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Special values of twisted symmetric square $L$-functions and the trace formula*

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Abstract. We use the trace formula to compute explicitly the trace, over a Hecke eigenbasis, of the algebraic part of the special values. The case of twisting holomorphic level one modular forms by a quadratic character modulo $q$ is considered. The result involves both class numbers of binary quadratic forms with discriminant depending on $q$, and also the number of points on certain elliptic curves reduced modulo $q$.

0. Introduction

We let $K = \mathbb{Q}(\sqrt{q})$, with ring of integers $\mathcal{O}$ for an odd prime $q \equiv 1 \mod 4$. We use $\chi$ to denote $(q/^\ast)$, the quadratic character modulo $q$. The integral kernel for the base change lifting $f \rightarrow \tilde{f}$ from $\text{SL}(2, \mathbb{Z})$ cusp forms to those of $\text{SL}(2, \mathcal{O})$ is denoted $\Omega(\tau, z, z')$ (see [12]).

Integrating $\Omega$ against a Hilbert modular form $F$ in the $(z, z')$ variable gives a linear map from $S_k(\text{SL}(2, \mathcal{O}))$ to $S_k(\text{SL}(2, \mathbb{Z}))$. Its easy to see this is the adjoint of the lift $\Omega$, so we denote this linear map $\Omega^*$. What can be said about this map? From a version of the Strong Multiplicity One theorem due to Ramakrishnan, we get that the lift is 1-1, and from this follows

EASY LEMMA. Every Hecke eigenform $f$ is also an eigenform of the map $\Omega^*\Omega$, with eigenvalue equal to $\langle \tilde{f}, \tilde{f} \rangle / \langle f, f \rangle$. If $F$ is a Hilbert modular eigenform which is not a lift, then $\Omega^*F = 0$.

As we will show below

$$\frac{\langle \tilde{f}, \tilde{f} \rangle}{\langle f, f \rangle} = \beta(f) \langle f, f \rangle, \quad \beta(f) \in \mathbb{Q}(f).$$

(Zagier did the analogous result for forms of nebentypus in [12], Corollary 1 to Theorem 5 by a different method.) Thus the map $B$, defined on eigenforms by

$$Bf = (f, f)^{-1} \Omega^*\Omega f = \beta(f)f,$$

(0.1)

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is in the Hecke algebra. In this paper we will use the trace formula to get an explicit 
computation of the trace of $B$, and more generally the trace of $T(n)B$ for any 
Hecke operator $T(n)$. The trace is a sum over $l$ of class numbers $H(q(l^2 - 4q^2n))$ and $H(q(l^2 - 4n))$ of orders in complex quadratic fields. These class numbers are 
weighted, respectively, by the character $\chi(l)$ and by the number of points

$$a(l, n) = \#E - q - 1, \quad E/\mathbb{F}_q: y^2 = x^3 + lx^2 + nx.$$ (0.2)

0.1. Why is this interesting?

The method of computing the trace of the period is one Zagier suggested in Section 5 
of [12]. He did the analogous trace formula for forms of level $D$, quadratic character 
in Section 6 of that paper; including the technically much more difficult case of 
weight 2 forms. (This is implicit in formulae (91) and (98). All that is missing is 
to subtract off an appropriate multiple of an Eisenstein series to get a cusp form.) 
His interest was in connection with intersection numbers of cycles on the Hilbert 
modular surface. Gordon [3] has considered higher weight analogs of intersection 
numbers; presumably $\beta(f)$ has an interpretation in terms of these intersection 
numbers.

The value at $s = k$ (in the standard normalization) of the twisted symmetric 
square $L$-function $D(s, f, \chi)$ is $\beta(f)(f, f)$. This is also the residue of the cor-
responding Asai $L$-function $L(s, f, \rho \otimes \chi)$. By the work of Harder, Langlands and 
Rapoport, the Asai $L$-functions occur as factors of the Hasse–Weil zeta function 
of the Hilbert modular surface $\mathcal{H}^2/\text{SL}(2, \mathcal{O})$ (for weight 2 forms). There are lots 
of conjectures on the arithmetic significance of the special values of such zeta 
functions, in the context of higher $K$-theory and regulators for algebraic varieties. 
See [8] and [9] for an exposition.

