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CONTROL NORMS FOR LARGE CONTROL TIMES*

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Abstract. A control system of the second order in time with control $u = u(t) \in L^2([0,T];U)$ is considered. If the system is controllable in a strong sense and u^T is the control steering the system to the rest at time T, then the L^2–norm of u^T decreases as $1/\sqrt{T}$ while the $L^1([0,T];U)$–norm of u^T is approximately constant. The proof is based on the moment approach and properties of the relevant exponential family. Results are applied to the wave equation with boundary or distributed controls.

Résumé. On considère un système du second ordre à contrôler en temps avec une fonction $u = u(t) \in L^2([0,T];U)$. Si le système est contrôlable au sens fort et u^T est la fonction de contrôle qui gouverne le système à partir du temps T, alors la norme L^2 de u^T décroît comme $1/\sqrt{T}$ tandis que sa norme $L^1([0,T];U)$ est approximativement constante. La preuve est basée sur la théorie des moments et les propriétés d'une famille appropriée d'exponentielles. Ces résultats sont ensuite appliqués aux équations d'onde avec contrôle au bord et contrôle distribué.

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1. Introduction

In numerical experiments of Glowinski *et al.* [7] and Asch and Lebeau [1], a regular behavior of the norms of the control u^T, steering the wave equation to the rest at time T, has been observed. To be more specific, for control systems governed by hyperbolic type equations the L^2–norm of u^T decreases as $1/\sqrt{T}$ and the L^1–norm is approximately constant. In [1] this effect has been numerically shown for control systems with various domains and control subsets of the boundary, both for exactly and approximately controllable systems. Dimensionality arguments have been discussed and an example of a controlled string has been also considered. The first result in this direction, namely the estimate $1/\sqrt{T}$ of the L^2–norm of u^T for the boundary control of a square plate, has been obtained by Krabs *et al.* in [10]; see also [9].

The main goal of this paper is to prove this behavior of control systems including, in particular, the systems considered in the papers cited above.

Let us consider a control system

$$\ddot{y} + Ay = Bu(t), \quad y(0) = y_0, \ \dot{y}(0) = y_1,$$

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where A is a selfadjoint operator with eigenfunctions φ_n and positive eigenvalues λ_n, $u \in L^2(0,T;U)$ is a control (for a fixed time the control belongs to the control space U), B is an operator acting from U.

We assume that the control system possesses a "perfect controllability" property: each initial state (y_0, y_1) from the appropriate state space may be steered to the rest at time $t = T > T_0$ by a control u^T and the map

$$(y_0, y_1) \mapsto u^T$$

is an isomorphism of the state space on its image. [We always denote by u^T the control with the minimal norm.] In this case we show that for $T > T_0$

$$\left\|u^T\right\|_{L^2(0,T;U)} \asymp \frac{1}{\sqrt{T}}, \tag{1.1}$$

$$\left\|u^T\right\|_{L^1(0,T;U)} \asymp 1. \tag{1.2}$$

Here and in what follows the sign $\asymp$ means that the ratio of the two sides lies between two positive constants.

The setting in which we can prove the same behavior (1.1, 1.2) assumes the initial state is a linear (finite) combination of eigenmodes and each eigenmode can be steered to rest at $t = T$. This type of controllability is called *spectral* or *M–controllability.*

In order to study norms of controls, we apply the moment approach, which reduces the control problem to a moment problem in $L^2(0,T;U)$ relative to an exponential family $\mathcal{E}$. This family has the form $\mathcal{E} = \{e_n\} = \{\eta_n e^{\pm i\omega_n t}\}$, where ω_n is an eigenfrequency of A, $\omega_n^2 = \lambda_n$, and η_n is an element of the control space U. For the case of the wave equation in a domain Ω with Dirichlet boundary control on $\Gamma \subset \partial\Omega$ we have

$$\eta_n = \partial_\nu \varphi_n|_\Gamma,$$

where ∂_ν is the normal derivative.

The proof of (1.1, 1.2) is based on deep relations between the types of controllability and properties of the exponential family $\mathcal{E}$. We refer the interested readers to the book of Avdonin and Ivanov [2]. In particular,

B–controllability is equivalent to $\mathcal{E}$ being a Riesz basis in the closure of its linear span in $L^2(0,T;U)$, i.e., for $\{c_k\} \in \ell^2$ we have $\|\sum c_k e_k\|^2 \asymp \sum |c_k|^2$.

M–controllability is equivalent to the minimal property of $\mathcal{E}$, i.e., no element $e_k \in \mathcal{E}$ belongs to the closed linear span of the others.

The main idea of the proof is that in both cases we can write the control u^T as a *converging* series (or a finite sum) in elements of the family $\Theta^T = \left\{\theta_k^T\right\}$, biorthogonal to $\mathcal{E}$ in $L^2(0,T;U)$. This representation is more explicit than that given by the Hilbert Uniqueness Method (HUM) (see [1, 7, 12]), and enables us to study u^T directly. The point is: elements of Θ^{NT} are simply related to elements of Θ^T. This fact was first observed in [10] for the control acting on one side of a homogeneous square plate. In this example the exponentials form an orthogonal family when the time is a multiple of $l/2\pi$ (l is the length of the side).

We thus can prove (1.1, 1.2) for abstract control systems in the context B– and M–controllability.

Let us consider now systems governed by Partial Differential Equation (PDE). To use the results obtained for the abstract control system, we have to prove that the system under question is B– or M–controllable. If we apply the moment approach, we need to study the basis or minimality properties of the corresponding exponential families.

Exponential families are well studied [2], Chapter II, in scalar and finite dimensional cases ($\dim U < \infty$). In these cases controls are scalar or finite dimensional functions of t. For systems governed by PDE with several spatial variables, such controls are too weak to have B– or even M–controllability [2], Chapter V.

There are no general results concerning exponential families with infinite dimensional U and, therefore, the moment approach does not work well is these cases. Nevertheless we have the estimates [1, 2], if we are able to prove controllability using other approaches. In this paper, we apply the "geometrical" conditions for the controllability of wave equations, obtained by Bardos *et al.* [5].

