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Artin-Nagata properties and Cohen-Macaulay associated graded rings

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1. Introduction

Let R be a Noetherian local ring with infinite residue field k, and let I be an R-ideal. The *Rees algebra* $\mathcal{R} = R[It] \cong \bigoplus_{i \geqslant 0} I^i$ and the *associated graded ring* $G = gr_I(R) = \mathcal{R} \otimes_R R/I \cong \bigoplus_{i \geqslant 0} I^i/I^{i+1}$ are two graded algebras that reflect various algebraic and geometric properties of the ideal I. For instance, $\operatorname{Proj}(\mathcal{R})$ is the blow-up of $\operatorname{Spec}(R)$ along $V(I)$, and $\operatorname{Proj}(G)$ corresponds to the exceptional fibre of the blow-up. One is particularly interested in when the 'blow-up algebras' $\mathcal{R}$ and G are Cohen-Macaulay or Gorenstein: Besides being important in its own right, either property greatly facilitates computing various numerical invariants of these algebras, such as the Castelnuovo-Mumford regularity, or the number and degrees of their defining equations (see, for instance, [29], [5], or Section 4 of this paper).

The relationship between the Cohen-Macaulayness of $\mathcal{R}$ and G is fairly well understood: For some time, it was known that G is Cohen-Macaulay if $\mathcal{R}$ has this property (at least in case R is Cohen-Macaulay and $I \not\subset \sqrt{0}$, [24]), but recently, various criteria have been found for the converse to hold as well (e.g., [29], [33], [5], [30]). This shifts the focus of attention, at least in principle, to studying the Cohen-Macaulayness of G, and similar remarks apply for the Gorenstein property (which is less interesting for $\mathcal{R}$ than for G).

When investigating $\mathcal{R}$ or G, one first tries to simplify I by passing to a reduction: Recall that an ideal $J \subset I$ is called a *reduction* of I if the extension of Rees algebras $\mathcal{R}(J) \subset \mathcal{R}(I)$ is module finite, or equivalently, if $I^{r+1} = JI^r$ for some $r \geqslant 0$ ([31]). The least such r is denoted by $r_J(I)$. A reduction is *minimal* if it is minimal with respect to inclusion, and the *reduction number* $r(I)$ of I is defined as $\min\{r_J(I) \mid J \text{ a minimal reduction of } I\}$. Finally, the *analytic spread* $\ell(I)$ of I is the Krull dimension of the fiber ring $\mathcal{R} \otimes_R k \cong G \otimes_R k$, or equivalently, the minimal number of generators $\mu(J)$ of any minimal reduction J of I ([31]). Philosophically speaking, J is a 'simplification' of I, with the reduction number $r(I)$ measuring

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how closely the two ideals are related. While the passage of algebraic properties from $\mathcal{R}(J)$ to $\mathcal{R}(I)$ is by no means a simple matter, one might hope for some success if $r(I)$ is 'small'. This line of investigation was initiated by S. Huckaba and C. Huneke, who were able to treat the case where $\ell(I) \leqslant \operatorname{grade} I + 2$ and $r(I) \leqslant 1$ ([20], [21]).

Now let I be an ideal with grade g, minimal number of generators n, analytic spread ℓ, and reduction number r. Under various additional assumptions G was shown to be Cohen-Macaulay when $\ell \leqslant g+2$ and $r \leqslant 1$ in [20], [21], when $r \leqslant 1$ in [41], [38], when $\ell \leqslant g+2$ and $r \leqslant 2$ in [11], [12], [4], [3], when $n \leqslant \ell+1$ and $r \leqslant \ell - g + 1$ in [33], when $r \leqslant \ell - g + 1$ (and sufficiently many powers of I have high depth) in [35], when $\ell \leqslant g+2$ and $r \leqslant 3$ in [2].

We are going to present a comparatively short and self-contained proof of a more general result that contains essentially all the above cases. Our main technique is to exploit the Artin-Nagata properties of the ideal. To explain this, we first recall the notion of residual intersection, which generalizes the concept of linkage to the case where the two 'linked' ideals may not have the same height.

DEFINITION 1.1. ([6], [28]). Let R be a local Cohen-Macaulay ring, let I be an R-ideal of grade g, let K be a proper R-ideal, and let $s \geqslant g$ be an integer.

(a) K is called an *s-residual intersection* of I if there exists an R-ideal $\mathfrak{a} \subset I$, such that $K = \mathfrak{a} : I$ and $\operatorname{ht} K \geqslant s \geqslant \mu(\mathfrak{a})$.
(b) K is called a *geometric s-residual intersection* of I, if K is an s-residual intersection of I and if in addition $\operatorname{ht} I + K > s$.

We now define what we mean by Artin-Nagata properties.

DEFINITION 1.2. ([38]). Let R be a local Cohen-Macaulay ring, let I be an R-ideal of grade g, and let s be an integer.

(a) We say that I satisfies AN_s if for every $g \leqslant i \leqslant s$ and every i-residual intersection K of I, R/K is Cohen-Macaulay.
(b) We say that I satisfies AN_s^- if for every $g \leqslant i \leqslant s$ and every geometric i-residual intersection K of I, R/K is Cohen-Macaulay.

It is known that any perfect ideal of grade 2, any perfect Gorenstein ideal of grade 3, or more generally, any ideal in the linkage class of a complete intersection satisfies AN_s for every s (at least if R is Gorenstein, [28]). (See Section 2 for more examples of ideals satisfying AN_s.)

In the context of the questions we are interested in, one usually assumes the local condition G_s of [6]: The ideal I satisfies G_s if $\mu(I_p) \leqslant \dim R_p$ for every $p \in V(I)$ with $\dim R_p \leqslant s-1$, and I satisfies G_∞ if G_s holds for every s. We are now ready to state our main results:

THEOREM 3.1. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction* *number r, let $k \geqslant 1$ be an integer with $r \leqslant k$, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-\max\{2,k\}}$, and that depth $R/I^j \geqslant d - \ell + k - j$ for $1 \leqslant j \leqslant k$.*

Then G is Cohen-Macaulay, and if $g \geqslant 2$, $\mathcal{R}$ is Cohen-Macaulay.