Notation. The prime $q \equiv 1 \mod 4$ is fixed throughout, as is a positive integer 
n indexing a Fourier coefficient. The weight $k$ of $\text{SL}(2, \mathbb{Z})$ cusp forms is fixed 
throughout, but $k$ is also used sometimes as a subscript in sums. $\delta(*)$ is 0 if the 
argument is not an integer, and $e(*)$ denotes $\exp(2\pi i *)$. $\Delta$ denotes a typical non-
primitive discriminant, either $l^2 - 4n$ or $l^2 - 4q^2n$, written as $Df^2$ with $D$ primitive. 
Subscripts $q$ on $L$-functions indicate local factors at $q$ omitted, while superscripts 
$L^q$ are those local factors.

1. Special values of $L$-series

It will be very useful to view the constant $\beta(f)$ of (0.1) in terms of $L$-series. Write

$$f(\tau) = \sum_n a(n)n^{(k-1)/2} \exp(2\pi i n\tau).$$

and let $f_{\chi}(\tau)$ be the quadratic twist (with level $q^2$ and trivial character.) Note we 
have a non-classical normalization of the Fourier coefficients. Letting $\alpha + \alpha' = a(p)$
and \( \alpha \alpha' = 1 \), we write \( L(s, f \times f) \) and \( L(s, f \times f_\chi) \) as degree four Euler products; a computation then gives the splitting formula \( L(s, f \times f)L(s, f \times f_\chi) = L(s, \tilde{f} \times \tilde{f}) \). The poles of the Eisenstein series produce poles in the Rankin convolutions
\[
\Gamma(k) \text{res}_{s=1} L(s, f \times f) = (4\pi)^{k-1} \langle f, f \rangle,
\]
\[
q^2 \Gamma(k) \text{res}_{s=1} L(s, \tilde{f} \times \tilde{f}) = (4\pi)^{2k-2} \langle \tilde{f}, \tilde{f} \rangle \text{res}_{s=1} \zeta_K(s),
\]
so \( L(s, f \times f_\chi) \) is entire at \( s = 1 \) and
\[
q^2 \Gamma(k) L(1, f \times f_\chi) = (4\pi)^{k-1} \langle \tilde{f}, \tilde{f} \rangle \text{res}_{s=1} \zeta_K(s).
\]
We can also consider the twisted symmetric square \( L \) function
\[
D(s, f, \chi) = \prod_p (1 - \chi(p) \alpha^2 p^{-s})^{-1} (1 - \chi(p) p^{-s})^{-1} (1 - \chi(p) \alpha' p^{-s})^{-1}.
\]
Thus \( L(s, \chi) D(s, f, \chi) = L(s, f \times f_\chi) \), and by the above
\[
D(1, f, \chi) = C_k \langle \tilde{f}, \tilde{f} \rangle \langle f, f \rangle,
\]
where \( C_k = (2\pi)^2 (4\pi)^{k-1} (q^2 \Gamma(k))^{-1} \). We can build a cusp form \( \Phi_s(\tau) \) depending also on \( s \) which satisfies
\[
\langle \Phi_s, f_i \rangle = C_k^{-1} D(s, f_i, \chi),
\]
for each eigenform \( f_i \). One the one hand, the eigenforms give an orthogonal basis, so
\[
\Phi_s(\tau) = C_k^{-1} \sum_i D(s, f_i, \chi) f_i(\tau).
\]
This implies that the Fourier expansion of \( \Phi_s(\tau) \) looks like
\[
\Phi_s(\tau) = C_k^{-1} \sum_{n=1}^{\infty} \left\{ \sum_i \frac{a_i(n)n^{(k-1)/2} D(s, f_i, \chi)}{\langle f_i, f_i \rangle} \right\} \exp(2\pi i n\tau).
\]
Plugging in \( s = 1 \) and using the above gives
\[
\Phi_1(\tau) = \sum_{n=1}^{\infty} \left\{ \sum_i \frac{a_i(n)n^{(k-1)/2} \langle \tilde{f}_i, \tilde{f}_i \rangle}{\langle f_i, f_i \rangle^2} \right\} \exp(2\pi i n\tau).
\]
On the other hand we can define the usual the Poincare series \( G_r(\tau) \) which satisfies
\[
\langle G_r, f \rangle = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} a(\tau)^{r(1-k)/2},
\]
and use the identity (formula (0.2) of [10])

\[ D(s, f, \chi) = \zeta(2s)q \sum_{m=1}^{\infty} \chi(m)a(m^2)m^{-s}. \]

