,T;\mathcal{L}^2(\Gamma)),\, \nu \cdot J(t) = 0 \;\text{ for a.a. } x \in \Gamma \text{ and a.a. } t \in (0,T)\}\cdot$$

Function spaces of $\mathcal{C}$-valued functions are denoted by capital roman letters, while function spaces of $\mathcal{C}^3$-valued functions are denoted by capital script letters. We use $\alpha \cdot \beta$ to denote the natural scalar product in $\mathcal{C}^3$, *i.e.*, $\alpha \cdot \beta = \sum_{j=1}^3 \alpha_j \overline{\beta}_j$, and write $\langle \cdot, \cdot \rangle$ for the natural scalar product in various function spaces such as $L^2(\Omega)$ and $\mathcal{L}^2(\Omega)$. A subscript may sometimes be added to avoid confusion. The spaces $L^2(\Omega)$ and $\mathcal{L}^2(\Omega)$ denote the usual spaces of Lebesque square integrable $\mathcal{C}$-valued functions and $\mathcal{C}^3$-valued functions, respectively.

Set $\mathcal{H} = \mathcal{L}^2(\Omega) \times \mathcal{L}^2(\Omega)$ with weight matrix $M = \operatorname{diag}(\varepsilon, \mu)$. Thus

$$\|(\phi, \psi)\|^2_{\mathcal{H}} = \langle \varepsilon\phi, \phi \rangle + \langle \mu\psi, \psi \rangle \cdot$$

It will be proved below that for $J \in \mathcal{U}$ and $(E_0, H_0) \in \mathcal{H}$, equation (1.2) has a unique solution with regularity $(E^\delta, H^\delta) \in C([0,T];\mathcal{H})$, $\nu \wedge E^\delta|_\Sigma \in \mathcal{L}^2_\tau(\Sigma)$, $\nu \wedge (H^\delta \wedge \nu)|_\Sigma \in \mathcal{L}^2_\tau(\Sigma)$. On the other hand, for the same data, the solution (E, H) of (1.1) has less spatial regularity. Thus there is a loss of regularity as $\delta \to 0$ and therefore we consider (1.2) as a *singular perturbation* of (1.1).

For a scalar function $a \in L^\infty(\Omega)$ we define

$$\mathcal{D}_{a,0}(\Omega) = \{\chi \in \mathcal{L}^2(\Omega) :\, \operatorname{div}(a\chi) = 0\},$$

and we set

$$\mathcal{H}_0 = \mathcal{D}_{\varepsilon,0}(\Omega) \times \mathcal{D}_{\mu,0}(\Omega),$$

which is a closed subspace of $\mathcal{H}$. We note that $(E_0, H_0) \in \mathcal{H}_0$ implies that $(E^\delta, H^\delta) \in C([0,T];\mathcal{H}_0)$ for $\delta > 0$. Consider the problem of exact controllability of the solution of (1.1) to the space $\mathcal{H}_0$ at time T: given fixed but arbitrary (E_0, H_0), $(E_1, H_1) \in \mathcal{H}_0$, find a control $J_0 \in \mathcal{U}$ such that the solution (1.1) satisfies

$$E(T) = E_1,\; H(T) = H_1. \tag{1.3}$$

Without loss of generality, one may assume that $E_0 = H_0 = 0$. It is known that the exact controllability problem has a solution if and only if $\mathcal{H}_0$ is *continuously observable*, that is, there is a constant $C^0_T > 0$ such that

$$\|(\phi_0, \psi_0)\|^2_{\mathcal{H}} \le C^0_T \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma, \quad \forall (\phi_0, \psi_0) \in \mathcal{F}_0, \tag{1.4}$$

where

$$\psi_\tau := \nu \wedge (\psi \wedge \nu) = \psi - (\psi \cdot \nu)\nu,$$

$$\mathcal{F}_0 = \mathcal{F} \cap \mathcal{H}_0, \quad \mathcal{F} = \{(\phi_0, \psi_0) \in \mathcal{H} :\, \psi_\tau|_\Sigma \in \mathcal{L}^2_\tau(\Sigma)\},$$

and where (ϕ, ψ) is the solution of the problem

$$\begin{gathered} \begin{cases} \varepsilon\phi_t - \operatorname{rot}\psi = 0 \\ \mu\psi_t + \operatorname{rot}\phi = 0 \quad \text{in } \mathcal{Q} \end{cases} \\ \nu \wedge \phi = 0 \text{ on } \Sigma \\ \phi(T) = \phi_0, \quad \psi(T) = \psi_0 \text{ in } \Omega. \end{gathered} \tag{1.5}$$

Indeed, it follows from Green's formula (2.1) below that formally

$$\langle (E(T), H(T)), (\phi_0, \psi_0)\rangle_{\mathcal{H}} = -\int_\Sigma J \cdot \psi_\tau \mathrm{d}\Sigma, \quad \forall (\phi_0, \psi_0) \in \mathcal{F}_0.$$

When (1.4) holds, the control of minimum norm in $\mathcal{L}^2_\tau(\Sigma)$ such that the state constraint (1.3) is satisfied is given by

$$J^0 = -\psi_\tau|_\Sigma \tag{1.6}$$

where (ϕ, ψ) is the solution of (1.5) with final data $(\phi_0, \psi_0) \in \mathcal{F}_0$ given by

$$\langle (E_1, H_1), (\phi_0, \psi_0)\rangle_{\mathcal{H}} = \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma. \tag{1.7}$$

Therefore, the *optimality system* for the problem of norm minimum control of the system (1.1) from the rest state $(0, 0)$ to the state (E_1, H_1) at time T is given by (1.1) and (1.5), where the final data is given by (1.7), and the norm minimum control J^0 is given by (1.6).

Similarly, consider the problem of exact controllability of the solution of (1.2) to the space $\mathcal{H}_0$ at time T: with $E_0 = H_0 = 0$ and given fixed but arbitrary $(E_1, H_1) \in \mathcal{H}_0$, find a control $J^\delta \in \mathcal{U}$ such that the solution (1.2) satisfies

$$E^\delta(T) = E_1,\ H^\delta(T) = H_1. \tag{1.8}$$

This problem has a solution if and only if there is a constant $C^\delta_T > 0$ such that

$$\|(\phi_0, \psi_0)\|^2_{\mathcal{H}} \le C^\delta_T \int_\Sigma |\psi^\delta_\tau|^2 \mathrm{d}\Sigma, \quad \forall (\phi_0, \psi_0) \in \mathcal{H}_0, \tag{1.9}$$

where $(\phi^\delta, \psi^\delta)$ is the solution of

$$\begin{gathered} \begin{cases} \varepsilon\phi^\delta_t - \operatorname{rot}\psi^\delta = 0 \\ \mu\psi^\delta_t + \operatorname{rot}\phi^\delta = 0 \quad \text{in } \mathcal{Q} \end{cases} \\ \nu \wedge \phi^\delta + \delta\psi^\delta_\tau = 0 \text{ on } \Sigma \\ \phi^\delta(T) = \phi_0, \quad \psi^\delta(T) = \psi_0 \text{ in } \Omega. \end{gathered} \tag{1.10}$$

Indeed, for the solution of (1.2) one has

$$\langle (E^\delta(T), H^\delta(T)), (\phi_0, \psi_0)\rangle_{\mathcal{H}} = -\int_\Sigma J \cdot \psi^\delta_\tau \mathrm{d}\Sigma, \quad \forall (\phi_0, \psi_0) \in \mathcal{H}_0. \tag{1.11}$$

Remark 1.1. From the easily verified equality

$$\|(\phi^\delta(t), \psi^\delta(t))\|^2_{\mathcal{H}} + 2\delta \int_t^T \int_\Gamma |\psi^\delta_\tau|^2 \mathrm{d}\Gamma \mathrm{d}t = \|(\phi_0, \psi_0)\|^2_{\mathcal{H}}, \forall (\phi_0, \psi_0) \in \mathcal{H},\ 0 \le t \le T, \tag{1.12}$$

one has the reverse inequality of (1.9)

$$\int_\Sigma |\psi^\delta_\tau|^2 \mathrm{d}\Sigma \le \frac{1}{2\delta} \|(\phi_0, \psi_0)\|^2_{\mathcal{H}}, \quad \forall (\phi_0, \psi_0) \in \mathcal{H}. \tag{1.13}$$

It follows from (1.11) and (1.13) that the control-to-state map $L_T^\delta J := (E^\delta(T), H^\delta(T))$ is *bounded* from $\mathcal{U}$ into $\mathcal{H}_0$, and one sees that (1.9) is equivalent to the bounded invertibility of $(L_T^\delta)^*$, which in turn is equivalent to $\mathrm{Rg}(L_T^\delta) = \mathcal{H}_0$.