As a first corollary, one obtains that G is Cohen-Macaulay, if R is Cohen-Macaulay and I is a strongly Cohen-Macaulay R-ideal satisfying G_ℓ and having $r \leqslant \ell - g + 1$. (See Section 3 for further applications and a discussion of the assumptions in Theorem 3.1.)

We also consider the question of when G is Gorenstein. Generalizing earlier work from [18], [11], [13], [33], [35], [16] (see Section 5 for precise references), we present two conditions for the Gorensteinness of G, a sufficient one, very much in the spirit of Theorem 3.1, and a necessary one. Combining both results one concludes that the Gorenstein property of G corresponds to the reduction number of I being 'very small':

COROLLARY 5.3. *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, let $k \geqslant 1$ be an integer, assume that I has no embedded associated primes, that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-\max\{2,k\}}$, and that depth $R/I^j \geqslant d - \ell + k - j + 1$ for $1 \leqslant j \leqslant k$. Any two of the following conditions imply the third one:*

(a) $r \leqslant k$.
(b) depth $R/I^j \geqslant d - g - j + 1$ for $1 \leqslant j \leqslant \ell - g - k$.
(c) *G is Gorenstein.*

We mention one simple obstruction for G to be Gorenstein, generalizing a result from [33]: Assume I is generically a complete intersection and the Koszul homology modules H_i of I are Cohen-Macaulay for $0 \leqslant i \leqslant \frac{\ell-g}{2}$, then the Gorensteinness of G implies that I satisfies G_∞ and is strongly Cohen-Macaulay. (See Section 5 for further applications along these lines.)

2. Residual Intersections

In this section we review some basic facts about residual intersections. We also prove two technical results (Lemmas 2.5 and 2.8) that will play a crucial role later in the paper.

Let $I = (a_1, \ldots, a_n)$ be an ideal in a local Cohen-Macaulay ring R of dimension d. Recall that $a_1, \ldots, a_n$ is a *d-sequence* if $[(a_1, \ldots, a_i) : (a_{i+1})] \cap I = (a_1, \ldots, a_i)$ for $0 \leqslant i \leqslant n - 1$. A *$d$-sequence* is called *unconditioned* in case every permutation of the elements forms a d-sequence. The ideal I is *strongly Cohen-Macaulay* if all Koszul homology modules $H_i = H_i(a_1, \ldots, a_n)$ of I are Cohen-Macaulay, and more generally, I satisfies *sliding depth* if depth $H_i \geqslant d - n + i$ for every

i. (These conditions are independent of the generating set.) Standard examples of strongly Cohen-Macaulay ideals include perfect ideals of grade 2 ([7]), perfect Gorenstein ideals of grade 3, or more generally, ideals in the linkage class of a complete intersection ([25]).

We begin by mentioning two results that guarantee Artin-Nagata properties and describe the canonical module $\omega_{R/K}$ of a residual intersection K.

THEOREM 2.1. ([19], cf. also [26]). *Let R be a local Cohen-Macaulay ring, and let I be an R-ideal satisfying G_s and sliding depth.*

Then I satisfies AN_s.

THEOREM 2.2. ([38]). *Let R be a local Gorenstein ring of dimension d, let I be an R-ideal of grade g, assume that I satisfies G_s and that depth $R/I^j \geqslant d-g-j+1$ whenever $1 \leqslant j \leqslant s-g+1$. Then:*

(a) *I satisfies AN_s.*
(b) *For every $g \leqslant i \leqslant s$ and every i-residual intersection $K = \mathfrak{a} : I$ of I, $\omega_{R/K} \cong I^{i-g+1}/\mathfrak{a}I^{i-g}$, where $\omega_{R/K} \cong I^{i-g+1} + K/K$ in case K is a geometric i-residual intersection.*

The above assumption that depth $R/I^j \geqslant d-g-j+1$ for $1 \leqslant j \leqslant s-g+1$, is automatically satisfied if I is a strongly Cohen-Macaulay ideal satisfying G_s, as can be easily seen from the Approximation Complex ([17, the proof of 5.1]).

The next two lemmas are refinements of results from [26] and [19].

LEMMA 2.3. ([38]). *Let R be a local Cohen-Macaulay ring with infinite residue field, let $\mathfrak{a} \subset I$ be (not necessarily distinct) R-ideals with $\mu(\mathfrak{a}) \leqslant s \leqslant \operatorname{ht} \mathfrak{a} : I$, and assume that I satisfies G_s.*

(a) *There exists a generating sequence $a_1, \ldots, a_s$ of $\mathfrak{a}$ such that for every $0 \leqslant i \leqslant s-1$ and every subset $\{\nu_1, \ldots, \nu_i\}$ of $\{1, \ldots, s\}$, $\operatorname{ht}(a_{\nu_1}, \ldots, a_{\nu_i}) : I \geqslant i$ and $\operatorname{ht} I + (a_{\nu_1}, \ldots, a_{\nu_i}) : I \geqslant i+1$.*
(b) *Assume that I satisfies AN_{s-2}^-. Then any sequence $a_1, \ldots, a_s$ as in (a) forms an unconditioned d-sequence.*
(c) *Assume that I satisfies AN_t^- for some $t \leqslant s-1$ and that $\mathfrak{a} \neq I$, write $\mathfrak{a}_i = (a_1, \ldots, a_i)$, $K_i = \mathfrak{a}_i : I$, and let '$^-$' denote images in R/K_i. Then for $0 \leqslant i \leqslant t+1$:*
   (i) *$K_i = \mathfrak{a}_i : (a_{i+1})$ and $\mathfrak{a}_i = I \cap K_i$, if $i \leqslant s-1$.*
   (ii) *depth $R/\mathfrak{a}_i = d-i$.*
   (iii) *K_i is unmixed of height i.*
   (iv) *$\bar{a}_{i+1}$ is regular on $\bar{R}$ and $\operatorname{ht} \bar{I} = 1$, if $g-1 \leqslant i \leqslant s-1$.*

LEMMA 2.4. *Let R be a local Cohen-Macaulay ring and let I be an R-ideal satisfying AN_t^-.*

(a) ([38, 1.10]) *Assume that I satisfies G_{t+1}, and let $p \in V(I)$; then I_p satisfies AN_t^-.*

(b) *Assume that $I \cap (0 : I) = 0$, and let '$^-$' denote images in $R/0 : I$; then $\bar{I}$ satisfies AN_t^-.*