This gives

\[ \Phi_s(\tau) = \frac{(k - 1)q^2}{(2\pi)^2} \zeta(2s)q \sum_{m=1}^{\infty} \chi(m)m^{-s-k-1}G_{m^2}(\tau). \] (1.1)

The series converges absolutely and uniformly on compact subsets of \( \{ \text{re}(s) > 1 \} \times \mathcal{H} \). Writing out the Fourier expansion of the Poincare series \( G_{m^2}(\tau) \), interchanging the order of summation, and continuing to \( s = 1 \) will give an explicit computation of the Fourier series coefficients

\[ \Phi_s(\tau) = \sum_{n=1}^{\infty} b(n, s) \exp(2\pi i n \tau), \] (1.2)

and \( b(n, 1) \) gives an explicit computation of the (algebraic) expression

\[ \sum_i \frac{a_i(n)n^{(k-1)/2}(\bar{f_i}, \bar{f_i})}{(f_i, f_i)^2} = \sum_i a_i(n)n^{(k-1)/2} \beta(f_i) = \text{trace}(T(n)B). \]

2. Poisson summation

In computing the Fourier expansion of \( \Phi_s(\tau) \), we will suppose first \( 1 < \text{re}(s) < k - 1 \) and then find analytic continuation to include \( \text{re}(s) = 1 \). It is well known the Fourier expansion of the Poincare series \( G_{r}(\tau) \) is

\[ \sum_{n=1}^{\infty} \delta_{r,n} + 2\pi(-1)^{k/2} \left( \frac{n}{r} \right)^{(k-1)/2} \left\{ \sum_{c=1}^{\infty} \frac{1}{c} \frac{4\pi \sqrt{rn}}{c} \right\} e(n\tau). \]

Plugging this into (1.1) with \( r = m^2 \) we get

\[ \sum \sum_{n=1}^{\infty} \chi(m)m^{-s-k-1} \left( \delta_{m^2,n} + 2\pi(-1)^{k/2} \left( \frac{n}{m^2} \right)^{(k-1)/2} \right. \]

\[ \times \left\{ \sum_{c=1}^{\infty} \frac{1}{c} \frac{4m\pi \sqrt{n}}{c} \right\} e(n\tau). \]

Letting

\[ S = 2 \sum_{c=1}^{\infty} \sum_{m=1}^{\infty} \frac{\chi(m)}{c} m^{-s} K_c(m^2, n) J_{k-1} \left( \frac{4m\pi \sqrt{n}}{c} \right), \]
we see the \( n \)th Fourier coefficient \( b(n, s) \) of \( \Phi_s(\tau) \) in (1.2) satisfies

\[
\frac{(2\pi)^2}{(k-1)q^2\zeta(2s)}b(n, s) = \pi(-1)^{k/2}n^{(k-1)/2}S + \delta(\sqrt{n})\chi(\sqrt{n})n^{(-s+k-1)/2}.
\]

**PROPOSITION.** The ‘main term’, \( S \) is equal

\[
S = \sum_{c \in \mathbb{Z}} \sum_{c \mid q} c^{-1}(cq)^{-s}I(l, nq^2, s) \left\{ \sum_{r(cq)} \chi(r)K_c(r^2, n)e \left( \frac{lr}{cq} \right) \right\} + \sum_{a=1}^{\infty} (cq^a)^{-1-s}I(l, n, s) \left\{ \sum_{r(cq^a)} \chi(r)K_{cq^a}(r^2, n)e \left( \frac{lr}{cq^a} \right) \right\}.
\]  

(2.1)

where the special function \( I(l, n, s) \) is defined by (2.2), and \( K_c(r^2, n) \) is a Kloosterman sum.