When (1.9) holds, the control of minimum norm in $\mathcal{L}^2_\tau(\Sigma)$ such that the state constraint (1.8) is satisfied is given by

$$J^\delta = -\psi^\delta_\tau|_\Sigma \tag{1.14}$$

where $(\phi^\delta, \psi^\delta)$ is the solution of (1.10) with final data $(\phi^\delta_0, \psi^\delta_0) \in \mathcal{H}_0$ given by

$$\langle (E_1, H_1), (\phi^\delta_0, \psi^\delta_0)\rangle_{\mathcal{H}} = \int_\Sigma |\psi^\delta_\tau|^2 \mathrm{d}\Sigma. \tag{1.15}$$

Therefore, the *optimality system* for the problem of norm minimum control of the system (1.2) from the rest state $(0, 0)$ to the state (E_1, H_1) at time T is given by (1.2) and (1.10), where the final data is given by (1.15), and the norm minimum control is given by (1.14).

The purpose of this paper is to examine the connection between the observability estimates (1.4) and (1.9), and between the corresponding optimality systems for small values of δ. Specifically, we shall prove the following results:

Theorem 1.1. *For $\delta \geq 0$, let $(\phi^\delta, \psi^\delta)$ be the solution of* (1.10) *if $\delta > 0$, or the solution of* (1.5) *if $\delta = 0$, where $(\phi_0, \psi_0) \in \mathcal{H}$ if $\delta > 0$, and $(\phi_0, \psi_0) \in \mathcal{F}$ if $\delta = 0$. The map $\delta \mapsto \|\psi^\delta_\tau\|_{\mathcal{L}^2_\tau(\Sigma)} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is nonincreasing.*

Corollary 1.1. *If* (1.9) *holds for some $\delta_0 > 0$, then it hold for all $\delta \in (0, \delta_0]$ with the same constant $C_T^{\delta_0}$, and* (1.4) *holds with $C_T^0 = C_T^{\delta_0}$.*

Theorem 1.2. *Assume that* (1.9) *holds for some $\delta_0 > 0$. Let $(E_1, H_1) \in \mathcal{H}_0$, and $(\phi^\delta, \psi^\delta)$ be the solution of* (1.10)*, where $(\phi^\delta_0, \psi^\delta_0) \in \mathcal{H}_0$ is given by* (1.15)*. Then as $\delta \to 0$,*

$$(\phi^\delta(\cdot), \psi^\delta(\cdot)) \to (\phi(\cdot), \psi(\cdot)) \text{ weakly* in } L^\infty(0, T; \mathcal{H})$$
$$(\phi^\delta_0, \psi^\delta_0) \to (\phi_0, \psi_0) \text{ weakly in } \mathcal{H},$$

where

$$\begin{cases} \varepsilon\phi' - \operatorname{rot}\psi = 0 \\ \mu\psi' + \operatorname{rot}\phi = 0 \quad \text{in } \mathcal{Q} \end{cases}$$
$$\nu \wedge \phi = 0 \text{ on } \Sigma$$
$$\phi(T) = \phi_0, \quad \psi(T) = \psi_0 \text{ in } \Omega.$$

Further, $(\phi_0, \psi_0) \in \mathcal{F}_0$, $\psi^\delta_\tau|_\Sigma \to \psi_\tau|_\Sigma$ strongly in $\mathcal{L}^2_\tau(\Sigma)$ and

$$\langle (E_1, H_1), (\phi_0, \psi_0)\rangle_{\mathcal{H}} = \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma.$$

Theorem 1.3. *Assume that* (1.9) *holds for some $\delta_0 > 0$. Let $E_0 = H_0 = 0$, $(E_1, H_1) \in \mathcal{H}_0$, and (E^δ, H^δ) be the solution of* (1.2) *with $J = -\psi^\delta_\tau|_\Sigma$ (thus* (1.8) *holds). Then $(E^\delta, H^\delta) \to (E, H)$ weakly* in $L^\infty(0, T; \mathcal{X}')$, where (E, H) is the solution of* (1.1) *with $J = -\psi_\tau|_\Sigma$ (thus* (1.3) *holds) and where $\mathcal{X} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{X}'$ is given by* (2.7) *below.*

Remark 1.2. The validity of (1.9) for some $\delta_0 > 0$ is equivalent to

$$\|(\phi^\delta(0), \psi^\delta(0))\|^2_{\mathcal{H}} \leq C_T^\delta \int_\Sigma |\psi^\delta_\tau|^2 \mathrm{d}\Sigma, \quad \forall (\phi_0, \psi_0) \in \mathcal{H}_0$$

for some $\delta > 0$. The latter is equivalent to the following *stability estimate* for the system (1.2) with $J = 0$:

$$\|(E^\delta(T), H^\delta(T))\|_{\mathcal{H}}^2 \leq C_T^\delta \int_\Sigma |H_\tau^\delta|^2 \mathrm{d}\Sigma, \quad \forall (E_0, H_0) \in \mathcal{H}_0. \tag{1.16}$$

Inequality (1.16) is equivalent to the uniform exponential stability in $\mathcal{H}_0$ of the system (1.2) with $J = 0$. Therefore, Corollary 1.1 implies that (1.1) is exactly controllable to $\mathcal{H}_0$ at time T if (1.2) with $J = 0$ is uniformly exponentially stable in $\mathcal{H}_0$ for some $\delta_0 > 0$. Of course, this implication may also be proved by using the "forward – backwards" argument of Russell [14], which is based on the contraction mapping principle. However, Theorem 1.1 shows that this conclusion follows *automatically* from the observability estimate (1.9) (or the stability estimate (1.16)). A similar observation has recently been made in a general framework by Ammari and Tucsnak [1] for a class of second order evolution equations; see also Tucsnak and Weiss [15].

Remark 1.3. Theorems 1.2 and 1.3 show that the solution of the optimality system for the problem of norm minimum control of the system (1.2) from the rest state $(0, 0)$ to the state (E_1, H_1) at time T converges in the sense described to the solution of the optimality system for the problem of norm minimum control of the system (1.1) from the rest state $(0, 0)$ to the state (E_1, H_1) at time T. In particular, the optimal control for (1.2) converges strongly in $\mathcal{L}_\tau^2(\Sigma)$ to the optimal control for (1.1). Further, the optimal trajectory $(E(\cdot), H(\cdot)) \in L^\infty(0, T; \mathcal{X}')$ that joins $(0, 0)$ to (E_1, H_1) at time T is approximated in $L^\infty(0, T; \mathcal{X}')$ by the more regular optimal trajectories $(E^\delta(\cdot), H^\delta(\cdot)) \in C([0, T]; \mathcal{H}_0)$.