Proof. To prove part (b), let $\operatorname{ht} \bar{I} \leqslant i \leqslant t$ and let $(\bar{a}_1, \ldots, \bar{a}_i) : \bar{I}$ be a geometric i-residual intersection of $\bar{I}$, where we may assume that $a_i \in I$. Now write $\mathfrak{a} = (a_1, \ldots, a_i)$ and $K = \mathfrak{a} : I$. Then $0 : I \subset K \subset (\mathfrak{a} + 0 : I) : I = [\mathfrak{a} + I \cap (0 : I)] : I = \mathfrak{a} : I = K$, which shows that $R/K \cong \bar{R}/\bar{\mathfrak{a}} : \bar{I}$ and $R/I + K \cong \bar{R}/\bar{I} + \bar{\mathfrak{a}} : \bar{I}$. Thus K is a geometric i-residual intersection of I, and therefore $\bar{R}/\bar{\mathfrak{a}} : \bar{I} \cong R/K$ is Cohen-Macaulay. □

Our next lemma generalizes a result from [38].

LEMMA 2.5. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let $\mathfrak{a} \subset I$ be R-ideals with $\mu(\mathfrak{a}) \leqslant s \leqslant \operatorname{ht} \mathfrak{a} : I$, let k and t be integers, assume that I satisfies G_s and AN_{s-3}^- locally in codimension $s - 1$, that I satisfies AN_t^-, and that depth $R/I^j \geqslant d - s + k - j$ whenever $1 \leqslant j \leqslant k$. Let $\mathfrak{a}_i$ and K_i be the ideals as defined in Lemma 2.3. Then*

(a) depth $R/\mathfrak{a}_i I^j \geqslant \min\{d - i, d - s + k - j\}$ *whenever* $0 \leqslant i \leqslant s$ *and* $\max\{0, i - t - 1\} \leqslant j \leqslant k$.

(b) $[\mathfrak{a}_i : (a_{i+1})] \cap I^j = \mathfrak{a}_i I^{j-1}$ *whenever* $0 \leqslant i \leqslant s-1$ *and* $\max\{1, i-t\} \leqslant j \leqslant k$.

(c) $K_i \cap I^j = \mathfrak{a}_i I^{j-1}$ *whenever* $0 \leqslant i \leqslant s$ *and* $\max\{1, i - t\} \leqslant j \leqslant k + 1$, *provided that* $\operatorname{ht} I + \mathfrak{a} : I \geqslant s + 1$ *and I satisfies AN_{s-2}^- locally in codimension s.*

Proof. We first show that if (a) holds for i, then so do (b) and (c). However, it suffices to check the equalities in (b) and (c) locally at every prime $p \in \operatorname{Ass}(R/\mathfrak{a}_i I^{j-1})$, where for (c) we may even assume that $p \in V(I)$. Now by (a), $\dim R_p \leqslant \max\{i, s - k + j - 1\}$. Thus in the situation of (b), $\dim R_p \leqslant s - 1$, hence $I_p = \mathfrak{a}_p \neq R_p$ and this ideal satisfies AN_{s-2}^- by Lemma 2.3 ((c)(iii)). On the other hand, with the assumption of (c), $\dim R_p \leqslant s$, and again $I_p = \mathfrak{a}_p \neq R_p$ satisfies AN_{s-2}^-. Now in either case, Lemma 2.3 (b) implies that the generators $a_1, \ldots, a_s$ of I_p form a d-sequence in R_p, and the assertions follow from [23, Theorem 2.1].

Thus it suffices to prove (a), which we are going to do by induction on i, $0 \leqslant i \leqslant s$. The assertion being trivial for $i = 0$, we may assume that $0 \leqslant i \leqslant s - 1$, and that (a) and hence (b) hold for i. We need to verify (a) for $i + 1$. But for $j = 0$ (which can only occur if $i + 1 \leqslant t + 1$), our assertion follows from Lemma 2.3 ((c)(ii)). Thus we may suppose that $j \geqslant 1$. But then by part (b) for i,

$$\begin{aligned} \mathfrak{a}_i I^j \cap a_{i+1} I^j &= a_{i+1}[(\mathfrak{a}_i I^j : (a_{i+1})) \cap I^j] \subset a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^j] \\ &= a_{i+1} \mathfrak{a}_i I^{j-1} \subset \mathfrak{a}_i I^j \cap a_{i+1} I^j. \end{aligned}$$

Hence, writing $\mathfrak{a}_{i+1} = \mathfrak{a}_i + (a_{i+1})$, we obtain an exact sequence

$$0 \to a_{i+1}\mathfrak{a}_i I^{j-1} \to \mathfrak{a}_i I^j \oplus a_{i+1} I^j \to \mathfrak{a}_{i+1} I^j \to 0. \tag{2.6}$$

On the other hand, by part (b) for $i = 0$, $[0 : (a_{i+1})] \cap \mathfrak{a}_i I^{j-1} \subset [0 : (a_{i+1})] \cap I^j = 0$, and therefore $a_{i+1}\mathfrak{a}_i I^{j-1} \cong \mathfrak{a}_i I^{j-1}$, $a_{i+1} I^j \cong I^j$. If $i = 0$, the latter isomorphism implies the required depth estimate for $R/\mathfrak{a}_{i+1} I^j$, whereas if $i \geqslant 1$, we use (2.6) together with part (a) for i. □

We will apply Lemma 2.5 by way of the following remark:

REMARK 2.7. ([38, 1.11]). Let R be a local Cohen-Macaulay ring with infinite residue field, let I be an R-ideal with analytic spread ℓ satisfying G_ℓ and $AN_{\ell-3}^-$ locally in codimension $\ell - 1$, and let J be a minimal reduction of I. Then $\operatorname{ht} J : I \geqslant \ell$.

A special case of our next result can be found in [35].

LEMMA 2.8. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, let k and t be integers with $r \leqslant k$ and $t \geqslant \ell - k - 1$, assume that I satisfies G_ℓ and $AN_{\ell-3}^-$ locally in codimension $\ell - 1$, that I satisfies AN_t^-, and that depth $R/I^j \geqslant d - \ell + k - j$ for $1 \leqslant j \leqslant k$. Let J be a minimal reduction of I with $r_J(I) = r$, write $G = gr_I(R)$, for $a \in I$ let a' denote the image of a in $[G]_1$, and for $\mathfrak{a} = J$, let $a_1, \ldots, a_\ell$ and $\mathfrak{a}_i$ be as defined in Lemma 2.3 (a). Then:*

(a) $\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$ *whenever* $0 \leqslant i \leqslant \ell - 1$ *and* $j \geqslant \max\{1, i - t\}$, *or* $i = \ell$ *and* $j \geqslant r + 1$.