Remark 1.1. For parabolic equations, norms of the control u^T decrease exponentially as $T \to \infty$. More interesting question is the behavior of the $\|u^T\|$ as $T \to 0$, see papers [15, 16] and [8].

The rest of the paper falls into two parts. In Section 2 we present general results for abstract control systems and prove the theorem describing the norms of controls. We also prove that u^T is continuous as a function of T:

$$\left\|u^{T_1} - u^{T_2}\right\|_{L^2(0,\infty;U)} \underset{T_1 \to T_2}{\to} 0.$$

Note that the regularity of $\|u^T\|_{L^p(0,T;U)}$ with respect to T and the initial state is studied in [8] in a more general situation. There, in particular, continuity of this norm was proved.

In Section 3 we apply these results to specific systems governed by hyperbolic type equations. We consider boundary and distributed controls assuming B– or M–controllability. A simple example is also given, in which the control system is not controllable in the weakest sense: the union $\underset{T>0}{\cup} R^T$ of the reachable sets R^T is not dense in the state space. In this case u^T may not depend on T for large T at all, that is $u^T = u^{T_0}$ for some T_0.

2. Abstract control systems

2.1. Control systems and exponential families

In this section we present the moment approach to abstract control systems. This approach is presented in detail in [2].

Let H be a Hilbert space, A be a positive definite selfadjoint operator with discrete spectrum $\{\lambda_n\}_{n=1}^{\infty}$ and orthonormalized eigenbasis $\{\varphi_n\}_{n=1}^{\infty}$. For $r \geq 0$, we introduce the spaces $W_r := D(A^{r/2})$. It is easy to see that

$$W_r = \left\{ f = \sum c_n \varphi_n \,\middle|\, \|f\| := \sum |c_n|^2 \lambda_n^r < \infty \right\}.$$

We retain this definition for $r < 0$.

Let U be a Hilbert space and B be a bounded operator from U into W_r. We introduce the control system

$$\begin{cases} \ddot{y} + Ay = Bu(t), \\ y(0) = y_0 \in W_{r+1},\ \dot{y}(0) = y_1 \in W_r, \end{cases} \tag{2.1}$$

where u belongs to the control space $\mathcal{U}^T := L^2(0,T;U)$. To make the results of this paper clear, we describe here how the control problem can be reduced to a moment problem in the Hilbert space $\mathcal{U}^T := L^2(0,T;U)$. Using the Fourier method we find the solution of (2.1) in the form

$$y(T) = \sum_{n=1}^{\infty} y_n(T)\varphi_n.$$

The initial data may be written as series in φ_n

$$y_0(x) = \sum_{n=1}^{\infty} y_n^0 \varphi_n(x), \quad y_1(x) = \sum_{n=1}^{\infty} y_n^1 \varphi_n(x),$$

and the inclusions $y_0 \in W_{r+1}$, $y_1 \in W_r$, imply

$$\sum_{n=1}^{\infty} |y_n^0|^2 \omega_n^{2(r+1)} < \infty, \quad \sum_{n=1}^{\infty} |y_n^1|^2 \omega_n^{2r} < \infty, \tag{2.2}$$

where $\omega_n := \sqrt{\lambda_n}$. Denote by $b_n(t)$ the Fourier coefficients of $Bu(t) \in W_r$. It is well known (see, *e.g.* [2]) that for the coefficients $y_n(T)$ we get

$$y_n(T) = y_n^0 \cos\omega_n T + \frac{1}{\omega_n} y_n^1 \sin\omega_n T + \int_0^T \frac{1}{\omega_n} \sin\omega_n(T-t) b_n(t)dt, \tag{2.3}$$

$$\dot{y}_n(T) = -y_n^0 \omega_n \sin\omega_n T + y_n^1 \cos\omega_n T + \int_0^T \cos\omega_n(T-t) b_n(t)dt. \tag{2.4}$$

Crude estimates of the sine and cosine functions in (2.3, 2.4) lead to

Proposition 2.1. *[2] For every initial data $(y_0, y_1) \in W_{r+1} \oplus W_r$ and $u \in \mathcal{U}^T$ there is the unique solution of (2.1) such that $(y, \dot{y}) \in C([0,T]; W_{r+1} \oplus W_r)$.*

This proposition is sharp in general, *i.e.*, for an arbitrary discrete set $\{\lambda_n\} \subset \mathbb{R}_+$ and arbitrary B. But solutions of the wave equation turn out to be smoother than it is provided by this proposition, see Section 3 below. Keeping this in mind, we will assume that Proposition 2.1 is true for a smoother space $W_{s+1} \oplus W_s$, with $s \geq r$:

$$(y(T), \dot{y}(T)) \in W_{s+1} \oplus W_s. \tag{2.5}$$

Let us introduce the exponential family $\mathcal{E} := \{e_k\}_{k\in\mathbb{K}}$, where $\mathbb{K} := \{\pm 1, \pm 2, \dots\}$ and

$$e_k(t) := |\omega_k|^s B^* \varphi_{|k|} e^{i\omega_k t},$$

with $\omega_k = -\omega_{|k|}$, for $k < 0$. By $\mathcal{E}^T = \left\{e_k^T\right\}$ we denote this family restricted to the interval $[0,T]$ and extended by zero for $t > T$, that is $\bigvee \mathcal{E}^T \subset \mathcal{U}^T$. Set

$$c_k^0 := i \ \mathrm{sign}\, k |\omega_k|^{s+1} y_{|k|}^0 + |\omega_k|^s y_{|k|}^1. \tag{2.6}$$

and

$$c_k(T) := i \ \mathrm{sign}\, k |\omega_k|^{s+1} y_{|k|}(T) + |\omega_k|^s \dot{y}_{|k|}(T).$$