**Proof.** Put

\[
A(x) = |x|^{-s}J_{k-1} \left( \frac{4\pi\sqrt{n|x|}}{c} \right), \quad A(0) = 0,
\]

then \( S \) equals

\[
\sum_{c=1}^{\infty} \sum_{m \in \mathbb{Z}} \frac{\chi(m)}{c} K_c(m^2, n)A(m).
\]

Before applying Poisson summation we need to relate the modulus \( q \) of \( \chi \) to the modulus \( c \) of the Kloosterman sum \( K_c \). We first break the sum on \( c \) into a double sum of \( c \) prime to \( q \) and a sum on prime powers \( q^a \)

\[
\sum_{(c,q)=1}^{\infty} \sum_{a=0}^{\infty} \sum_{m \in \mathbb{Z}} (cq^a)^{-1-s} \chi(m)K_{cq^a}(m^2, n)A(m).
\]

Now \( \chi(m)K_{cq^a}(m^2, n) \) depends only on \( m \) modulo \( cq^{a+\varepsilon} \) with

\[
\varepsilon = \begin{cases} 
1 & \text{if } a = 0, \\
0 & \text{if } a \geq 1.
\end{cases}
\]

We can then replace the sum on \( m \) with a double sum over \( r \) modulo \( cq^{a+\varepsilon} \) and \( m \) in \( \mathbb{Z}, m \equiv r \). We then consider

\[
\sum_{m \equiv r} A(m) = (cq^{a+\varepsilon})^{-s} \sum_{j \in \mathbb{Z}} \left| j + \frac{r}{cq^{a+\varepsilon}} \right|^{-s}J_{k-1} \left( 4\pi\sqrt{nq^\varepsilon \left| j + \frac{r}{cq^{a+\varepsilon}} \right|} \right),
\]

and let

\[
B(x) = \sum_{j \in \mathbb{Z}} |j + x|^{-s}J_{k-1}(4\pi\sqrt{nq^\varepsilon|j+x|}).
\]
Write the Fourier expansion of $B(x)$ as $\sum_{l} c_l e(lx)$. By Poisson summation we have
\[
c_l = \int_{-\infty}^{\infty} |x|^{-s} J_{k-1}(4\pi \sqrt{nq^c} |x|) e(-lx) \, dx = (\text{definition}) \, I(l, nq^{2c}, s).
\] (2.2)
Thus
\[
\sum_{m \equiv r} A(m) = (cq^{a+\varepsilon})^{-s} B\left(\frac{r}{cq^{a+\varepsilon}}\right)
= (cq^{a+\varepsilon})^{-s} \sum_{l \in \mathbb{Z}} I(l, nq^{2c}, s) e\left(\frac{lr}{cq^{a+\varepsilon}}\right).
\]
which proves the proposition. $\Box$

3. Character sums

We will work towards getting the character sums in braces in (2.1) into a closed form, to realize the $n$th Fourier coefficient $b(n, s)$ as an infinite sum over $l$ of Dirichlet series times the special functions $I(l, n, s)$. Consider first when $a \geq 1$
\[
\sum_{r(cq^a)} \chi(r) \sum_{(x, cq^a) = 1} e\left(\frac{r^2 x^{-1} + nx}{cq^a}\right) e\left(\frac{lr}{cq^a}\right).
\] (3.1)
After interchanging the sums and changing variables $r \rightarrow rx$, one sees the character sum depends on the behavior of the two counting functions (choosing either + or − and fixed $t$)
\[
\# \left\{ r \mod cq^a \mid \chi(r) = \pm 1 \quad \text{and} \quad r^2 + lr + n \equiv t \mod cq^a \right\},
\]
and particularly their difference. Each of these counting functions can be written as a product of a term depending only on $c$ and a term depending only on $a$. In particular the former can be written
\[
N(t, x^2 + lx + n, c) = \# \left\{ r \mod c \mid r^2 + lr + n \equiv t \mod c \right\},
\]
while for the latter, the relevant term is the difference
\[
R(t, q^a) = \# \left\{ r \mod q^a \mid \chi(r) = +1 \quad \text{and} \quad r^2 + lr + n \equiv t \mod q^a \right\}
- \# \left\{ r \mod q^a \mid \chi(r) = -1 \quad \text{and} \quad r^2 + lr + n \equiv t \mod q^a \right\},
\]
and in this case the dependence on the quadratic is suppressed in the notation. By direct computation one sees that the character sum (3.1) simplifies to
\[
cq^{a-1/2} \sum_{d \mid c} \chi\left(\frac{c}{d}\right) \mu\left(\frac{c}{d}\right) \chi(d) N(0, x^2 + lx + n, d)
\times \sum_{j=1}^{q} \chi(j) R(jq^{a-1}, q^a).
\]
The character sum \( \sum_{j=1}^{q} \chi(j) R(j, q) \) is equal to \( q - 1 \), where \( q \) is the number of projective points over \( \mathbb{Z}/q\mathbb{Z} \) of the curve \( E: y^2 = x^3 + lx^2 + nx \) with discriminant \( (4n)^2(l^2 - 4n) \). We denote this character sum \( a(l, n) \). In particular if the curve is singular, the character sum is \( -\chi(l/2) \). For the general \( n \), it is convenient to introduce the generating function \( F_{l,n}(T) \) defined by