(b) $a'_1, \ldots, a'_g$ *form a G-regular sequence, and* $[(a'_1, \ldots, a'_i) :_G (a'_{i+1})]_j = [(a'_1, \ldots, a'_i)]_j$ *whenever* $g \leqslant i \leqslant \ell - 2$ *and* $j \geqslant \max\{1, i - t\}$, *or* $i = \ell - 1$ *and* $j \geqslant \max\{1, \ell - t - 1, r - 1\}$.

Proof. (a): If $i = \ell$, our claim is clear since $j \geqslant r + 1$ and therefore $I^j = JI^{j-1} = \mathfrak{a}_\ell I^{j-1}$. Furthermore, if $0 \leqslant i \leqslant \ell - 1$ and $1 \leqslant j \leqslant k$, then the assertion follows from Lemma 2.5 (b) with $s = \ell$. Thus we may assume that $j \geqslant k + 1$. In this case, we are going to prove by decreasing induction on i, $0 \leqslant i \leqslant \ell$, that $\mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$. This equality being clear for $i = \ell$, we may suppose that $0 \leqslant i \leqslant \ell - 1$.

Since $i - t \leqslant k$, and since $\mathfrak{a}_i \cap I^\nu = \mathfrak{a}_i I^{\nu-1}$ is already known for $\max\{1, i - t\} \leqslant \nu \leqslant k$ and for $\nu = 1$, we have that the desired equality holds for $\nu = \max\{1, i - t\}$. Thus we may assume that $j \geqslant \max\{2, i - t + 1\}$ and that by increasing induction on j,

$$\mathfrak{a}_i \cap I^{j-1} = \mathfrak{a}_i I^{j-2}. \tag{2.9}$$

Furthermore, by decreasing induction on i,

$$\mathfrak{a}_{i+1} \cap I^j = \mathfrak{a}_{i+1} I^{j-1}. \tag{2.10}$$

Finally, Lemma 2.3 ((c)(i)) (if $k = 0$) and Lemma 2.5 (b) (if $k \geqslant 1$) imply that $(\mathfrak{a}_i : (a_{i+1})) \cap I^{\max\{1,i-t\}} \subset \mathfrak{a}_i$, which upon intersection with I^{j-1} yields

$$(\mathfrak{a}_i : (a_{i+1})) \cap I^{j-1} = \mathfrak{a}_i \cap I^{j-1}. \tag{2.11}$$

Now we obtain

$$\begin{aligned}
\mathfrak{a}_i \cap I^j &= \mathfrak{a}_i \cap \mathfrak{a}_{i+1} \cap I^j = \mathfrak{a}_i \cap \mathfrak{a}_{i+1} I^{j-1} && \text{by (2.10)}\\
&= \mathfrak{a}_i \cap (\mathfrak{a}_i I^{j-1} + a_{i+1} I^{j-1})\\
&= \mathfrak{a}_i I^{j-1} + \mathfrak{a}_i \cap a_{i+1} I^{j-1}\\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}[(\mathfrak{a}_i : (a_{i+1})) \cap I^{j-1}]\\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}(\mathfrak{a}_i \cap I^{j-1}) && \text{by (2.11)}\\
&= \mathfrak{a}_i I^{j-1} + a_{i+1}\mathfrak{a}_i I^{j-2} && \text{by (2.9)}\\
&= \mathfrak{a}_i I^{j-1}.
\end{aligned}$$

(b): We first show that $a'_1, \ldots, a'_g$ form a regular sequence. If $\ell = g$, then $k \leqslant 1$ and R/I is Cohen-Macaulay, hence the assertion follows from [41, 2.4]. Thus we may assume that $g \leqslant \ell - 1$. We may further suppose that $t \geqslant g - 1$. But then part (a) and [40, 2.6] imply that $a'_1, \ldots, a'_g$ form a G-regular sequence.

Now let $u \in [(a'_1, \ldots, a'_i) : (a'_{i+1})]_j$. Picking an element $x \in I^j$ with $x + I^{j+1} = u$, we have $a_{i+1}x \in \mathfrak{a}_i + I^{j+2}$, and therefore by part (a), $a_{i+1}x \in \mathfrak{a}_{i+1} \cap (\mathfrak{a}_i + I^{j+2}) = \mathfrak{a}_i + \mathfrak{a}_{i+1} \cap I^{j+2} = \mathfrak{a}_i + \mathfrak{a}_{i+1} I^{j+1} = \mathfrak{a}_i + a_{i+1} I^{j+1}$. Thus $a_{i+1}(x - y) \in \mathfrak{a}_i$ for some $y \in I^{j+1}$. Since $x - y + I^{j+1} = x + I^{j+1} = u$, we may replace x by $x - y$ to assume that $a_{i+1}x \in \mathfrak{a}_i$. But then by Lemma 2.5 (b) and by part (a), $x \in [\mathfrak{a}_i : (a_{i+1})] \cap I^j = \mathfrak{a}_i \cap I^j = \mathfrak{a}_i I^{j-1}$, which implies $u \in (a'_1, \ldots, a'_i)$. □

3. Conditions for the Cohen-Macaulayness of the Associated Graded Ring

Throughout, R will be a local Cohen-Macaulay ring, I will be a proper R-ideal, G and $\mathcal{R}$ will denote the associated graded ring and the Rees algebra of I.

So far, the Cohen-Macaulayness of G has been shown under suitable assumptions in [20, 2.9], [21, 4.1], [41, 4.5 and 4.13], [38, 4.1 and 4.9], [11, 1.5], [12, 1.4], [4, 8.2], [3, 3.1], [33, 4.10], [35, 5.5 and 5.8], and [2, 2.10]. The goal of the present section is to give a self-contained proof of a general theorem, which essentially contains these results as special cases (Theorem 3.1). After this paper was written, it came to our attention that other generalizations have been recently proved by Goto, Nakamura, Nishida ([14]), and by Aberbach ([1]).