Combining (2.2) and assumption (2.5) it follows that $c^0 = \left\{c_k^0\right\}$ and $c = \{c_k\}$ belong both to ℓ^2. We rewrite the integrals in (2.3, 2.4) as

$$\int_0^T \left(u(t), \frac{1}{\omega_n} \sin\omega_n(T-t) B^*\varphi_n\right)_U dt,$$

and

$$\int_0^T (u(t), \cos\omega_n(T-t) B^*\varphi_n)_U \, dt,$$

where $(\cdot,\cdot)_U$ is the inner product in U. Then, multiplying (2.3) by $\pm i\omega_n^{s+1}$, multiplying (2.4) by ω_n^s, and adding, we obtain

$$c_k(T) = c_k^0 e^{i\omega_k T} + \int_0^T (u(t), e_{-k}(T-t))_U \, dt.$$

Thus, for the control u^T driving the system to the zero state in time $t = T$, we have a moment problem in the space $\mathcal{U}^T$: *given $\left\{c_k^0\right\} \in \ell^2$ find $u^T \in \bigvee \mathcal{E}^T$ such that*

$$\widetilde{c}_k^0 := -c_k^0 = \left(u^T, e_k\right)_{L^2(0,T;U)}, \quad k \in \mathbb{K}. \tag{2.7}$$

We relate the moment operator

$$J^T : \mathcal{U}^T \mapsto \ell^2, \ J^T f = \{(f, e_k)_{\mathcal{U}^T}\}_k \,,$$

to the moment problem. We denote the restriction of J^T to $\bigvee \mathcal{E}^T$ by J_0^T. In contrast to J^T, the latter operator is invertible for any elements $\{e_k\}$.

Let us suppose that the family $\mathcal{E}^T$ is minimal in $\mathcal{U}^T$ and $\Theta^T := \left\{\theta_k^T\right\}$ is the (unique) biorthogonal family belonging to $\bigvee \mathcal{E}^T$:

$$\left(\theta_k^T, e_p\right)_{\mathcal{U}^T} = \delta_p^k.$$

Then the formal solution of (2.7) has the form

$$u^T = \sum_k \widetilde{c}_k^0 \theta_k^T. \tag{2.8}$$

If this series converges weakly, then its limit u^T is the control steering the system to the rest in time T and this u^T has the minimal norm among such controls. Thus (2.8) coincides with the control obtained by the HUM [1,12].

Let us introduce the notions of B, M of controllability [2] for system (2.1). It is convenient to formulate the definitions in terms of the reachable set R^T (the set of all states $(y(T), \dot{y}(T))$ for the zero initial data and all controls).

Definition 2.2. System (2.1) is called B–controllable in time T relative to $W_{s+1} \oplus W_s$ if R^T coincides with $W_{s+1} \oplus W_s$.

Definition 2.3. System (2.1) is called M–controllable (spectral controllable) in time T, if R^T contains the eigenmodes $(y_0, y_1) = (\frac{1}{i\omega_n}\varphi_n, \pm\varphi_n)$, for all $n \in \mathbb{N}$.

Remark 2.4. It is easy to see that for B-controllable systems the map

$$(y_0, y_1) \mapsto u^T$$

is a bounded and boundedly invertible operator (an isomorphism) from $W_{s+1} \oplus W_s$ onto $\bigvee \mathcal{E}^T$.

The equalities (2.7) establish an isomorphism between those states in $W_{s+1} \oplus W_s$ which can be steered to zero, and the set of sequences $\left\{c_k^0\right\}$ in ℓ^2, for which the moment problem (2.7) can be solved. The study of the abstract moment problem leads to the relationship between the controllability of the control system and the geometrical properties (types of "linear independence") of the exponential family.

Proposition 2.5. *[2] (i) System (2.1) is B–controllable in $W_{s+1} \oplus W_s$ in time T, if and only if the exponential family $\mathcal{E}^T$ forms a Riesz basis in the closure of its span.*

(ii) System (2.1) is M–controllable in time T, if and only if the exponential family $\mathcal{E}^T$ is minimal in $\mathcal{U}^T$.

In terms of the moment operator, Proposition 2.5 (i) means that $J_0^{T_0}$ is an isomorphism between the spaces $\bigvee \mathcal{E}^{T_0}$ and ℓ^2 if and only if the control system is B–controllable. The second statement means that the range of $J_0^{T_0}$ contains all finite sequences, if and only if the control system is M–controllable.

2.2. Norms of u^T

Theorem 2.6. *Assume that control system (2.1) is in the state (y_0, y_1) at time $t = 0$ and that one of the following two conditions is fulfilled:*

(i) *the system is B–controllable in the space $W_{s+1} \oplus W_s$ in time T_0 and $(y_0, y_1) \in W_{s+1} \oplus W_s$,*
(ii) *the system is M–controllable in time T_0 and the initial state is a linear (finite) combination of the eigenmodes $\left(\frac{1}{i\omega_n}\varphi_n, \pm\varphi_n\right)$.*

Let u^T be the control with the minimal L^2–norm steering the system to the rest at $t = T > T_0$.

Then
(i) the L^2–norm $\|u^T\|_{L^2(0,T;U)}$ is a nonincreasing function of T with

$$\|u^T\|_{L^2(0,T;U)} \asymp 1/\sqrt{T}, \tag{2.9}$$

(ii) the L^1–norm $\|u^T\|_{L^1(0,T;U)}$ is bounded from above and away from zero:

$$\|u^T\|_{L^1(0,T;U)} \asymp 1. \tag{2.10}$$

In the proof we use an explicit expression for the biorthogonal family Θ^T in terms of Θ^{T_0} for times $T = NT_0$, multiples of T_0.