Remark 1.4. When ε and μ are positive scalars, the estimate (1.9) was (implicitly) established by multiplier methods for $\delta = 1$ by Komornik [7], who showed that (1.2) with $J = 0$ is exponentially stable provided Γ is star-shaped with respect to some point $x_0 \in \Omega$ and T is suitably large depending on the geometry of Ω. These results were greatly extended by Phung [13], who used results on the propagation of singularities of electromagnetic fields to obtain (1.9) for general regions. Very recently Eller [3] has established (1.9) in the case of $C^\infty(\overline{\Omega})$ positive scalar functions ε and μ, provided Ω is simply connected, T is suitably large, and ε, μ satisfy the technical condition $M \cdot \nabla(1/\varepsilon\mu) \leq 0$ for all $x \in \Omega$, where $M(x) = x - x_0$ and x_0 is some point in $\mathbb{R}^3$. On the other hand, when ε and μ are positive scalars the observability estimate (1.4) was established in [8] by multiplier methods provided Γ is star-shaped with respect to some point $x_0 \in \Omega$ and T is sufficiently large. These results were later extended by Nalin [12] and, especially, by Phung [13] to general regions, and then by Eller [4] to the case of $C^\infty(\overline{\Omega})$ positive scalar functions ε and μ under the same conditions mentioned above.

Remark 1.5. The proofs of Theorems 1.1 and 1.2 extend with only minor changes to the Maxwell system

$$\varepsilon E_t - \operatorname{rot} H + \sigma E = 0, \quad \mu H_t + \operatorname{rot} E = 0,$$

where σ is a nonnegative Hermitian matrix with $L^\infty(\Omega)$ entries that represent the resistivity of the electromagnetic material. However, the author is unaware of any result that establishes either observability estimate (1.4) or (1.9) for this system, even when σ is a nonnegative constant.

Remark 1.6. Although the above results are stated in the context of the Maxwell system, our arguments apply equally to many other singular perturbations problems. One may consider, for example, the elasticity system with traction boundary conditions:

$$\begin{gathered} w_{i,tt} - \sigma_{ij,j} = 0 \text{ in } \mathcal{Q} \\ \sigma_{ij}\nu_j = f_i \text{ on } \Sigma \\ w_i(0) = w_{i0}, \quad w_{i,t}(0) = w_{i1} \text{ in } \Omega, \end{gathered}$$

and its singular perturbation

$$\begin{gathered} w_{i,tt}^\delta - \sigma_{ij,j}^\delta = 0 \text{ in } \mathcal{Q} \\ \sigma_{ij}^\delta\nu_j + \delta w_{i,t}^\delta = f_i \text{ on } \Sigma,\ \delta > 0 \\ w_i^\delta(0) = w_{i0}, \quad w_{i,t}^\delta(0) = w_{i1} \text{ in } \Omega, \end{gathered}$$

where $i = 1, 2, 3$,

$$\sigma_{ij}^\delta = a_{ijkl}\varepsilon_{kl}^\delta, \quad \varepsilon_{kl}^\delta = \frac{1}{2}(w_{k,l}^\delta + w_{l,k}^\delta), \quad \delta \geq 0, \quad w_i^0 := w_i,$$

and $\{a_{ijkl}\}$ is the elasticity tensor. Assume that each $a_{ijkl} \in L^\infty(\Omega)$. Then if $f_i \in L^2(\Sigma)$ and $(w_{i0}, w_{i1}) \in H^1(\Omega) \times L^2(\Omega)$, for $\delta > 0$ one has the *a priori* estimate (*cf.* Lem. 2.3 below)

$$\mathcal{E}^\delta(T) + \sum_{i=1}^{3} \int_\Sigma \left[\frac{1}{\delta}|\sigma_{ij}^\delta \nu_j|^2 + \delta |w_{i,t}^\delta|^2\right] \mathrm{d}\Sigma = \mathcal{E}(0) + \frac{1}{\delta}\sum_{i=1}^{3} \int_\Sigma |f_i|^2 s\Sigma,$$

where

$$\mathcal{E}^\delta(t) = \int_\Omega [|\mathbf{w}_{,t}^\delta|^2 + \sigma_{ij}^\delta \varepsilon_{ij}^\delta] \mathrm{d}x,$$

thus $(w_i^\delta, w_{it}^\delta) \in C([0,T]; H^1(\Omega) \times L^2(\Omega))$, $\sigma_{ij}^\delta \nu_j|_\Sigma \in L^2(\Sigma)$, $w_{i,t}^\delta|_\Sigma \in L^2(\Sigma)$. On the other hand, for such data the solution has less spatial regularity when $\delta = 0$; in general $(w_i(t), w_{i,t}(t)) \notin H^1(\Omega) \times L^2(\Omega)$ and $w_{i,t}|_\Sigma \notin L^2(\Sigma)$. All of the results for the Maxwell system stated above have analogs for the elasticity system (and for many others), with similar proofs.

Remark 1.7. Apropos to the last remark, after reading this manuscript Lasiecka has written in a private communication that techniques developed in Hendrickson and Lasiecka [5, 6] for purposed entirely different than those of the present paper may be employed to prove a general result closely related to Theorem 1.2. Consider the second order system

$$x_{tt}^\delta + \mathcal{A}x^\delta + \delta\mathcal{B}\mathcal{B}^* x_t^\delta = 0, \quad \delta > 0, \tag{1.17}$$

where $\mathcal{A}$ is a positive self-adjoint operator in a Hilbert space $\mathcal{H}$ and $\mathcal{B} : U \mapsto \mathcal{D}(\mathcal{A}^{1/2})'$ is bounded. Write (1.17) as the first order system in $H := \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{H}$

$$y_t^\delta = Ay^\delta - \delta BB^* y^\delta := A_\delta y^\delta$$

with the standard definitions of A, B. Suppose that (A_{δ_0}, B) is exactly controllable to H at time T for some $\delta_0 > 0$. (It is then easy to see that (A_δ, B) is exactly controllable to H at time T for $0 \leq \delta \leq \delta_0$.) One then has the following result: if u_δ is the minimum $L^2(0,T;U)$ norm control corresponding to the dynamics (A_δ, B) that steers the origin to $y_0 \in H$ at time T, then $u_\delta \to u$ strongly in $L^2(0,T;U)$, where u is the minimum $L^2(0,T;U)$ norm control corresponding to the dynamics (A, B) that steers the origin to y_0 at time T.

Theorems 1.1–1.3 are proved in Section 3. Well-posedness of problems (1.1) and (1.2) is considered in the next section.