THEOREM 3.1. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ and reduction number r, let $k \geqslant 1$ be an integer with $r \leqslant k$, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in* *codimension* $\ell-1$, *that* I *satisfies* $AN^-_{\ell-\max\{2,k\}}$, *and that depth* $R/I^j \geqslant d-\ell+k-j$ *for* $1 \leqslant j \leqslant k$.

Then G *is Cohen-Macaulay.*

We are first going to discuss the assumptions of Theorem 3.1. Notice that the Artin-Nagata condition and the depth assumption on the powers both involve the parameter k: Increasing k has the effect of weakening the former condition and of strengthening the latter, or vice versa. On the other hand, both assumptions are not completely independent, as we will see below.

The condition depth $R/I^j \geqslant d - \ell + k - j$ for $1 \leqslant j \leqslant k$, gives a linearly decreasing bound on the depths of the powers of I so that depth $R/I^k \geqslant d - \ell$, where the latter inequality is necessary for G to be Cohen-Macaulay (e.g. [9, 3.3]). Also notice that if $\ell = d$, then it suffices to require depth $R/I^j \geqslant d - \ell + k - j$ in the range $1 \leqslant j \leqslant k - 1$. Moreover, for any strongly Cohen-Macaulay ideal I satisfying G_ℓ, one has depth $R/I^j \geqslant d - g - j + 1$ for $1 \leqslant j \leqslant \ell - g + 1$. Finally, the depth assumptions in Theorem 3.1 imply that $k \leqslant \ell - g + 1$, which can be seen by setting $j = 1$.

As to the Artin-Nagata properties, notice that these assumptions automatically hold if $\ell = g + 2$ and $k = 3$, or if I is $AN^-_{\ell-2}$ (Lemma 2.4 (a)). On the other hand, $AN^-_{\ell-2}$ is always satisfied if $\ell \leqslant g + 1$, or if I satisfies G_ℓ and sliding depth (Theorem 2.1), or if R is Gorenstein, I satisfies G_ℓ, and depth $R/I^j \geqslant d-g-j+1$ for $1 \leqslant j \leqslant \ell - g - 1$ (Theorem 2.2 (a)). In particular, any reference to the Artin-Nagata property can be omitted in Theorem 3.1 if $\ell \leqslant g + 1$, or if R is Gorenstein and $k = \ell - g + 1$.

This discussion shows that the results mentioned at the beginning of the section are indeed special cases of Theorem 3.1; it also gives the following application:

COROLLARY 3.2. *Let* R *be a local Cohen-Macaulay ring with infinite residue field, let* I *be a strongly Cohen-Macaulay* R*-ideal with grade* g, *analytic spread* ℓ, *and reduction number* r, *assume that* I *satisfies* G_ℓ *and that* $r \leqslant \ell - g + 1$.

Then G *is Cohen-Macaulay.*

We want to list several other consequences of Theorem 3.1, before turning to its proof.

COROLLARY 3.3. *Let* R *be a local Gorenstein ring of dimension* d *with infinite residue field, let* I *be an* R*-ideal with grade* g, *analytic spread* ℓ, *and reduction number* r, *let* $k \geqslant 1$ *be an integer with* $r \leqslant k$, *assume that* I *satisfies* G_ℓ *and sliding depth locally in codimension* $\ell - 1$ *and that depth* $R/I^j \geqslant d - g - j + 1$ *for* $1 \leqslant j \leqslant \max\{k, \ell - g + 1 - k\}$.

Then G *is Cohen-Macaulay.*

Proof. The assertion follows from Theorem 3.1, in conjunction with Theorems 2.1 and 2.2 (a). □

COROLLARY 3.4. *With the assumptions of Theorem 3.1, Corollary 3.2, or Corollary 3.3, $\mathcal{R}$ is Cohen-Macaulay if and only if $g \geqslant 2$, or $g = 1$ and $r \neq \ell$, or I is nilpotent.*

Proof. We already know that G is Cohen-Macaulay and that $r \leqslant \ell - g + 1$. By the latter condition, $\mathcal{R}$ is Cohen-Macaulay if I is nilpotent. On the other hand, if I is not nilpotent, then $\mathcal{R}$ is Cohen-Macaulay if and only if $g > 0$ and $r < \ell$ ([33, 3.6], cf. also [29] and [5]). □

COROLLARY 3.5. *Let R be a local Cohen-Macaulay ring with infinite residue field, let I be a perfect R-ideal of grade 2 with analytic spread ℓ and reduction number r, and assume that I satisfies G_ℓ. The following are equivalent:*

(a) $r < \ell$.
(b) $r = 0$ *or* $r = \ell - 1$
(c) $\mathcal{R}$ *is Cohen-Macaulay.*

Proof. (a) $\Leftrightarrow$ (c): This follows from Corollary 3.4 and [33, 3.6] (or [29], [5]).

(a) $\Rightarrow$ (b): If $r \neq 0$, then $r \geqslant \ell - 1$, as can be seen from the Approximation Complex (cf., e.g., [39, the proof of 2.5], or Proposition 4.10). □

EXAMPLE 3.6. (cf. also [38, 2.12 and 4.12]). Let k be a local Gorenstein ring with infinite residue field, let X be an alternating 5 by 5 matrix of variables, let Y be a 5 by 1 matrix of variables, set $R = k[X, Y]$ (possibly localized at the irrelevant maximal ideal), and let $I = Pf_4(X) + I_1(XY)$ be the ideal generated by the 4 by 4 Pfaffians of X and the entries of the product matrix XY.

This example has already played some role in the study of minimal free resolutions ([34]). Furthermore, R/I is itself the associated graded ring of the ideal $Pf_4(X)$ in $k[X]$ ([25, 2.2]). From the latter description one concludes that grade $I = 5$, that I is a complete intersection in codimension 9 ([24, the proof of Proposition 2.1]), and that R/I is Cohen-Macaulay ([25, 2.2]). Furthermore, $\ell(I) = 9$, and a computation using MACAULAY shows that R/I^j is Cohen-Macaulay for $2 \leqslant j \leqslant 3$. Thus I satisfies AN_7 (Theorem 2.2 (a)), and therefore $r(I) = 1$ (Proposition 4.7).