\[
\sum_{a=1}^{\infty} \sum_{j=1}^{q} \chi(j) R(jq^{\alpha-1}, q^\alpha) T^a =
\begin{cases}
T & \text{if } q \nmid \Delta \\
T - \sum_{\beta=1}^{\alpha-1} (q - 1)q^{\beta} T^{2\beta+1} + q^{\alpha} T^{2\alpha+1} & \text{if } q \nmid D, \alpha > 0 \\
T - \sum_{\beta=1}^{\alpha} (q - 1)q^{\beta} T^{2\beta+1} + \chi \left( \frac{D}{q} \right) q^{\alpha+1} T^{2\alpha+2} & \text{if } q | D \\
T - \sum_{\beta=1}^{\infty} (q - 1)q^{\beta} T^{2\beta+1} & \text{if } \Delta = 0.
\end{cases}
\]

Here the discriminant of the quadratic \( l^2 - 4n \) is written \( \Delta = Df^2 \) with \( D \) a fundamental discriminant. In the first three cases \( q^{\alpha} \parallel f \), while in the last three cases the factor \( a(l, n) \) reduces to \( -\chi(l/2) \). By summing the geometric series one can write this in the alternative form (with \( \chi = \chi(D/q) \) when \( q | D \))

\[
F_{l,n}(T) = \frac{a(l, n)T}{1 - qT^2} \times \left\{ \begin{array}{l}
1 - qT^2, \\
1 - q^2 T^2 + q^{\alpha+1} T^{2\alpha}(1 - T^2), \\
1 - q^2 T^2 - \chi q^{\alpha+1} T^{2\alpha+1}(1 + \chi T)(1 - T q), \\
1 - q^2 T^2.
\end{array} \right. \tag{3.2}
\]

For the other sum in braces in (2.1), coming from \( 'a = 0' \), similar methods show that

\[
\sum_{r(qc)} \chi(r) K_c(r^2, n) e \left( \frac{lr}{cq} \right) = \chi(l) q^{1/2} \sum_{d | c} \chi \left( \frac{c}{d} \right) \mu \left( \frac{c}{d} \right) \chi(d) N(0, qx^2 + lx + qn, d). \tag{3.3}
\]

4. Zeta functions

We now sum the Dirichlet series

\[
\sum_{(c,q)=1} c^{-s} \sum_{d | c} \chi \left( \frac{c}{d} \right) \mu \left( \frac{c}{d} \right) \chi(d) N(0, qx^2 + lx + qn, d), \tag{4.1}
\]
where $\epsilon$ is 0 or 1. This is a convolution of two Dirichlet series, so we have

$$L(s, \chi)^{-1} \sum_{c=1}^{\infty} \chi(c) N(0, q^c x^2 + lx + q^c n, c) c^{-s}.$$  