2. Existence and uniqueness of solutions

We set

$$\mathcal{H}_\tau^1(\Omega) = \{\phi \in \mathcal{H}^1(\Omega) : \nu \cdot \phi|_\Gamma = 0\}$$
$$\mathcal{R} = \{\phi \in \mathcal{L}^2(\Omega) : \operatorname{rot}\phi \in \mathcal{L}^2(\Omega)\}$$
$$\|\phi\|_{\mathcal{R}}^2 = \int_\Omega (|\phi|^2 + |\operatorname{rot}\phi|^2)\,\mathrm{d}x.$$

It is well known ([2], Lem. VII.4.2) that $\mathcal{C}^1(\overline{\Omega})$ is dense in $\mathcal{R}$ and that the map $\phi \mapsto \nu \wedge \phi|_\Gamma : \mathcal{C}^1(\overline{\Omega}) \mapsto \mathcal{C}^1(\Gamma)$ extends by continuity to a continuous linear map $\mathcal{R} \mapsto \mathcal{H}_\tau^{-1/2}(\Gamma) := (\mathcal{H}_\tau^{1/2}(\Gamma))'$. That is to say, for each $\phi \in \mathcal{R}$ there exists a $g \in \mathcal{H}_\tau^{-1/2}(\Gamma)$ such that

$$\langle \phi, \operatorname{rot} \chi\rangle = \langle \operatorname{rot} \phi, \chi\rangle - \langle g, \chi\rangle_\Gamma, \quad \forall \chi \in \mathcal{H}_\tau^1(\Omega),$$

and the map $\phi \mapsto g := \nu\wedge\phi|_\Gamma$ is linear and continuous, where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the pairing in the $\mathcal{H}_\tau^{-1/2}(\Gamma)$–$\mathcal{H}_\tau^{1/2}(\Gamma)$ duality. Set

$$V = \{\phi \in \mathcal{L}^2(\Omega) : \operatorname{rot} \phi \in \mathcal{L}^2(\Omega),\, \nu \wedge \phi \in \mathcal{L}_\tau^2(\Gamma)\},$$
$$\|\phi\|_V^2 = \int_\Omega (|\phi|^2 + |\operatorname{rot} \phi|^2)\mathrm{d}x + \int_\Gamma |\nu \wedge \phi|^2 \mathrm{d}\Gamma,$$

and define

$$A = M^{-1} \begin{pmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{pmatrix},$$
$$D(A) = \{(\phi, \psi) \in V \times V : \nu \wedge \phi - \delta\, \psi_\tau = 0 \text{ on } \Gamma\}\cdot$$

Lemma 2.1. *If $\delta > 0$, A is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$. If $\delta = 0$, A is the infinitesimal generator of a C_0 unitary group on $\mathcal{H}$.*

Proof. The conclusion for $\delta = 0$ is well-known; see, *e.g.* [11] (Chap. 8). Suppose that $\delta > 0$. The linear operator A is densely defined, and from the Green's formula

$$\langle \psi, \operatorname{rot} \phi\rangle = \langle \operatorname{rot} \psi, \phi\rangle - \int_\Gamma (\nu \wedge \psi) \cdot \phi_\tau \,\mathrm{d}\Gamma = \langle \operatorname{rot} \psi, \phi\rangle + \int_\Gamma \psi_\tau \cdot (\nu \wedge \phi)\,\mathrm{d}\Gamma, \quad (\phi, \psi) \in V \times V, \tag{2.1}$$

we obtain

$$\begin{aligned}\langle AU, U\rangle_{\mathcal{H}} &= -\int_\Gamma \psi_\tau \cdot (\nu \wedge \phi)\,\mathrm{d}\Gamma + 2\sqrt{-1}\,\operatorname{Im}\langle \operatorname{rot} \psi, \phi\rangle \\ &= -\delta \int_\Gamma |\psi_\tau|^2 \mathrm{d}\Gamma + 2\sqrt{-1}\,\operatorname{Im}\langle \operatorname{rot} \psi, \phi\rangle, \quad \forall U = \begin{pmatrix}\phi\\ \psi\end{pmatrix} \in D(A),\end{aligned}$$

so $\operatorname{Re}\langle AU, U\rangle_{\mathcal{H}} \le 0$.

Let $(f, g) \in \mathcal{H}$ and let ϕ be the unique solution in V of the variational equation

$$\langle \varepsilon\phi, \chi\rangle + \langle \mu^{-1} \operatorname{rot} \phi, \operatorname{rot} \chi\rangle + \frac{1}{\delta}\int_\Gamma (\nu \wedge \phi) \cdot (\nu \wedge \chi)\,\mathrm{d}\Gamma = \langle g, \operatorname{rot} \chi\rangle + \langle \varepsilon f, \chi\rangle, \quad \forall \chi \in V. \tag{2.2}$$

Set $\psi = g - \mu^{-1} \operatorname{rot} \phi \in \mathcal{L}^2(\Omega)$. Then (2.2) reads

$$\langle \psi, \operatorname{rot} \chi\rangle = \langle \varepsilon\phi, \chi\rangle - \langle \varepsilon f, \chi\rangle + \frac{1}{\delta}\int_\Gamma (\nu \wedge \phi) \cdot (\nu \wedge \chi)\,\mathrm{d}\Gamma, \quad \forall \chi \in V.$$

It follows that $\operatorname{rot} \psi \in \mathcal{L}^2(\Omega)$ and that

$$\varepsilon\phi - \operatorname{rot} \psi = \varepsilon f \text{ in } \Omega, \quad \delta\, \psi_\tau = \nu \wedge \phi \text{ on } \Gamma.$$

Therefore $\begin{pmatrix}\phi\\ \psi\end{pmatrix} \in D(A)$ and $(I - A)\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}f\\ g\end{pmatrix}$. □

Now consider the problem

$$\begin{cases} \varepsilon\Phi_t^\delta - \operatorname{rot}\Psi^\delta = \varepsilon f \\ \mu\Psi_t^\delta + \operatorname{rot}\Phi^\delta = \mu g \quad \text{in } \mathcal{Q} \end{cases}$$
$$\nu \wedge \Phi^\delta + \delta\,\Psi_\tau^\delta = 0 \ \text{ on } \Sigma \tag{2.3}$$
$$\Phi^\delta(T) = \Phi_0, \quad \Psi^\delta(T) = \Psi_0 \text{ in } \Omega.$$

As a consequence of Lemma 2.1 we have the following result:

Corollary 2.1. *Let* $\delta \geq 0$.

(1) If $(\Phi_0, \Psi_0) \in \mathcal{H}$ *and* $(f, g) \in L^1(0, T; \mathcal{H})$*, then* (2.3) *has a unique mild solution* $(\Phi^\delta, \Psi^\delta) \in C([0, T]; \mathcal{H})$ *and*

$$\|(\Phi^\delta, \Psi^\delta)\|_{L^\infty(0,T;\mathcal{H})} \leq C\big(\|(\Phi_0, \Psi_0)\|_{\mathcal{H}} + \|(f, g)\|_{L^1(0,T;\mathcal{H})}\big).$$

(2) If $(\Phi_0, \Psi_0) \in D(A)$ *and* $(f, g) \in C^1([0, T]; \mathcal{H})$*, then* $(\Phi^\delta, \Psi^\delta) \in C([0, T]; D(A))$.

Lemma 2.2. *Suppose that* $\delta > 0$. *Let* $(\Phi_0, \Psi_0) \in \mathcal{H}$ *and* $(f, g) \in L^1(0, T; \mathcal{H})$*. Then the solution of* (2.3) *satisfies* $\nu \wedge \Phi^\delta|_\Sigma \in \mathcal{L}_\tau^2(\Sigma)$.

Proof. First suppose that $(\Phi_0, \Psi_0) \in D(A)$ and $(f, g) \in C^1([0, T]; \mathcal{H})$. We then have

$$\begin{aligned} &-\frac{1}{2}\|(\Phi^\delta(t), \Psi^\delta(t))\|_{\mathcal{H}}^2 + \frac{1}{2}\|(\Phi_0, \Psi_0)\|_{\mathcal{H}}^2 + \int_t^T \int_\Gamma \Psi_\tau^\delta \cdot (\nu \wedge \Phi^\delta)\,\mathrm{d}\Gamma\mathrm{d}t \\ &- 2\sqrt{-1}\,\mathrm{Im}\int_t^T \langle \Psi^\delta, \operatorname{rot}\Phi^\delta\rangle\,\mathrm{d}t = \int_t^T \langle (f, g), (\Phi^\delta, \Psi^\delta)\rangle_{\mathcal{H}}\mathrm{d}t. \end{aligned}$$

It follows easily that

$$\|(\Phi^\delta, \Psi^\delta)\|_{L^\infty(0,T;\mathcal{H})}^2 + \frac{1}{\delta}\int_\Sigma |\nu \wedge \Phi^\delta|^2\mathrm{d}\Sigma \leq C\left\{\|(\Phi_0, \Psi_0)\|_{\mathcal{H}}^2 + \|(f, g)\|_{L^1(0,T;\mathcal{H})}^2\right\}.$$

The result now follows by density. □

By transposition, we have:

Theorem 2.1. *If* $(E_0, H_0) \in \mathcal{H}$ *and* $J \in \mathcal{L}_\tau^2(\Sigma)$*,* (1.2) *has a unique solution* $(E^\delta, H^\delta) \in C([0, T]; \mathcal{H})$.