Lemma 2.7. *(i) The family $\mathcal{E}^{[\alpha,\alpha+T_0]} := \{e_k|_{[\alpha,\alpha+T_0]}\}_{k\in\mathbb{K}}$ is minimal in the space $L^2(\alpha,\alpha+T_0;U)$ and the biorthogonal family $\Theta^{[\alpha,\alpha+T_0]} = \left\{\theta_k^{[\alpha,\alpha+T_0]}\right\}_{k\in\mathbb{K}}$ belonging to $\bigvee \mathcal{E}^{[\alpha,\alpha+T_0]}$ has the form*

$$\theta_k^{[\alpha,\alpha+T_0]}(t) = e^{i\omega_k\alpha}\theta_k^{T_0}(t-\alpha).$$

(ii) The family $\Theta^{NT_0} = \left\{\theta_k^{NT_0}\right\}_{k\in\mathbb{K}}$, biorthogonal to $\mathcal{E}^{NT_0}$ and belonging to $\bigvee \mathcal{E}^{NT_0}$, has the form

$$\theta_k^{NT_0} = \frac{1}{N}\begin{cases} \theta_k^{T_0}(t), & t\in[0,T_0],\\ e^{i\omega_k T_0}\theta_k^{T_0}(t-T_0), & t\in[T_0,2T_0],\\ \dots & \dots\\ e^{i\omega_k (N-1)T_0}\theta_k^{T_0}(t-(N-1)T_0), & t\in[(N-1)T_0,NT_0].\end{cases}$$

Proof of the lemma. The assertion (i) is immediately verified:

$$\begin{aligned}\left(e_k, e^{i\omega_p\alpha}\theta_p^{T_0}(\cdot-\alpha)\right)_{L^2(\alpha,\alpha+T;U)} &= \int_\alpha^{\alpha+T} e^{i\omega_k t}e^{-i\omega_p\alpha}\left(B^*\varphi_k,\theta_p^{T_0}(t-\alpha)\right)_U dt\\ &= e^{i(\omega_k-\omega_p)\alpha}\left(e_k,\theta_p^{T_0}\right)_{L^2(0,T_0;U)} = \delta_p^k.\end{aligned}$$

Since $\theta_p^T \in \bigvee_{L^2(0,T_0;U)} e_k$, we have $\theta_p^{[\alpha,\alpha+T_0]} \in \bigvee_{L^2(\alpha,\alpha+T_0;U)} e_k$.

Now (ii) follows from (i) by direct calculations. □

Proof. We continue the proof of the theorem. To illustrate the main idea, we start with the simplest case when the initial state is an eigenmode:

$$y_0 = -\frac{1}{2i\omega_n}\varphi_{|k|}, \quad y_1 = -\frac{1}{2}\varphi_{|k|}.$$

For such data the sequence c^0 has only one nonzero coefficient $c_k^0 = -1$. Consequently, the control u^{NT_0} is just $\theta_k^{NT_0}$ and from Lemma 2.7 (ii) we conclude that for $T = NT_0$

$$\begin{aligned}\|u^T\|^2_{L^2(0,T;U)} &= \left\|\theta_k^{NT_0}\right\|^2_{L^2(0,T;U)} = \sum_{j=0}^{N-1}\int_{jT_0}^{(j+1)T_0}\left\|\theta_k^{NT_0}(t)\right\|_U^2 dt\\ &= \frac{1}{N^2}\sum_{j=0}^{N-1}\int_{jT_0}^{(j+1)T_0}\left\|e^{i\omega_k jT_0}\theta_k^{T_0}(t-jT_0)\right\|_U^2 dt\\ &= \frac{1}{N}\left\|\theta_k^{T_0}\right\|^2_{L^2(0,T_0;U)} = \frac{T_0}{T}\left\|u^{T_0}\right\|^2_{L^2(0,T_0;U)}.\end{aligned}$$

For L^1 we have

$$\begin{aligned}\left\|u^T\right\|_{L^1(0,NT_0;U)} &= \frac{1}{N}\sum_{j=0}^{N-1}\int_{jT_0}^{(j+1)T_0}\left\|e^{i\omega_k jT_0}\theta_k^{T_0}(t-jT_0)\right\|_U^2\,dt\\ &= \left\|\theta_k^{T_0}\right\|_{L^1(0,T_0;U)} = \left\|u^{T_0}\right\|_{L^1(0,T_0;U)}.\end{aligned}$$

Thus, for this simple case the required estimates (2.9, 2.10) are valid.

Let the initial state (y_0, y_1) be an arbitrary element in $W_{s+1}\oplus W_s$ or a linear combination of eigenmodes for the cases of B–controllability and for M–controllability respectively. Then the minimal norm control u^T has the form (2.8)

$$u^T = \sum_k \widetilde{c}_k^0\theta_k^T, \tag{2.11}$$

where the $\widetilde{c}_k^0$ are calculated in (2.6, 2.7). The series (2.11) converges in $\mathcal{U}^T$ under B–controllability and is a finite sum under M–controllability.

Clearly, $\left\|u^T\right\|_{\mathcal{U}^T}$ can not increase in T; it is easy to see that for $T_1 < T_2$

$$u^{T_2} = \sum_k \widetilde{c}_k^0\theta_k^{T_2} = \mathcal{P}_{\bigvee\mathcal{E}^{T_2}}\sum_k \widetilde{c}_k^0\theta_k^{T_1} = \mathcal{P}_{\bigvee\mathcal{E}^{T_2}}u^{T_1}.$$

Take $T = NT_0$, $N\in\mathbb{N}$. Using Lemma 2.7, we have

$$\left\|u^T\right\|_{\mathcal{U}^T}^2 = \frac{1}{N^2}\sum_{j=0}^{N-1}\left\|\sum_k \widetilde{c}_k^0\varepsilon_k^{(j)}\theta_k^{T_0}\right\|_{\mathcal{U}^{T_0}}^2, \quad \varepsilon_k^{(j)} := \exp(i\omega_k jT_0). \tag{2.12}$$

Consider the expression

$$q(\varepsilon) = \left\|\sum_k \widetilde{c}_k^0\varepsilon_k\theta_k^{T_0}\right\|_{\mathcal{U}^{T_0}}^2$$

as a function of the variables ε_k. For a finite number of ε_k, *i.e.*, for the case of M–controllability, q is a continuous function of several variables. Therefore, on the compact set $\{|\varepsilon_k| = 1\}$, linear independence of Θ^{T_0} implies

$$0 < q_{\min} \le q(\varepsilon) \le q_{\max} < \infty. \tag{2.13}$$

The same estimates are true for the B–controllable system. To prove this we use Proposition 2.5. This proposition states that $\mathcal{E}^{T_0}$ forms a Riesz basis in $\bigvee\mathcal{E}^{T_0}$. Then the biorthogonal family is also a Riesz basis implying

$$\left\|\sum_k \widetilde{c}_k\varepsilon_k^{(j)}\theta_k^{T_0}\right\|_{\mathcal{U}^{T_0}}^2 \asymp \sum_k |\widetilde{c}_k|^2.$$

Now (2.12) and the last estimates lead to the inequalities

$$\left\|u^T\right\|_{\mathcal{U}^T}^2 \asymp \frac{1}{N}\sum|c_k^0|^2 \asymp \frac{1}{T}\left\|u^{T_0}\right\|^2$$

for times multiple to T_0. Since $\left\|u^T\right\|_{\mathcal{U}^T}$ is a monotone function, the same inequalities are also true for intermediate points. We have proved the estimates for the L^2–norm.