We will write $N_{c, \Delta}$ instead of $N(0, q^c x^2 + lx + q^c n, c)$, where $\Delta = l^2 - 4q^2 n$ is the discriminant. Elementary considerations tell us that in this case $N_{c, \Delta}$ is multiplicative. And for $k > 0$, $N_{p^k, \Delta} = N_{p, \Delta} = 1 + (\Delta/p)$ whenever the discriminant $\Delta$ is prime to $p$. More generally write $\Delta = Df^2$ with $D$ the discriminant of $Q(\sqrt{\Delta})$. One can compute

$$L(s, \chi)^{-1} \sum_{c=1}^{\infty} \chi(c) N_{c, \Delta} c^{-s} = \begin{cases} 
\frac{L(s, (D/\ast)\chi)}{\zeta(2s) q} \sum_{c \mid f} \mu(c) \left( \frac{D}{c} \right) \chi(c) c^{-s} \sigma_{1-2s}(fc^{-1}) \\
\zeta(2s-1) q \\
\zeta(2s) q 
\end{cases},$$

if $\Delta \neq 0$ (resp. $\Delta = 0$).

The term $\zeta(2s) q^{-1}$ in the lemma will be canceled out by the corresponding $\zeta(2s) q$ in (1.1). What remains will simplify if for each $\Delta = Df^2$, we write $q\Delta = D'f'^2$. The term on the right side above is very nearly the one associated to the non-fundamental discriminant $q\Delta$

$$L(s, q\Delta) = (\text{definition}) \ L \left( s, \left( \frac{D'}{\ast} \right) \right) \sum_{c \mid f'} \mu(c) \left( \frac{D'}{c} \right) c^{-s} \sigma_{1-2s}(f'c^{-1}), \quad (4.2)$$

the only local factor missing is the one at $q$, $L^q(s, q\Delta)$. If $(q, \Delta) = 1$, this local factor is 1. We can always assume this is the case for $\Delta$ of the form $l^2 - 4q^2 n$, since otherwise $q \mid \Delta$ implies $q \mid l$. The relevant character sum (3.3) is then zero; these terms will disappear from the trace. In the other case $\Delta = l^2 - 4n$; the missing Euler factor will be obtained from the sum on $a$ in (2.1). We introduce a fudge factor $\Gamma(s, \Delta)$ (which turns out to be related to the $q$-adic $\Gamma$ function) so that

$$F_{l, n}(q^{-s}) = a(l, n) q^{-s} \Gamma(s, \Delta) L^q(s, q\Delta). \quad (4.3)$$

Then (3.2) implies

(i) If $(q, \Delta) = 1$, $\Gamma(s, \Delta) = 1$ for all $s$.
(ii) If $q \mid D$ and $\chi(Dq^{-1}) = 1$, then $\Gamma(s, \Delta)$ vanishes at $s = 1$.
(iii) If $\Delta = 0$, $\Gamma(s, \Delta) = 1 - q^{2-2s}$ vanishes at $s = 1$. 

5. Special functions

After a change of variables the special function

\[ I(l, n, s) = 2^s \pi^{s-1} \int_0^\infty x^{-s} J_{k-1}(2\sqrt{n}x) \cos(|l|x) \, dx, \]

is found in the tables to have an analytic continuation to \( \frac{1}{2} < \text{re}(s) < k \). For example, if \( n \) is a square, \( \Delta \) will be 0 when \( l^2 = 4n \) or \( l^2 = 4q^2n \), and we see from [1], vol. 2 (19.2.24) on p. 342 that \( I(2\sqrt{n}, n, s) \) is a quotient of Gamma functions, and has a simple zero at \( s = 1 \).

The general case of this special function is found in [4], (6.561.14) on p. 684 and (6.699.2) on p. 747, in terms of hypergeometric functions. In particular if \( l^2 > 4n \), [2] vol. 2, 2.8 (6) shows that \( I(l, n, s) \) has a simple zero at \( s = 1 \). If \( 4n > l^2 \) we can get the value at \( s = 1 \) from [7], (1.13.12), p. 67

\[ I(l, n, 1) = (-1)^{(k-2)/2} \frac{(4n - l^2)^{1/2}}{(k - 1)\sqrt{n}} C_{k-2}^1 \left( \frac{|l|}{(\sqrt{4n})} \right). \]  

(5.1)

Here \( C_{k-2}^1 \) is a Gegenbauer polynomial.