Proof. By using Green's formula one finds that (E^δ, H^δ) formally satisfies

$$\begin{aligned} \langle (E^\delta(T), H^\delta(T)), (\Phi_0, \Psi_0)\rangle_{\mathcal{H}} \quad &- \int_{\mathcal{Q}} \langle (E^\delta, H^\delta), (f, g)\rangle_{\mathcal{H}}\mathrm{d}x\mathrm{d}t = \langle (E_0, H_0), (\Phi^\delta(0), \Psi^\delta(0))\rangle_{\mathcal{H}} \\ &- \int_\Sigma J \cdot \Psi_\tau^\delta\,\mathrm{d}\Sigma, \forall (\Phi_0, \Psi_0) \in \mathcal{H},\ \forall (f, g) \in L^1(0, T; \mathcal{H}), \end{aligned} \tag{2.4}$$

where $(\Phi^\delta, \Psi^\delta)$ is the solution of (2.3). From Lemma 2.2, for each $(E_0, H_0) \in \mathcal{H}$ and $J \in \mathcal{L}_\tau^2(\Sigma)$ the right side of (2.4) is a continuous linear functional on $\mathcal{H} \times L^1(0, T; \mathcal{H})$. Thus there are unique pairs $(E_T^\delta, H_T^\delta) \in \mathcal{H}$, $(E^\delta, H^\delta) \in L^\infty(0, T; \mathcal{H})$, such that

$$\begin{aligned} \langle (E_T^\delta, H_T^\delta), (\Phi_0, \Psi_0)\rangle_{\mathcal{H}} \quad &- \int_{\mathcal{Q}} \langle (E^\delta, H^\delta), (f, g)\rangle_{\mathcal{H}}\mathrm{d}x\mathrm{d}t = \langle (E_0, H_0), (\Phi^\delta(0), \Psi^\delta(0))\rangle_{\mathcal{H}} \\ &- \int_\Sigma J \cdot \Psi_\tau^\delta\,\mathrm{d}\Sigma, \forall (\Phi_0, \Psi_0) \in \mathcal{H},\ \forall (f, g) \in L^1(0, T; \mathcal{H}). \end{aligned}$$

By definition, we set $(E^\delta(T), H^\delta(T)) = (E^\delta_T, H^\delta_T)$. This is justified by the fact that it holds if $(E^\delta, H^\delta) \in C([0,T];\mathcal{H})$ rather than just $L^\infty(0,T;\mathcal{H})$ (*cf.* [9], Prop. 2.3). The passage from $L^\infty(0,T;\mathcal{H})$ to $C([0,T];\mathcal{H})$ is somewhat tedious but standard, and we omit this part of the argument which, in any case, is inessential to what follows. □

Lemma 2.3. *If $(E_0, H_0) \in \mathcal{H}$ and $J \in \mathcal{L}^2_\tau(\Sigma)$, the solution of (1.2) satisfies $\nu \wedge E^\delta|_\Sigma \in \mathcal{L}^2_\tau(\Sigma)$ and*

$$\|(E^\delta(t), H^\delta(t))\|^2_{\mathcal{H}} + \int_0^t \int_\Gamma \left(\frac{1}{\delta}|\nu \wedge E^\delta|^2 + \delta |H^\delta_\tau|^2 \right) \mathrm{d}\Gamma \mathrm{d}t = \|(E_0, H_0)\|^2_{\mathcal{H}} + \frac{1}{\delta}\int_0^t \int_\Gamma |J|^2 \mathrm{d}\Gamma \mathrm{d}t. \qquad (2.5)$$

Proof. First assume that (E_0, H_0) and J are such that $(E^\delta, H^\delta) \in C([0,T]; V \times V)$. This will hold if, for example $(E_0, H_0) \in V \times V$, $J = \nu \wedge \hat{J}|_\Sigma$, where $\hat{J} \in C^2([0,T];\mathcal{L}^2(\Omega)) \cap C^1([0,T];\mathcal{H}^1(\Omega))$, and $\nu \wedge E_0 - H_{0\tau} = J(0)$ on Γ. As may be seen by making the change of variables $\hat{E} = E^\delta - \hat{J}$, $\hat{H} = H^\delta$, one has $(\hat{E}, \hat{H}) \in C([0,T]; D(A))$ and therefore $(E^\delta, H^\delta) \in C([0,T]; V \times V)$. By calculating as in Lemma 2.2 we obtain

$$\frac{1}{2}\|(E^\delta(t), H^\delta(t))\|^2_{\mathcal{H}} + \frac{1}{\delta}\int_0^t \int_\Gamma |\nu \wedge E^\delta|^2 \mathrm{d}\Gamma \mathrm{d}t = \frac{1}{2}\|(E_0, H_0)\|^2_{\mathcal{H}} + \frac{1}{\delta} \mathrm{Re} \int_0^t \int_\Gamma J \cdot (\nu \wedge E^\delta)\, \mathrm{d}\Gamma \mathrm{d}t.$$

From the boundary condition we have

$$\mathrm{Re}\, J \cdot (\nu \wedge E^\delta) = \frac{1}{2}\big(|\nu \wedge E^\delta|^2 + |J|^2 - \delta^2 |H^\delta_\tau|^2\big),$$

which leads to (2.5). The result now follows by density. □

The solution of (1.1) is also defined by transposition. To that end, consider the system (2.3) with $\delta = 0$:

$$\begin{gathered} \begin{cases} \varepsilon \Phi_t - \mathrm{rot}\, \Psi = \varepsilon f \\ \mu \Psi_t + \mathrm{rot}\, \Phi = \mu g \quad \text{in } \mathcal{Q} \end{cases} \\ \nu \wedge \Phi = 0 \quad \text{on } \Sigma \\ \Phi(T) = \Phi_0, \quad \Psi(T) = \Psi_0 \text{ in } \Omega. \end{gathered} \qquad (2.6)$$

By Corollary 2.1, if $(\Phi_0, \Psi_0) \in \mathcal{H}$ and $(f, g) \in L^1(0,T;\mathcal{H})$, (2.4) has a unique solution $(\Phi, \Psi) \in C([0,T];\mathcal{H})$. Also, in this case the domain of the generator is

$$D(A) = \mathcal{R}^0 \times \mathcal{R},$$

where

$$\mathcal{R}^0 = \{\chi \in \mathcal{R} : \nu \wedge \chi|_\Gamma = 0\}\cdot$$

Thus, if $(\Phi_0, \Psi_0) \in \mathcal{R}^0 \times \mathcal{R}$ and $(f, g) \in L^1(0,T;\mathcal{R}^0 \times \mathcal{R})$, then $(\Phi, \Psi) \in C([0,T];\mathcal{R}^0 \times \mathcal{R})$ and

$$\|(\Phi, \Psi)\|_{L^\infty(0,T;\mathcal{R}^0 \times \mathcal{R})} \le C\big(\|(\Phi_0, \Psi_0)\|_{\mathcal{R}^0 \times \mathcal{R}} + \|(f, g)\|_{L^1(0,T;\mathcal{R}^0 \times \mathcal{R})}\big).$$

Set

$$\mathcal{D}_\mu = \{\chi \in \mathcal{L}^2(\Omega) : \mathrm{div}(\mu\chi) \in \mathcal{L}^2(\Omega)\}$$
$$\mathcal{D}^0_\mu = \{\chi \in \mathcal{D} : \nu \cdot (\mu\chi)|_\Gamma = 0\}\cdot$$

Remark 2.1. It is known that the map $\chi \mapsto \nu \cdot (\mu\chi) : \mathcal{C}^1(\Omega) \mapsto C^1(\Gamma)$ extends by continuity to a continuous linear mapping $\mathcal{D}_\mu \mapsto H^{-1/2}(\Gamma)$ ([2], Lem. VII.5.2).