The L^1–estimate from above follows from the Cauchy–Bunyakovski inequality and the estimate for the L^2–norm:

$$\left\|u^T\right\|_{L^1(0,T;U)} \leq \sqrt{T}\left\|u^T\right\|_{L^2(0,T;U)} \overset{(2.9)}{\prec} 1. \tag{2.14}$$

Let us check (2.10) from below. For any T we take an integer N in such a way that $NT_0 < T \leq (N+1)T_0$. Then $T_1 := T/N$. Replacing T_0 by T_1, we derive, similarly to (2.12),

$$\left\|u^{T_1}\right\|_{L^1(0,T;U)} = \frac{1}{N}\sum_{j=0}^{N-1}\left\|\sum_k \widetilde{c}_k^0 \varepsilon_k^{(j)}\theta_k^{T_1}\right\|_{L^1(0,T_1;U)}, \tag{2.15}$$

with $\varepsilon_k^{(j)} := \exp(i\omega_k jT_1)$. In order to estimate this sum, we use an analog of (2.13).

Lemma 2.8. *For any unimodular sequence ε_k, fixed $\widetilde{c}^0$, and $T_1 \geq T_0$, the expression*

$$Q_\varepsilon(T) := \sum_k \widetilde{c}_k^0 \varepsilon_k \theta_k^{T_1} \tag{2.16}$$

is bounded away from zero: $\|Q_\varepsilon(T_1)\|_{L^1(0,T_1;U)} \geq c > 0$.

Proof of the lemma. The series Q_ε converges in $L^2(0,T_1;U)$ for any unimodular ε_k. Calculating the inner product in U^{T_1} for (2.16) and e_k, we have

$$\varepsilon_k \widetilde{c}_k^0 = (Q_\varepsilon, e_k)_{\mathcal{U}^T},$$

and thus

$$|\widetilde{c}_k^0| = \left|\int_0^{T_1}\big(Q_\varepsilon(t), e_k(t)\big)_U dt\right| \leq \int_0^{T_1}\|Q_\varepsilon(t)\|_U\,\|e_k(t)\|_U\,dt \leq d_k\,\|Q_\varepsilon\|_{L^1(0,T_1;U)}$$

with $d_k := |\omega_k|^s\left\|B^*\varphi_{|k|}\right\|_U$. Clearly,

$$\|Q_\varepsilon\|_{L^1(0,T_1;U)} \geq \max_k\left(|\widetilde{c}_k^0|/d_k\right) \geq c > 0,$$

uniformly in T. The lemma is proved.

Now (2.15) and this lemma imply the estimate (2.10) from below which completes the proof of the theorem. $\square$

Remark 2.9. Generally speaking, $\left\|u^T\right\|_{L^2(0,T;U)}$ may be constant as a function of T on intervals, see Section 3.4 below.

Remark 2.10. It is seen from the estimates obtained above, that for B–controllable systems we may replace (2.9) by sharper estimates

$$\left\|u^T\right\|_{\mathcal{U}^T} \asymp \left(1/\sqrt{T}\right)\left\|u^T\right\|_{\mathcal{U}^{T_0}}.$$

That is, for any $T > T_0$ there exist positive constants c_1 and c_2 such that for any initial data (y_0, y_1) from $W_{s+1} \oplus W_s$ one has estimates

$$\left(c_1/\sqrt{T}\right)\left\|u^{T_0}\right\|_{\mathcal{U}^{T_0}} \leq \left\|u^T\right\|_{\mathcal{U}^T} \leq \left(c_2/\sqrt{T}\right)\left\|u^{T_0}\right\|_{\mathcal{U}^{T_0}}.$$

Whether these inequalities are true for M–controllable systems is an open question. Also we do not know whether or not the analogous estimates

$$c_1 \left\|u^{T_0}\right\|_{L^1(0,T_0;U)} \leq \left\|u^T\right\|_{L^1(0,T;U)} \leq c_2 \left\|u^{T_0}\right\|_{L^1(0,T_0;U)}$$

hold in L^1.

2.3. **Continuity of controls**

In the sequel we extend controls u^T by zero for $t > T$. Then u^T may be considered in $L^2(0,\infty;U)$ and it turns out that u^T is continuous as a function of T.

Theorem 2.11. *If the control system (2.1) is B–controllable in time T_0 in the space $W_{s+1} \oplus W_s$, then*

$$\left\|u^{T_1} - u^{T_2}\right\|_{L^2(0,\infty;U)} \underset{T_1 \to T_2}{\to} 0, \quad T_1, T_2 > T_0.$$

We will prove the theorem using the Gram matrix of the exponential family. Denote by $\Gamma(T)$ an infinite matrix with entries

$$\Gamma_{pk}(T) := (e_p, e_k)_{\mathcal{U}^T}.$$

We see from the definition, that for a finite sequence $a = \{a_k\}$

$$(\Gamma(T)a, a)_{\ell^2} = \left\|\sum a_k e_k\right\|^2_{L^2(0,T;U)}. \tag{2.17}$$

The Gram matrix describes the "geometrical" properties of the family. More precisely, it determines the family up to an isometric mapping. In the following we need a proposition

Proposition 2.12. *(see, e.g. [13]) The Gram matrix generates an isomorphism in ℓ^2, if and only if $\mathcal{E}$ forms a Riesz basis*[1].