Hence we may apply Corollary 3.4 to conclude that $\mathcal{R}$ is Cohen-Macaulay.

We now turn to the proof of Theorem 3.1. The statement of this theorem was somewhat inspired by [35], the proof we present however, is quite different. It is based on the next proposition, which provides a general criterion for a homogeneous ring to be Cohen-Macaulay. As a matter of notation, we write $[M]_{\geqslant i}$ for the truncated submodule $\bigoplus_{j \geqslant i} M_j$ of a graded module $M = \bigoplus_j M_j$.

PROPOSITION 3.7. *Let S be a homogeneous Noetherian ring of dimension d with S_0 local, write $I = S_+$, let $b_1, \ldots, b_\ell$ be linear forms in S, set $\mathfrak{b}_i = (b_1, \ldots, b_i)$ for $-1 \leqslant i \leqslant \ell$ (where $(\emptyset) = 0$), $J = \mathfrak{b}_\ell$, and let g be an integer with $0 \leqslant g \leqslant \ell$.*

Further let $H^\bullet(-)$ denote local cohomology with support in the irrelevant maximal ideal of S.

Assume that $I^{k+1} \subset J$ (i.e., J is a reduction of I with $r_J(I) \leqslant k$), that $[\mathfrak{b}_i : (b_{i+1})]_{\geqslant i-g+1} = [\mathfrak{b}_i]_{\geqslant i-g+1}$ for $0 \leqslant i \leqslant \ell - 1$, that depth $[S/\mathfrak{b}_i]_{i-g+1} \geqslant d-i-1$ for $g-1 \leqslant i \leqslant \ell-1$, and that depth $[S/J]_j \geqslant d-\ell$ for $\ell-g+1 \leqslant j \leqslant k$.

Then S is a Cohen-Macaulay ring. Furthermore, socle$(H^d(S))$ is concentrated in degrees at least $-g$ and at most $\max\{-g, k-\ell\}$.

Proof. To simplify notation we factor out $\mathfrak{b}_g$ and assume $g = 0$ (notice that $b_1, \ldots, b_g$ form an S-regular sequence and that $[S/\mathfrak{b}_{g-1}]_0 = S_0 = [S/\mathfrak{b}_{-1}]_0$). Now $[\mathfrak{b}_i : (b_{i+1})]_{\geqslant i+1} = [\mathfrak{b}_i]_{\geqslant i+1}$ whenever $0 \leqslant i \leqslant \ell - 1$, depth $[S/\mathfrak{b}_{i-1}]_i \geqslant d - i$ whenever $0 \leqslant i \leqslant \ell$, and depth $[S/J]_j \geqslant d - \ell$ whenever $\ell + 1 \leqslant j \leqslant k$.

For $0 \leqslant i \leqslant \ell$ consider the graded S-modules $M_{(i)} = [S/\mathfrak{b}_i]_{\geqslant i+1} = I^{i+1}/\mathfrak{b}_i I^i$, and $N_{(i)} = I^i/\mathfrak{b}_{i-1}I^{i-1} + b_i I^i$ (where $I^{-1} = I^0 = S$). Notice that $[N_{(i)}]_{\geqslant i+1} = M_{(i)}$ and $[N_{(i)}]_i = [S/\mathfrak{b}_{i-1}]_i$, which yields exact sequences

$$0 \to M_{(i)} \to N_{(i)} \to [S/\mathfrak{b}_{i-1}]_i \to 0. \tag{3.8}$$

On the other hand, if $0 \leqslant i \leqslant \ell - 1$, then $N_{(i+1)} = M_{(i)}/b_{i+1}M_{(i)}$, and since $[\mathfrak{b}_i : (b_{i+1})]_{\geqslant i+1} = [\mathfrak{b}_i]_{\geqslant i+1}$ it follows that $0 :_{M_{(i)}} (b_{i+1}) = 0$. Thus, in the range $0 \leqslant i \leqslant \ell - 1$, we have the exact sequences

$$0 \to M_{(i)}(-1) \xrightarrow{b_{i+1}} M_{(i)} \to N_{(i+1)} \to 0. \tag{3.9}$$

Also notice that $N_{(0)} = S$. Hence it suffices to prove by decreasing induction on i, $0 \leqslant i \leqslant \ell$, that $\operatorname{depth}_S N_{(i)} \geqslant d - i$, and that socle$(H^{d-i}(N_{(i)}))$ is concentrated in degrees at least i and at most $\max\{i, k - \ell + i\}$.

If $i = \ell$, then $N_{(\ell)} = [S/\mathfrak{b}_{\ell-1}]_\ell \oplus \bigoplus_{j=\ell+1}^k [S/J]_j$ has depth at least $d - \ell$ as an S_0-module, and hence as an S-module as well. Furthermore, $H^{d-\ell}(N_{(\ell)})$ is concentrated in degrees at least ℓ and at most $\max\{\ell, k\}$ (see, e.g., [10, 2.2]).

So let $0 \leqslant i \leqslant \ell - 1$ and suppose that our assertions hold for $i + 1$. From (3.9), since b_{i+1} is regular on $M_{(i)}$, we see that

$$\operatorname{depth} M_{(i)} \geqslant d - i. \tag{3.10}$$

Then (3.9) yields an exact sequence

$$0 \to H^{d-i-1}(N_{(i+1)}) \to H^{d-i}(M_{(i)})(-1) \xrightarrow{b_{i+1}} H^{d-i}(M_{(i)}),$$

which implies

$$\operatorname{socle}(H^{d-i}(M_{(i)}) \cong \operatorname{socle}(H^{d-i-1}(N_{(i+1)}))(1). \tag{3.11}$$

On the other hand, $\operatorname{depth}_S[S/\mathfrak{b}_{i-1}]_i = \operatorname{depth}_{S_0}[S/\mathfrak{b}_{i-1}]_i \geqslant d - i$. Hence (3.8) and (3.10) show that $\operatorname{depth}_S N_{(i)} \geqslant d - i$. Furthermore, (3.8) induces an exact sequence

$$0 \to H^{d-i}(M_{(i)}) \to H^{d-i}(N_{(i)}) \to H^{d-i}([S/\mathfrak{b}_{i-1}]_i). \tag{3.12}$$

Now taking socles and using (3.12) as well as (3.11), we conclude from our induction hypothesis that socle$(H^{d-i}(N_{(i)}))$ is concentrated in degrees at least i and at most $\max\{i, k-\ell+i\}$. $\square$

We are now ready to prove a special case of Theorem 3.1, and to provide some information about the canonical module ω_G, which will be needed in the last section of this paper.