Set

$$\mathcal{X} = \mathcal{R}^0 \times (\mathcal{R} \cap \mathcal{D}_\mu^0)$$

$$\begin{aligned} \|(\chi,\zeta)\|_{\mathcal{X}}^2 &= \int_\Omega (|\chi|^2 + |\zeta|^2 + |\operatorname{rot}\chi|^2 + |\operatorname{rot}\zeta|^2 + |\operatorname{div}(\mu\zeta)|^2)\,\mathrm{d}x \\ &= \|(\chi,\zeta)\|_{\mathcal{R}^0\times\mathcal{R}}^2 + \int_\Omega |\operatorname{div}(\mu\zeta)|^2 \mathrm{d}x. \end{aligned} \tag{2.7}$$

Lemma 2.4. *Assume that* $(\Phi_0, \Psi_0) \in \mathcal{X}$ *and* $(f,g) \in L^1(0,T;\mathcal{X})$. *Then* $(\Phi,\Psi) \in C([0,T];\mathcal{X})$ *and*

$$\|(\Phi,\Psi)\|_{L^\infty(0,T;\mathcal{X})} \le C\big(\|(\Phi_0,\Psi_0)\|_{\mathcal{X}} + \|(f,g)\|_{L^1(0,T;\mathcal{X})}\big). \tag{2.8}$$

In fact, from the second equation in (2.6)

$$\operatorname{div}(\mu\Psi(t)) = \operatorname{div}(\mu\Psi_0) - \int_t^T \operatorname{div}(\mu g(s))\mathrm{d}s$$

and

$$\nu \cdot (\mu\Psi(t))|_\Gamma = \nu \cdot (\mu\Psi_0)|_\Gamma - \int_t^T \nu \cdot (\mu g(s))|_\Gamma \mathrm{d}s = 0$$

since $\nu \cdot \operatorname{rot}\Phi|_\Gamma$ is a *tangential* differentiation on Γ of $\nu \wedge \Phi$ (see [2], p. 358).

Lemma 2.5. $\mathcal{R} \cap \mathcal{D}_\mu^0 \hookrightarrow \mathcal{H}^1(\Omega)$.

Proof. See Leis [11] (Th. 8.6). ☐

We identify $\mathcal{H}$ with its dual space, so that $\mathcal{X} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{X}'$. The scalar product in the $\mathcal{X}' - \mathcal{X}$ duality is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{X}}$. It follows from Lemmas 2.4 and 2.5 that for $(\Phi_0,\Psi_0) \in \mathcal{X}$ and $(f,g) \in L^1(0,T;\mathcal{X})$ the solution of (2.6) satisfies

$$\|(\Phi(0),\Psi(0))\|_{\mathcal{X}}^2 + \int_\Sigma |\Psi_\tau|^2 \le C\big(\|(\Phi_0,\Psi_0)\|_{\mathcal{X}} + \|(f,g)\|_{L^1(0,T;\mathcal{X})}\big). \tag{2.9}$$

By duality, we then have the following result:

Theorem 2.2. *If* $(E_0,H_0) \in \mathcal{X}'$ *and* $J \in \mathcal{L}_\tau^2(\Sigma)$, *(1.1) has a unique solution* $(E,H) \in C([0,T];\mathcal{X}')$ *defined by*

$$\langle (E(T),H(T)),(\Phi_0,\Psi_0)\rangle_{\mathcal{X}} - \int_Q \langle (E,H),(f,g)\rangle_{\mathcal{X}} \mathrm{d}x\mathrm{d}t = \langle (E_0,H_0),(\Phi(0),\Psi(0))\rangle_{\mathcal{X}} - \int_\Sigma J\cdot\Psi_\tau\,\mathrm{d}\Sigma \tag{2.10}$$

for all $(\Phi_0,\Psi_0) \in \mathcal{X}$ *and* $(f,g) \in L^1(0,T;\mathcal{X})$, *where* (Φ,Ψ) *is the solution of* (2.6).

Indeed, from (2.9) there exists unique $(E,H) \in L^\infty(0,T;\mathcal{X}')$, $(E(T),H(T)) \in \mathcal{X}'$ satisfying (2.10). One may use a lifting theorem of Lasiecka and Triggiani [10] to obtain $(E,H) \in \mathcal{C}([0,T];\mathcal{X}')$, and one may prove that the value (E,H) at $t = T$ is exactly $(E(T),H(T))$ (see [9], Prop. 2.3).

3. Proofs of Theorems 1.1–1.3

Proof of Theorem 1.1. Let $0 \leq \delta_1 < \delta_2$ and set

$$\Phi = \phi^{\delta_1} - \phi^{\delta_2}, \quad \Psi = \psi^{\delta_1} - \psi^{\delta_2}.$$

Assume that $(\phi_0, \psi_0) \in \mathcal{H}$ if $\delta_1 > 0$, else that $(\phi_0, \psi_0) \in \mathcal{F}$, where

$$\mathcal{F} = \text{completion of } \mathcal{X} \text{ in the norm } \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma$$

(ϕ, ψ) denoting the solution of (1.5). Then $\psi_\tau^{\delta_1}|_\Sigma \in \mathcal{L}_\tau^2(\Sigma)$ and (Φ, Ψ) satisfy

$$\begin{gathered}\begin{cases}\varepsilon \Phi_t - \operatorname{rot} \Psi = 0 \\ \mu \Psi_t + \operatorname{rot} \Phi = 0 \quad \text{in } \mathcal{Q}\end{cases} \\ \nu \wedge \Phi + \delta_2 \Psi_\tau = (\delta_2 - \delta_1)\psi_\tau^{\delta_1} \text{ on } \Sigma \\ \Phi(T) = \Psi(T) = 0 \text{ in } \Omega.\end{gathered}$$

From Lemma 2.3 we have

$$\|(\Phi(0), \Psi(0))\|_{\mathcal{H}}^2 + \int_\Sigma \left(\frac{1}{\delta_2}|\nu \wedge \Phi|^2 + \delta_2 |\Psi_\tau|^2\right) \mathrm{d}\Sigma = \frac{(\delta_2 - \delta_1)^2}{\delta_2} \int_\Sigma |\psi_\tau^{\delta_1}|^2 \mathrm{d}\Sigma.$$

On Σ we have

$$\psi_\tau^{\delta_2} = -\Psi_\tau + \psi_\tau^{\delta_1} = \frac{1}{\delta_2}\nu \wedge \Phi - \frac{\delta_2 - \delta_1}{\delta_2}\psi_\tau^{\delta_1} + \psi_\tau^{\delta_1} = \frac{1}{\delta_2}\nu \wedge \Phi + \frac{\delta_1}{\delta_2}\psi_\tau^{\delta_1}.$$