6. Evaluate at \( s = 1 \)

We return to consideration of the Fourier expansion. Combine the proposition in Section 2 with formula (4.1) and (4.3) to see that the Fourier coefficient \( b(n, s) \) is written in closed form as

**THEOREM.**

\[
\frac{(-1)^{k/2}(2\pi)^2 b(n, s)}{(k - 1)q^{3/2}} = \pi \sum_{l \in \mathbb{Z}} \chi(l)q^{(1-s)n}l^{(k-1)/2}I(l, q^2n, s)L(s, q(l^2 - 4q^2n)) \\
+ \pi \sum_{l \in \mathbb{Z}} a(l, n)q^{-s}n^{(k-1)/2}I(l, n, s)L(s, q(l^2 - 4n))\Gamma(s, l^2 - 4n) \\
+ \delta(\sqrt{n})(-1)^{k/2}\sqrt{q}\zeta(2s)q\chi(\sqrt{n})n^{(-s+k-1)/2}. 
\]

For the reader inclined to skim, we recall that \( L(s, q(l^2 - 4n)) \) is defined by formula (4.2) to be the Dirichlet series associated to a (non-fundamental) discriminant, and \( a(l, n) \) is defined by (0.2). The special functions \( I(l, n, s) \) are defined by (2.2), and \( \Gamma(s, l^2 - 4n) \) by (4.3).

The infinite sum converges uniformly and absolutely in the strip \( \frac{1}{2} < \text{re}(s) < k - 1 \), which by Fubini justifies changing the order of summation in Section 2.
This is as far as we will go with the variable $s$ present; the expression will simplify when we evaluate at $s = 1$. In terms of the Eichler–Selberg trace formula, this is the ‘Selberg principle’ that the orbital integrals coming from hyperbolic conjugacy classes should not contribute to the trace.

**PROPOSITION.** The terms in the theorem with $\Delta \geq 0$ all vanish at $s = 1$.

**Proof.** The discriminant $\Delta$ can only be zero if $n$ is a square. We observed in Section 5 that $I(2\sqrt{n}, n, s)$ has a simple zero at $s = 1$, while $L(s, 0)$ has a simple pole at $s = 1$. However if $l^2 - 4q^2n$ is 0, then $l$ is 0 modulo $q$ and so $\chi(l) = 0$. And (4.3 (iii)) shows that $\Gamma(s, 0)$ has a simple zero at $s = 1$. This completes the $\Delta = 0$ case. We know from Section 5 that $I(l, n, 1) = 0$ when $\Delta = l^2 - 4n > 0$. These terms contribute nothing unless $L(s, q\Delta)$ has a pole at $s = 1$. This happens exactly when $\Delta = qf^2$ for some $f \neq 0$, then $L(s, q^2f^2)$ is $\zeta(s)$ times a Dirichlet polynomial. If $l^2 - 4q^2n = qf^2$, then $\chi(l) = 0$; these terms drop out. On the other hand $\Delta = l^2 - 4n = qf^2$ terms are in one to one correspondence with divisors of $n = \nu\nu', \nu \in \mathcal{O}, \nu \notin \mathbb{Z}$. (There will be infinitely many such, if any.) But the simple zero of $I(l, n, s)$ will cancel the pole of $\zeta(s)$, and (4.3 (ii)) shows that $\Gamma(s, qf^2)$ has a simple zero at $s = 1$, since $Dq^{-1}$ is a square in this case. 

There are finitely many $l$ such that $\Delta < 0$. We use (5.1) in this case to evaluate the special function $I(l, n, s)$ at $s = 1$ in terms of Gegenbauer polynomials (see also [2], vol. 1, Sect. 3.15.1)

$$n^{(k-2)/2}C_{k-2}^1 \left( \frac{|l|}{\sqrt{4n}} \right) = \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}}, \quad \rho + \bar{\rho} = |l|, \quad \rho \bar{\rho} = n,$$

which is $P_{k,1}(l, n)$ in the notation of Zagier ([12] formula (18).) The value of $L(1, q\Delta) = \pi H(q\Delta)/\sqrt{q\Delta}$ is classical, $H(q\Delta)$ being the number of equivalence classes of binary quadratic forms $\phi$ of discriminant $q\Delta$ weighted by $1/\text{Aut}(\phi)$. Equivalently, $H(q\Delta)$ is the sum of class numbers of orders of discriminant $q\Delta/f^2$ in $\mathbb{Q}(\sqrt{q\Delta})$.