PROPOSITION 3.13. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, assume that I satisfies G_ℓ and $AN_{\ell-3}^-$ locally in codimension $\ell-1$, that* depth $R/I^j \geqslant d-g-j+1$ *for* $1 \leqslant j \leqslant \ell-g+1$, *and that* $r \leqslant \ell-g+1$.

Then $G = gr_I(R)$ is Cohen-Macaulay. Furthermore, ω_G (in case it exists) is generated in degrees $\min\{g, \ell-r\}$ *and g.*

Proof. Let J be a minimal reduction of I with $r_J(I) = r$, let $a_1, \ldots, a_\ell$ and $\mathfrak{a}_i$ be as defined in Lemma 2.8, and let a_i' denote the image of a_i in $[G]_1$. We wish to apply Proposition 3.7 with $k = \max\{\ell-g, r\}$ to the ring $S = G$ and the linear forms $b_i = a_i'$, $1 \leqslant i \leqslant \ell$. From Lemma 2.8 (b) with $t = g-1$ we already know that $[\mathfrak{b}_i : (b_{i+1})]_{\geqslant i-g+1} = [\mathfrak{b}_i]_{\geqslant i-g+1}$ for $0 \leqslant i \leqslant \ell-1$. Thus it suffices to verify that depth $[S/\mathfrak{b}_i]_{i-g+1} \geqslant d-i-1$ for $g-1 \leqslant i \leqslant \ell-1$, and that depth $[S/J]_{\ell-g+1} \geqslant d-\ell$.

Since $[\mathfrak{b}_i : (b_{i+1})]_{\geqslant i-g+1} = [\mathfrak{b}_i]_{\geqslant i-g+1}$ for $0 \leqslant i \leqslant \ell-1$, there are exact sequences

$$0 \to [S/\mathfrak{b}_i]_j \to [S/\mathfrak{b}_i]_{j+1} \to [S/\mathfrak{b}_{i+1}]_{j+1} \to 0 \tag{3.14}$$

whenever $0 \leqslant i \leqslant \ell-1$ and $j \geqslant i-g+1$. On the other hand by our assumption, depth $[S/\mathfrak{b}_0]_j$ = depth $[S]_j \geqslant d-g-j$ for $j \leqslant \ell-g$. Hence using (3.14), we can see by induction on i that depth $[S/\mathfrak{b}_i]_j \geqslant d-g-j$ whenever $0 \leqslant i \leqslant \ell-1$ and $i-g+1 \leqslant j \leqslant \ell-g$. In particular, depth $[S/\mathfrak{b}_i]_{i-g+1} \geqslant d-i-1$ for $0 \leqslant i \leqslant \ell-1$. As to $[S/J]_{\ell-g+1}$, notice that this module is $I^{\ell-g+1}/JI^{\ell-g} + I^{\ell-g+2} = I^{\ell-g+1}/JI^{\ell-g} + JI^{\ell-g+1} = I^{\ell-g+1}/JI^{\ell-g}$, which has the required depth by Lemma 2.5 (a).

Now Proposition 3.7 and local duality imply that G is Cohen-Macaulay, and that ω_G is generated in degrees $\min\{g, \ell-k\} = \min\{g, \ell-r\}$ and g. $\square$

We will need the following special case of [20, 2.9], which we prove using an argument from [41]:

PROPOSITION 3.15. *Let R be a local Cohen-Macaulay ring of dimension d, let I be an R-ideal with $I^2 = aI$ for some $a \in I$, assume that $I_\mathfrak{p} = 0$ for every associated prime $\mathfrak{p}$ of R containing I, and that* depth $R/I \geqslant d-1$.

Then $G = gr_I(R)$ is Cohen-Macaulay.

Proof. The R-homomorphism from the polynomial ring $R[T]$ to the Rees algebra $\mathcal{R} = R[It]$ sending T to at, induces a homomorphism of $R[T]$-modules $\varphi : IR[T] \to IR[It]$. Now φ is surjective because $I^2 = aI$, and φ is injective because $I_p = 0$ for every $p \in V(I) \cap \mathrm{Ass}(R)$ and hence $[0 : (a^j)] \cap I = 0$ for every $j \geqslant 1$. Thus $IR[It] \cong IR[T]$ has depth at least $d + 1$. Now a depth chase using the two exact sequences

$$0 \to I\mathcal{R}(-1) \to \mathcal{R} \to R \to 0,$$
$$0 \to I\mathcal{R} \to \mathcal{R} \to G \to 0$$

shows that G has depth at least d. □

We are now ready to complete the proof of Theorem 3.1. The main idea is to deduce this theorem from Proposition 3.13, by factoring out a suitable link of the ideal and thereby lowering the analytic deviation (this method has been employed by other authors, e.g., [12] and [35], or earlier, but in a different context, [22], [26], [19]).

The Proof of Theorem 3.1. Write $g =$ grade I and $\delta = \delta(I) = \ell - g + 1 - k$, and recall that $\delta \geqslant 0$. We are going to induct on δ, the case $\delta = 0$ being covered by Proposition 3.13. Thus we may assume that $\delta \geqslant 1$ and that the assertion holds for smaller values of δ. Now $\ell \geqslant g + k \geqslant g + 1$.

We adopt the notation of Lemma 2.8. By that lemma, $\mathfrak{a}_g \cap I^j = \mathfrak{a}_g I^{j-1}$ for $j \geqslant 1$, and, equivalently, $a'_1, \ldots a'_g$ from a G-regular sequence. Thus we do not change our assumptions and the conclusion if we factor out $\mathfrak{a}_g$ to assume that $g = 0$ (thereby d and ℓ decrease by g, whereas k may be taken to remain unchanged). Now $\ell \geqslant k \geqslant 1$, and in particular, I satisfies G_1 and therefore $I_p = 0$ for every $p \in V(I) \cap \mathrm{Ass}(R)$. Thus if $\ell = 1$, then our assertion follows from Proposition 3.15.