In view of this proposition and the B–controllability of the control system, $\Gamma(T_0)$ is a selfadjoint positive definite operator with a bounded inverse. We will show that the same is true for $\Gamma(T)$ for $T > T_0$. As in the case of u^T, we will see a monotone behavior of the Gram matrix.

Lemma 2.13. *(i) For any $T \geq T_0$ the Gram matrix is an isomorphism in ℓ^2, $\|\Gamma(T)\|$ is a nondecreasing function and $\left\|\Gamma^{-1}(T)\right\|$ is a nonincreasing one,*
(ii) $\Gamma(T)$ and $\Gamma^{-1}(T)$ are both strongly continuous in T for $T > T_0$.

Given this lemma, we are able to prove the theorem.

Proof of the theorem. First, we express the control u^T through $\Gamma(T)$. The biorthogonal elements θ_k^T lie, by definition, in $\bigvee \mathcal{E}^T$ and, therefore, can be expressed as a converging series in e_k. It is easy to see that

$$\theta_k^T = \sum_p \left(\Gamma^{-1}(T)\right)_{pk} e_p^T.$$

Then

$$u^T = \sum_p \left(\Gamma^{-1}(T)\widetilde{c}^0\right)_p e_p^T. \tag{2.18}$$

[1]This fact obviously is not limited to exponentials and has a general character.

Using this equality, we represent $u^{T_1} - u^{T_2}$ as

$$\begin{aligned} u^{T_1} - u^{T_2} &= \sum_p \left(\Gamma^{-1}(T_1)\tilde{c}^0\right)_p e_p^{T_1} - \sum_p \left(\Gamma^{-1}(T_2)\tilde{c}^0\right)_p e_p^{T_2} \\ &= \sum_p \left(\left[\Gamma^{-1}(T_1) - \Gamma^{-1}(T_2)\right]\tilde{c}^0\right)_p e_p^{T_1} + \sum_p \left(\Gamma^{-1}(T_2)\tilde{c}^0\right)_p \left(e_p^{T_1} - e_p^{T_2}\right) \\ &=: S_1 + S_2. \end{aligned} \tag{2.19}$$

We will prove continuity of u^T from the left, that is for $T_1 \nearrow T_2$; the opposite case is similar.

Let us estimate the first sum S_1. For $b := \left[\Gamma^{-1}(T_1) - \Gamma^{-1}(T_2)\right]\tilde{c}^0$, (2.17) implies the inequality

$$\|S_1\|^2_{\mathcal{U}^{T_2}} = (\Gamma(T_1)b, b)_{\ell^2} \le \|\Gamma(T_1)\|_{\ell^2} \|b\|^2_{\ell^2}.$$

From Lemma 2.13, we conclude that $\|b\|_{\ell^2} \underset{T_1 \to T_2}{\to} 0$ and that $\|\Gamma(T_1)\|$ is uniformly bounded in a neighborhood of T_2. Therefore, $S_1 \to 0$.

To estimate the second sum S_2 in (2.19), we set

$$f(t) := \sum_p \left(\Gamma^{-1}(T_2)\tilde{c}^0\right)_p e_p(t).$$

In view of Lemma 2.13 (i), this function is square integrable on every compact interval in $\mathbb{R}_+$ and $S_2 = \chi_{[T_1,T_2]} f(t)$. This implies $S_2 \to 0$.

In order to complete the proof of Theorem 2.11, it remains to verify Lemma 2.13.

Proof of the lemma. (i) As may be seen from (2.17), the form $(\Gamma(T)a, a)$ does not decrease in T for any finite sequence a. Therefore, $\|\Gamma(T)\|$ is also a nondecreasing function. To prove that $\Gamma(T)$ is bounded for all $T > T_0$, we check it for $T = NT_0$. Let $\Gamma(\alpha, \beta)$ denote the Gram matrix for exponentials on the interval $[\alpha, \beta]$. Then

$$\begin{aligned} \Big(\Gamma(NT_0)a, a\Big) = \left\|\sum a_k e_k\right\|^2_{L^2(0,NT_0;U)} &= \textstyle\sum_{j=1}^N \|a_k e_k\|^2_{L^2((j-1)T_0, jT_0;U)} \\ &= \textstyle\sum_{j=1}^N \left(\Gamma((j-1)T_0, jT_0))a, a\right). \end{aligned} \tag{2.20}$$

Since

$$(e_p, e_k)_{L^2((j-1)T_0, jT_0;U)} = e^{i(\omega_p - \omega_k)(j-1)T_0}(e_p, e_k)_{L^2(0,T_0;U)},$$

the operators $\Gamma((j-1)T_0, jT_0)$ are all unitary equivalent to $\Gamma(T_0)$:

$$\Gamma((j-1)T_0, jT_0) = \operatorname{diag}\left[e^{i\omega_k(j-i)T_0}\right]\Gamma(T_0)\operatorname{diag}\left[e^{-\omega_k(j-i)T_0)}\right].$$

Hence,

$$\|\Gamma((j-1)T_0, jT_0)\| = \|\Gamma(T_0)\|,$$

and (2.20) gives

$$\|\Gamma(NT_0)\| \le N\,\|\Gamma(T)\| < \infty.$$

The fact that $\left\|\Gamma^{-1}(T)\right\|$ can not increase is a consequence of the equalities

$$\left\|\Gamma^{-1}(T_0)\right\|^{-1} = \inf_{\|a\|=1}(\Gamma(T)a, a) = \inf_{\|a\|=1}\left\|\sum a_k e_k\right\|_{\mathcal{U}^T}.$$

The part (i) of the lemma is thus proved.