Therefore

$$\|\psi_\tau^{\delta_2}\|_{\mathcal{L}_\tau^2(\Sigma)} \leq \frac{1}{\delta_2}\|\nu \wedge \Phi\|_{\mathcal{L}_\tau^2(\Sigma)} + \frac{\delta_1}{\delta_2}\|\psi_\tau^{\delta_1}\|_{\mathcal{L}_\tau^2(\Sigma)} \leq \frac{\delta_2 - \delta_1}{\delta_2}\|\psi_\tau^{\delta^1}\|_{\mathcal{L}_\tau^2(\Sigma)} + \frac{\delta_1}{\delta_2}\|\psi_\tau^{\delta_1}\|_{\mathcal{L}_\tau^2(\Sigma)} = \|\psi_\tau^{\delta_1}\|_{\mathcal{L}_\tau^2(\Sigma)}.$$

Proof of Theorem 1.2. For $0 < \delta \leq \delta_0$ we have

$$\|(\phi_0^\delta, \psi_0^\delta)\|_{\mathcal{H}}^2 \leq C_T^{\delta_0} \int_\Sigma |\psi_\tau^\delta|^2 \mathrm{d}\Sigma. \tag{3.1}$$

Thus, for $\alpha > 0$,

$$\begin{aligned}\int_\Sigma |\psi_\tau^\delta|^2 \mathrm{d}\Sigma &\leq \|(E_1, H_1)\|_{\mathcal{H}} \|(\phi_0^\delta, \psi_0^\delta)\|_{\mathcal{H}} \leq \frac{1}{2\alpha}\|(E_1, H_1)\|_{\mathcal{H}}^2 + \frac{\alpha}{2}\|(\phi_0^\delta, \psi_0^\delta)\|_{\mathcal{H}}^2 \\ &\leq \frac{1}{2\alpha}\|(E_1, H_1)\|_{\mathcal{H}}^2 + \frac{\alpha}{2} C_T^{\delta_0} \int_\Sigma |\psi_\tau^\delta|^2 \mathrm{d}\Sigma.\end{aligned}$$

By choosing α sufficiently small it follows that

$$\int_\Sigma |\psi_\tau^\delta|^2 \mathrm{d}\Sigma \leq C\|(E_1, H_1)\|_{\mathcal{H}}^2. \tag{3.2}$$

Since $\|(\phi^\delta(t),\psi^\delta(t))\|_{\mathcal{H}} \le \|(\phi_0^\delta,\psi_0^\delta)\|_{\mathcal{H}}$ we then obtain

$$\begin{aligned}(\phi^\delta(\cdot),\psi^\delta(\cdot)) &\text{ is bounded in } L^\infty(0,T;\mathcal{H})\\ (\phi_0^\delta,\psi_0^\delta) &\text{ is bounded in } \mathcal{H},\\ \psi_\tau^\delta|_\Sigma &\text{ is bounded in } \mathcal{L}_\tau^2(\Sigma).\end{aligned}$$

Thus, on a sequence $\delta=\delta_n$ tending towards zero we have

$$\begin{aligned}(\phi^\delta(\cdot),\psi^\delta(\cdot)) &\to (\phi(\cdot),\psi(\cdot)) \text{ weakly* in } L^\infty(0,T;\mathcal{H})\\ (\phi_0^\delta,\psi_0^\delta) &\to (\phi_0,\psi_0) \text{ weakly in } \mathcal{H}\\ \psi_\tau^\delta|_\Sigma &\to g \text{ weakly in } \mathcal{L}_\tau^2(\Sigma),\end{aligned} \tag{3.3}$$

for some $(\phi_0,\psi_0)\in\mathcal{H}_0$ and $g\in\mathcal{L}_\tau^2(\Sigma)$.

Let $(\chi,\zeta)\in C^\infty(\overline{\Omega}\times[0,T])$ such that $\chi(0)=\zeta(0)=0$. We have

$$\begin{aligned}0 \;=\; &\int_0^T (\langle\varepsilon\phi_t^\delta-\operatorname{rot}\psi^\delta,\chi\rangle+\langle\mu\psi_t^\delta+\operatorname{rot}\phi^\delta,\zeta\rangle)\,\mathrm{d}t = -\int_0^T (\langle\phi^\delta,\varepsilon\chi_t-\operatorname{rot}\zeta\rangle+\langle\psi^\delta,\mu\zeta_t+\operatorname{rot}\chi\rangle]\,\mathrm{d}t\\ &+\langle(\phi_0^\delta,\psi_0^\delta),(\chi(T),\zeta(T))\rangle_{\mathcal{H}}+\int_\Sigma[\psi_\tau^\delta\cdot(\nu\wedge\chi)+(\nu\wedge\phi^\delta)\cdot\zeta_\tau]\,\mathrm{d}\Sigma.\end{aligned}$$

Upon passing to the limit through $\delta=\delta_n$ we obtain

$$\int_0^T[\langle\phi,\varepsilon\chi'-\operatorname{rot}\zeta\rangle+\langle\psi,\mu\zeta'+\operatorname{rot}\chi\rangle]\,\mathrm{d}t=\langle(\phi_0,\psi_0),(\chi(T),\zeta(T))\rangle_{\mathcal{H}}+\int_\Sigma g\cdot(\nu\wedge\chi)\,\mathrm{d}\Sigma,$$

It follows that (ϕ,ψ) satisfy

$$\begin{gathered}\begin{cases}\varepsilon\phi_t-\operatorname{rot}\psi=0\\ \mu\psi_t+\operatorname{rot}\phi=0 \quad \text{in } \mathcal{Q}\end{cases}\\ \nu\wedge\phi=0 \text{ on } \Sigma\\ \phi(T)=\phi_0,\quad \psi(T)=\psi_0 \text{ in } \Omega\end{gathered}$$

and that $\psi_\tau|_\Sigma=g$. Thus $(\phi_0,\psi_0)\in\mathcal{F}_0:=\mathcal{F}\cap\mathcal{H}_0$, and from (1.15) we have

$$\langle(E_1,H_1),(\phi_0,\psi_0)\rangle_{\mathcal{H}}=\lim_{n\to\infty}\int_\Sigma|\psi_\tau^{\delta_n}|^2\mathrm{d}\Sigma\ge\int_\Sigma|\psi_\tau|^2\mathrm{d}\Sigma. \tag{3.4}$$

Now consider the system

$$\begin{gathered}\begin{cases}\varepsilon\Phi_t-\operatorname{rot}\Psi=0\\ \mu\Psi_t+\operatorname{rot}\Phi=0 \quad \text{in } \mathcal{Q}\end{cases}\\ \nu\wedge\Phi=0 \text{ on } \Sigma\\ \Phi(T)=\Phi_0,\quad \Psi(T)=\Psi_0 \text{ in } \Omega,\end{gathered} \tag{3.5}$$

where $(\Phi_0,\Psi_0)\in\mathcal{F}_0$. Corollary 1.1 implies that the problem (1.1, 1.3) with $E_0=H_0=0$ has, for any $(E_1,H_1)\in\mathcal{H}_0$ a solution $J\in\mathcal{L}_\tau^2(\Sigma)$. Further, the control J of minimum $\mathcal{L}_\tau^2(\Sigma)$ norm that steers $(0,0)$ to (E_1,H_1) in time T is given by

$$J=-\Psi_\tau|_\Sigma,$$

where (Φ, Ψ) is the solution of (3.5) with final data given by

$$\langle (E_1, H_1), (\Phi_0, \Psi_0)\rangle_{\mathcal{F}_0} = \int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma = \|(\Phi_0, \Psi_0)\|^2_{\mathcal{F}_0}. \tag{3.6}$$