**THEOREM.** For the map $B$ defined in (0.1), the trace of the Hecke operator $T(n)B$ is $b(n, 1)$ where

$$b(n, 1) = -\frac{1}{4} \sum_{l^2 < 4q^2n} \chi(l)q^{2-k}P_{k,1}(l, q^2n)H(q(l^2 - 4q^2n))$$

$$-\frac{1}{4} \sum_{l^2 < 4n} a(l, n)P_{k,1}(l, n)H(q(l^2 - 4n))\Gamma(1, l^2 - 4n)$$

$$+\delta(\sqrt{n})\chi(\sqrt{n})n^{(k-2)/2} \frac{k-1}{24} (q^2 - 1).$$

**COROLLARY.** Let $\beta(f) = \langle \tilde{f}, \tilde{f} \rangle / \langle f, f \rangle ^2$, then $\beta(f) \in \mathbb{Q}(f)$. 
Proof. This follows from the fact that the Fourier coefficients \( b(n, 1) \) of \( \Phi_1(\tau) \) are rational in the theorem above, and Lemma 4, p. 792 of [11].

7. Examples

To convince ourselves the computations above are correct, we did some examples using Mathematica. In the case of weight \( k = 10 \), there are no cusp forms so \( \Phi_s(\tau) = 0 \) for all \( s \). With \( q = 5 \) we verified \( b(n, 1) = 0 \) for \( 1 \leq n \leq 20 \). In the case of weight \( k = 12 \), the space of cusp forms is spanned by the discriminant cusp form, so the Fourier coefficient \( b(n, 1) = \tau(n)b(1, 1) \), with \( \tau(n) \) Ramanujan’s tau function, and again the above relation holds for \( 1 \leq n \leq 20 \). The coefficient \( b(1, 1) = \beta \) was then computed (still weight \( k = 12 \)) for some small primes \( q \)

\[
\begin{align*}
q &= 5 & \beta &= 2^{12} \cdot 3^6 \cdot 7 / 5^8, \\
q &= 13 & \beta &= 2^{12} \cdot 3^9 \cdot 5^3 \cdot 7 \cdot 563 / 13^{10}, \\
q &= 17 & \beta &= 2^{19} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 2389 / 17^{10}, \\
q &= 29 & \beta &= 2^{12} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 13 \cdot 17 \cdot 317 / 29^{10}, \\
q &= 37 & \beta &= 2^{12} \cdot 3^{10} \cdot 5^6 \cdot 7 \cdot 89 \cdot 3889 / 37^{10}, \\
q &= 41 & \beta &= 2^{23} \cdot 3^6 \cdot 5^4 \cdot 7^2 \cdot 117413 / 41^{10}.
\end{align*}
\]

The formula for \( b(n, s) \) also reduces to a finite sum for \( s = 3, 5, \ldots, k - 1 \), with a different Gegenbauer polynomial and Cohen’s function instead of the Hurwitz–Kronecker class number. The computations are analogous to those in [12]. In this case the Dirichlet series \( D(s, f, \chi) \) converges absolutely, and we computed the first 100 terms of the series with \( s = 3, 5, 7 \)

\[
\begin{align*}
\frac{D(3, f, \chi)}{C_k(f, f)} &\approx 22.5795 & b(1, 3) &= \frac{2^{12} \cdot 3^2 \cdot 7 \cdot 2851}{5^{12} \cdot 13} \pi^4 \\
&\approx 22.5794729896, \\
\frac{D(5, f, \chi)}{C_k(f, f)} &\approx 20.30838094 & b(1, 5) &= \frac{2^{15} \cdot 3 \cdot 1511599}{5^{17} \cdot 13} \pi^8 \\
&\approx 20.3083809367, \\
\frac{D(7, f, \chi)}{C_k(f, f)} &\approx 19.903417716 & b(1, 7) &= \frac{2^{13} \cdot 3^2 \cdot 521 \cdot 295387}{5^{22} \cdot 13 \cdot 17} \pi^{12} \\
&\approx 19.9034177155.
\end{align*}
\]

Here again \( q = 5 \) and \( k = 12 \). The value \( 1.03536205679 \times 10^{-6} \) for the square of the norm of the discriminant function was taken from [12].
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