(ii) First, we check that $\Gamma(T)a$ is continuous in ℓ^2 for any $a \in \ell^2$ (as a function of T). Take $T_1 < T_2$, and calculate the k-th component of $[\Gamma(T_1) - \Gamma(T_2)]a$:

$$([\Gamma(T_1) - \Gamma(T_2)]a)_k = \sum_p [(e_k, e_p)_{\mathcal{U}^{T_1}} - (e_k, e_p)_{\mathcal{U}^{T_2}}]a_p = \sum_p (e_k, \chi_{[T_1,T_2]} e_p)_{\mathcal{U}^{T_2}} a_p = (\Gamma(T_1, T_2)a)_k\,.$$

That is

$$\Gamma(T_1) - \Gamma(T_2) = \Gamma(T_1, T_2).$$

Since, similarly to (i),

$$\|\Gamma(T_1, T_2)\| \le \|\Gamma(0, T_2)\|\,,$$

and

$$\begin{aligned} \|\Gamma(T_1,T_2)a\|^2 \le \left\|\Gamma^{1/2}(T_1,T_2)\right\|^2 \left\|\Gamma^{1/2}(T_1,T_2)a\right\|^2 &= \left\|\Gamma^{1/2}(T_1,T_2)\right\|^2 (\Gamma(T_1,T_2)a, a) \\ &= \left\|\Gamma^{1/2}(T_1,T_2)\right\|^2 \left\|\sum a_k e_k\right\|^2_{L^2(T_1,T_2;U)}, \end{aligned}$$

we have

$$\Gamma(T_1)a \underset{T_1 \to T_2}{\rightarrow} \Gamma(T_2)a.$$

The case $T_1 \ge T_2$ may be treated in the same manner. Thus, we have proved strong continuity of $\Gamma(T)$.

To demonstrate this property for $\Gamma^{-1}(T)$, we use the resolvent identity

$$\Gamma^{-1}(T_1) - \Gamma^{-1}(T_2) = \Gamma^{-1}(T_1)\,[\Gamma(T_2) - \Gamma(T_1)]\,\Gamma^{-1}(T_2).$$

$\Gamma^{-1}(T)$ is uniformly bounded for $T \ge T_0$ and, as was shown,

$$[\Gamma(T_2) - \Gamma(T_1)]\,b \to 0,\ \ b := \Gamma^{-1}(T_2)a.$$

Thus, we obtain strong continuity of $\Gamma^{-1}(T)$ □

Corollary 2.14. *The control u^T is continuous in $L^1(0,\infty;U)$ as a function of T.*

Indeed, on a finite time interval the metric of L^1 is weaker than the metric of L^2.

3. Examples

Let us give some applications of Theorem 2.6 to control systems governed by the wave equation. As already noted, an equation for a vibrating rectangular plate was considered in [9, 10].

The examples presented below deal with L^2–controls (in time). Examples of B–controllability with control from Sobolev spaces may be found in the paper [4] of Avdonin *et al.* It is possible to write analogues of the main theorem for such controls.

3.1. B–controllable wave equations with boundary controls

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. We consider the initial boundary value problem

$$y_{tt} = \mathcal{A}y \ \text{ in } \Omega \times (0,T),\ \ y|_{t=0} = y_0,\ \ y_t|_{t=0} = y_1, y|_{\partial\Omega} = u, \tag{3.1}$$

where $\mathcal{A}$ is a second order elliptic operator with smooth time independent coefficients. Let the controls act on $\Gamma \subset \partial\Omega$, $\operatorname{supp} u \subset \Gamma$, and $(y_0, y_1) \in L^2(\Omega) \oplus H^{-1}(\Omega)$.

The control system (3.1) may be treated as the system (2.1) if we set $A = -\mathcal{A}$ with Dirichlet boundary conditions (DBC), and $W_{r+1} \oplus W_r = L^2(\Omega) \oplus H^{-1}(\Omega)$, *i.e.*, $r = -1$, and if B is given by

$$B^*\varphi = \frac{\partial}{\partial\nu}\varphi|_\Gamma,$$

where $\frac{\partial}{\partial\nu}$ is the normal derivative associated with the operator $\mathcal{A}$. This form of B may be obtained from the integral inequality equivalent to (3.1).

For $r > 3/2$, the operator B^* acts continuously from W_r into U (we note that the space $W_r = \mathcal{D}(\mathcal{A}^{r/2})$ coincides with the Sobolev space H^r up to boundary conditions). Thus, B is bounded from U into W_{-r} and Proposition 2.1 implies that the state $(y(T), y_t(T))$ belongs to $W_{-1/2+\varepsilon} \oplus W_{-3/2+\varepsilon}$, $\varepsilon > 0$. In fact, the solution is smoother [11]:

$$(y(T), y_t(T)) \in W_0 \oplus W_{-1} = L^2(\Omega) \oplus H^{-1}(\Omega). \tag{3.2}$$

The exponential family, corresponding to the problem has the form

$$\mathcal{E} = \left\{\partial_\nu\varphi_n\big|_\Gamma e^{\pm i\omega_n t}\right\}_{n\in\mathbb{N}},$$

where ω_n are the eigenfrequencies of $\mathcal{A}$ with DBC and φ_n are the normalized eigenfunctions. In order to apply Theorem 2.6, the system has to be B– or M–controllable. Very little is known about $\mathcal{E}$ for $\dim\Omega > 1$ and a direct study of $\mathcal{E}$ has been performed only for $\mathcal{A} = \Delta$ and for Ω permitting separation of variables. For the general case sharp conditions of exact controllability have been proved by Bardos *et al.* [5]. Roughly speaking, they proved that: if every ray, starting at any point of Ω, hits a point of the control boundary Γ within the time T_{exact}, then the system is exactly controllable in $L^2(\Omega) \oplus H^{-1}(\Omega)$ in time $T > T_{\text{exact}}$. In view of (3.2), we see that this result gives the conditions for B–controllability in $W_0 \oplus W_{-1}$. Thus, we have

Theorem 3.1. *Let the "geometrical control conditions" of [5] be fulfilled in time T_0. Then the control u^T, steering the system (3.1) to the rest, satisfies the asymptotics (2.9, 2.10).*

3.2. B–controllable wave equations with distributed controls

Let us consider a hyperbolic type system with controls supported on a subdomain $\omega \subset \Omega$

$$y_{tt} = -\mathcal{A}y + u \quad \text{in } \Omega \times (0,T), \quad y|_{t=0} = y_0 \in H_0^1(\Omega), \quad y_t|_{t=0} = y_1 \in L^2(\Omega), y|_{\partial\Omega} = 0, \tag{3.3}$$

Set $U = L^2(\omega)$. Then B is the embedding operator from U onto $L^2(\Omega)$ and B^* acts as the orthoprojector from $L^2(\Omega)$ into $L^2(\omega)$. It is well known that

$$(y(T), y_t(T)) \in W_1 \oplus W_0 = H_0^1(\Omega) \oplus L^2(\Omega),$$

what also follows from Proposition 2.1.