From (3.6) we have

$$(E_1, H_1) = \Lambda_0(\Phi_0, \Psi_0)$$

where Λ_0 is the Riesz isomorphism of $\mathcal{F}_0$ onto $\mathcal{F}_0{}'$ and $\mathcal{F}_0'$ is the dual space of $\mathcal{F}_0$ with respect to $\mathcal{H}_0$, and

$$\|(E_1, H_1)\|_{\mathcal{F}_0{}'} = \|(\Phi_0, \Psi_0)\|_{\mathcal{F}_0} = \left\{\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma\right\}^{1/2}.$$

We now show that

$$\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma = \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma. \tag{3.7}$$

From (3.4) we have

$$\int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma \leq \langle (E_1, H_1), (\phi_0, \psi_0)\rangle_{\mathcal{H}} \leq \|(E_1, H_1)\|_{\mathcal{F}_0{}'} \|(\phi_0, \psi_0)\|_{\mathcal{F}_0} \leq \left\{\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma\right\}^{1/2} \left\{\int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma\right\}^{1/2}$$

and therefore

$$\int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma \leq \int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma.$$

On the other hand, $J = -\Psi_\tau|_\Sigma$ is the control of minimum $\mathcal{L}^2_\tau(\Sigma)$ norm such that the solution of (1.1) with $E_0 = H_0 = 0$ satisfies (1.3), while $\delta H^\delta_\tau + \psi^\delta_\tau|_\Sigma$ is another $\mathcal{L}^2_\tau(\Sigma)$ control that has this property. Thus

$$\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma \leq \int_\Sigma |\delta H^\delta_\tau + \psi^\delta_\tau|^2 \mathrm{d}\Sigma = \int_\Sigma |\nu \wedge E^\delta|^2 \mathrm{d}\Sigma \leq \int_\Sigma |\psi^\delta_\tau|^2 \mathrm{d}\Sigma = \langle (E_1, H_1), (\phi^\delta_0, \psi^\delta_0)\rangle_{\mathcal{H}},$$

where we have used Lemma 2.3 and (1.15). Upon passing to the limit through $\delta = \delta_n$ we obtain

$$\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma \leq \langle (E_1, H_1), (\phi_0, \psi_0)\rangle_{\mathcal{H}} \leq \left\{\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma\right\}^{1/2} \left\{\int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma\right\}^{1/2},$$

hence

$$\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma \leq \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma,$$

which proves (3.7).

It follows from (3.7) that

$$\langle (E_1, H_1), (\phi_0, \psi_0)\rangle_{\mathcal{H}} = \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma. \tag{3.8}$$

Indeed, if (3.4) holds with strict inequality, we immediately see that

$$\int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma < \int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma.$$

It now follows that

$$(\phi_0, \psi_0) = \Lambda_0^{-1}(E_1, H_1) = (\Phi_0, \Psi_0),$$

that the convergence in (3.3) is through all $\delta \to 0$, and that

$$\lim_{\delta\to 0} \int_\Sigma |\psi_\tau^\delta|^2 \mathrm{d}\Sigma = \int_\Sigma |\psi_\tau|^2 \mathrm{d}\Sigma.$$

Since also $\psi_\tau^\delta|_\Sigma \to \psi_\tau|_\Sigma$ weakly in $\mathcal{L}_\tau^2(\Sigma)$, it follows that the convergence is in the strong topology as well. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. From the definitions (2.10) and (2.4) of the solutions of (1.1) and (1.2) with $E_0 = H_0 = 0$ and with $J = -\psi_\tau|_\Sigma$ and $J = -\psi_\tau^\delta|_\Sigma$, respectively, we obtain

$$\int_0^T \langle (E^\delta - E, H^\delta - H), (f,g)\rangle_{\mathcal{X}} \mathrm{d}t = \int_\Sigma [\psi_\tau \cdot \Psi_\tau - \psi_\tau^\delta \cdot \Psi_\tau^\delta]\, \mathrm{d}\Sigma, \forall (f,g) \in L^1(0,T;\mathcal{X}),\ \forall (\Phi_0, \Psi_0) \in \mathcal{X},$$

where $(\Phi^\delta, \Psi^\delta)$ and (Φ, Ψ) are the solutions of (2.3) and (2.6), respectively, and $\langle\cdot,\cdot\rangle_{\mathcal{X}}$ denotes the scalar product in the $\mathcal{X}' - \mathcal{X}$ duality. To complete the proof it suffices to show that

$$\lim_{\delta\to 0} \int_\Sigma \psi_\tau^\delta \cdot \Psi_\tau^\delta \, \mathrm{d}\Sigma = \int_\Sigma \psi_\tau \cdot \Psi_\tau \, \mathrm{d}\Sigma.$$

Since $\psi_\tau^\delta|_\Sigma \to \psi_\tau|_\Sigma$ strongly in $\mathcal{L}_\tau^2(\Sigma)$, it is sufficient to prove that $\Psi_\tau^\delta|_\Sigma \to \Psi_\tau|_\Sigma$ weakly in $\mathcal{L}_\tau^2(\Sigma)$. In fact, we shall show convergence even in the strong topology. Set

$$\hat\Phi^\delta = \Phi^\delta - \Phi, \quad \hat\Psi^\delta = \Psi^\delta - \Psi.$$

These satisfy

$$\begin{gathered}\begin{cases}\varepsilon\hat\Phi_t^\delta - \operatorname{rot}\hat\Psi^\delta = 0 \\ \mu\hat\Psi_t^\delta + \operatorname{rot}\hat\Phi^\delta = 0 \quad \text{in } \mathcal{Q}\end{cases} \\ \nu\wedge\hat\Phi^\delta + \delta\hat\Psi_\tau^\delta = -\delta\Psi_\tau \text{ on } \Sigma \\ \hat\Phi^\delta(T) = \hat\Psi^\delta(T) = 0 \text{ in } \Omega.\end{gathered}$$

Because of our hypotheses on the data, we have $\Psi_\tau|_\Sigma \in \mathcal{L}_\tau^2(\Sigma)$. We apply Lemma 2.3 to this system to obtain

$$\delta\|(\hat\Phi^\delta(t), \hat\Psi^\delta(t))\|_{\mathcal{H}}^2 + \int_t^T \int_\Gamma \{|\nu\wedge\hat\Phi^\delta|^2 + \delta^2|\hat\Psi_\tau^\delta|^2\}\mathrm{d}\Gamma\mathrm{d}t = \int_t^T \int_\Gamma \delta^2|\Psi_\tau|^2 \mathrm{d}\Gamma\mathrm{d}t.$$

It follows that $\hat\Psi_\tau^\delta|_\Sigma$, and hence $\Psi_\tau^\delta|_\Sigma$, is bounded in $\mathcal{L}_\tau^2(\Sigma)$, that

$$(\Phi^\delta, \Psi^\delta) \to (\Phi, \Psi) \text{ strongly in } L^\infty(0,T;\mathcal{H}),$$

and that $\Psi_\tau^\delta|_\Sigma \to h$ weakly in $\mathcal{L}_\tau^2(\Sigma)$ through a sequence of δ's tending to zero where, in fact, $h = \Psi_\tau|_\Sigma$ (see the argument in the proof of Th. 1.2). In addition,

$$\int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma \leq \liminf_{\delta\to 0} \int_\Sigma |\Psi_\tau^\delta|^2 \leq \int_\Sigma |\Psi_\tau|^2 \mathrm{d}\Sigma,$$

where the last inequality is from Theorem 1.1. Hence $\Psi_\tau^\delta|_\Sigma \to \Psi_\tau|_\Sigma$ strongly in $\mathcal{L}_\tau^2(\Sigma)$.

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