As in the case of boundary control, we can not check directly the Riesz basis property of the exponential family

$$\mathcal{E} = \left\{\varphi_n\big|_\omega e^{\pm i\omega_n t}\right\}_{n\in\mathbb{N}}.$$

Nevertheless, just as in the previous example, this property follows from controllability conditions proved in [5]: if every ray, starting at any point of Ω hits a point of the control region ω during time T_{exact}, then the system is B–controllable in $T > T_{\text{exact}}$. Thus, we have the theorem

Theorem 3.2. *Let the "geometrical control conditions" of [5] be fulfilled for system (3.3) in time T_0. Then the control u^T, steering the system to the rest, satisfies the asymptotics (2.9, 2.10).*

Remark 3.3. If we consider the control on the whole domain $\omega = \Omega$, then it is possible to study $\|u^T\|$ without the use of Proposition 2.5 and Theorem 2.6. The point is that the exponential family $\mathcal{E} = \{\varphi_n\big|_\omega e^{\pm i\omega_n t}\}$ may be represented as the union of orthogonal $2d$ subfamilies $\{\varphi_n e^{i\omega_n t}, \varphi_n e^{-i\omega_n t}\}$, what allows us to describe and study the biorthogonal family Θ^T explicitly.

3.3. M–controllable wave equations

Let us consider a control system (3.1) under the assumption that there exists a ray which does not meet the control region $\bar{\Gamma}$ at any time. As examples of such systems we may take the following:

(i) a homogeneous circular membrane with Γ being an arc less than a semicircle;

(ii) a homogeneous annular membrane with Γ equal to the inner circle [3];

(iii) a homogeneous rectangular membrane with control acting on a side [6].

We assume additionally that at time $T_{\min}$ we can cancel every eigenmode $\left(\frac{1}{i\omega_n}\varphi_n, \pm\varphi_n\right)$, *i.e.*, we have M–controllability for the control time $T_{\min}$.

Remark 3.4. None of the control systems (i–iii) can be exactly controllable in any time in view of [5]. Nevertheless, these systems are approximately controllable in time $T > T_{\text{appr}} = 2T_1$, where T_1 is the minimal time needed for the wave generated by the sources on Γ to fill the whole domain (the Holmgren–John theorem [14,17]).

In contrast to the critical time of approximate (T_{appr}) and exacte (T_{exacte}) controllability, general sufficient conditions on a part of the boundary and the critical time are unknown for M-controllability. In particular, whether the control system in (i) is M–controllable for some T is an open problem (to our knowledge). In examples (ii) and (iii) it is possible, using separation of variables, to prove the M–controllability in time $T \geq T_{\text{appr}}$. The natural conjecture is that this takes place in the general case (some arguments may be found in [2] Sect. 5.2.1).

With the M–controllability assumptions, the control system (3.1) satisfies the conditions of Theorem 2.6. Therefore, if the initial data are linear combinations of eigenmodes, we obtain the behavior (2.9, 2.10).

In the cases (ii, iii) we can prove more: if arbitrary initial data (not necessarily a finite sum of eigenmodes) may be steered to zero in finite time, then the asymptotics (2.9, 2.10) are valid. The proof uses the Fourier method and separation of spatial variables. The corresponding exponential families can be written as an "orthogonal sum of scalar families", and each scalar family forms a Riesz basis in the closure of its span for $T \geq T_{\text{appr}}$. This allows us to make the following conjecture for the control system (3.1)

Conjecture. *If the initial data (y_0, y_1) may be steered to zero, then for the control u^T with minimal L^2–norm estimates (2.9, 2.10) are valid.*

3.4. Lack of approximate controllability

In the case of a control system without approximate controllability, the control u^T may be independent of T for large T. As the simplest example we take a homogeneous semi–infinite string

$$y_{tt} = y_{xx}, \quad x, t > 0, \tag{3.4}$$

with the Dirac δ–function as initial data

$$y|_{t=0} = \delta(x - x_0), \;\; y_t|_{t=0} = \delta'(x - x_0), \;\; x_0 > 0.$$

The initial wave, supported at x_0, moves to left and arrives the boundary at $t = x_0$. Suppose that we try to cancel this wave by the boundary controller

$$y|_{x=0} = u(t).$$

This is possible, of course, only for $T > x_0$ and such a control is unique

$$u(t) = \delta(t - x_0). \tag{3.5}$$

Indeed, for the system with homogeneous DBC the reflected wave is $-\delta(t-x-x_0)$ (for $t > x_0$), while the control (3.5) generates the wave $\delta(t-x-x_0)$. We see that the control cancelling the vibration does not depend on T for $T > T_0$.

The control system under question does not have the form described in Section 1, since the elliptic operator $\mathcal{A}$ does not possess an eigenbasis and its spectrum is continuous. If we take a finite string with zero DBC at the right end, say, l, then the control u^T has the same form (3.5) for $T < 2l + x_0$. At $T > 2l + x_0$ we have the second reflection at the origin and we may take u^T as $\frac{1}{2}\delta(t-x_0) + \frac{1}{2}\delta(t-2l-x_0)$ and so on (see [1], Sect. 5). Thus, we have piecewise constant control (in T), decreasing as $1/\sqrt{T}$ in the H^{-1} norm[2]. We note that for $T \geq 2l$ this control system is B–controllable in the state space $H^{-1}(0,l) \times H^{-2}(0,l)$ with the control space $H^{-1}(0,T)$.

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