Hence we may assume that $\ell \geqslant \max\{2, k\}$, in which case I satisfies AN_0^-. Write $K = 0 : I$ and let '$^-$' denote images in $\bar{R} = R/K$. Now $\bar{R}$ is Cohen-Macaulay since I satisfies AN_0^-, and by Lemmas 2.3 (c) and 2.4 (b), $I \cap K = 0$, grade $\bar{I} = 1$, $\bar{I}$ still satisfies G_ℓ and $AN_{\ell-3}^-$ locally in codimension $\ell - 1$, and $\bar{I}$ is $AN_{\ell-\max\{2,k\}}^-$. Furthermore, $\dim \bar{R} = \dim R = d$; and since $I \cap K = 0$, we have $\ell(\bar{I}) = \ell(I) = \ell$ and thus k may be taken to remain unchanged. Therefore $\delta(\bar{I}) = \ell -$ grade $\bar{I} + 1 - k < \ell + 1 - k = \delta(I)$. Again, as $I \cap K = 0$, we have an exact sequence

$$0 \to K \to \mathrm{gr}_I(R) \to \mathrm{gr}_{\bar{I}}(\bar{R}) \to 0, \tag{3.16}$$

where depth $K = d$ since depth $\bar{R} = d$. Now by (3.16), depth $\bar{R}/\bar{I} \geqslant \min\{\text{depth } K - 1, \text{depth } R/I\} \geqslant \min\{d-1, d-\ell+k-1\} = d-\ell+k-1$, where the latter equality holds because $\ell \geqslant k$. Furthermore, again by (3.16), $\bar{I}^{j-1}/\bar{I}^j \cong I^{j-1}/I^j$ for $j \geqslant 2$, and we conclude that depth $\bar{R}/\bar{I}^j \geqslant d - \ell + k - j$ whenever $j \leqslant k$. Thus we may apply our induction hypothesis to conclude that $\mathrm{gr}_{\bar{I}}(\bar{R})$ is Cohen-Macaulay, and hence by (3.16), $\mathrm{gr}_I(R)$ has the same property. □

4. The Defining Equations for $\mathcal{R}$ and G

We are now going to use the results of Section 2, in particular Lemmas 2.5 (a) and 2.8 (b), to investigate the reduction number of I and to study the defining equations and the resolution of $\mathcal{R}$ and of G.

Let S be a homogeneous Noetherian ring with $A = S_0$ local, present $S \cong B/Q$ as an epimorphic image of a (standard graded) polynomial ring $B = A[T_1, \dots, T_n]$, and let $F_\bullet$ with $F_i = \bigoplus_j B(-n_{ij})$ be a homogeneous minimal free B-resolution of S. Writing $a_i(S_+, S) = \max\{j \mid [H^i_{S_+}(S)]_j \neq 0\}$, one defines the *Castelnuovo-Mumford regularity* $\mathrm{reg}(S)$ of S as $\max\{a_i(S_+, S) + i \mid i \geqslant 0\}$. It turns out that $\mathrm{reg}(S) = \max\{n_{ij} - i \mid i \geqslant 0 \text{ and } j \text{ arbitrary}\}$ ([32], [8]). On the other hand, if $S = \mathcal{R}$ (and $n \geqslant 2$), then the maximal degree occurring in a homogeneous minimal generating set of the defining ideal Q is called the *relation type* of I and is denoted by $rt(I)$. Notice that by the above discussion, $rt(I) \leqslant \mathrm{reg}(\mathcal{R}) + 1$.

We begin by comparing the Castelnuovo-Mumford regularities of $\mathcal{R}$ and G.

PROPOSITION 4.1. *Let R be a Noetherian local ring and let I be a (proper) R-ideal.*

Then $\mathrm{reg}(\mathcal{R}) = \mathrm{reg}(G)$.

Proof. We look at the usual exact sequences from [24],

$$0 \to \mathcal{R}_+ \to \mathcal{R} \to R \to 0, \tag{4.2}$$

$$0 \to \mathcal{R}_+(1) \to \mathcal{R} \to G \to 0. \tag{4.3}$$

First notice that

$$H^i_{\mathcal{R}_+}(R) = H^i_0(R) = \begin{cases} R & \text{for } i = 0 \\ 0 & \text{for } i \neq 0 \end{cases}.$$

Thus by (4.2),

$$[H^i_{\mathcal{R}_+}(\mathcal{R}_+)]_j \cong [H^i_{\mathcal{R}_+}(\mathcal{R})]_j \quad \text{for } i \geqslant 2 \text{ or } j \neq 0. \tag{4.4}$$

On the other hand, (4.3) gives rise to an exact sequence

$$H^i_{\mathcal{R}_+}(\mathcal{R}_+)(1) \to H^i_{\mathcal{R}_+}(\mathcal{R}) \to H^i_{\mathcal{R}_+}(G) \to H^{i+1}_{\mathcal{R}_+}(\mathcal{R}_+)(1),$$

which, when combined with (4.4), yields

$$[H^i_{\mathcal{R}_+}(\mathcal{R})]_{j+1} \to [H^i_{\mathcal{R}_+}(\mathcal{R})]_j \to [H^i_{\mathcal{R}_+}(G)]_j \to [H^{i+1}_{\mathcal{R}_+}(\mathcal{R})]_{j+1}, \tag{4.5}$$

provided that $i \geqslant 2$ or $j \neq -1$. Also notice that $H^i_{\mathcal{R}_+}(G) \cong H^i_{G_+}(G)$.

Now write $s = \operatorname{reg}(\mathcal{R})$ and $t = \operatorname{reg}(G)$, and note that $s \geqslant 0$, $t \geqslant 0$. By (4.5), if $i \geqslant 2$ then $H^i_{G_+}(G)$ is concentrated in degrees $\leqslant s-i$, and if $i \leqslant 1$ then $H^i_{G_+}(G)$ is concentrated in degrees $\leqslant \max\{s-i, -1\} = s-i$. Thus $\operatorname{reg}(G) \leqslant s$. On the other hand, again by (4.5), if $i \geqslant 2$ then $H^i_{\mathcal{R}_+}(\mathcal{R})$ is concentrated in degrees $\leqslant t-i$, and if $i \leqslant 1$ then $H^i_{\mathcal{R}_+}(\mathcal{R})$ is concentrated in degrees $\leqslant \max\{t-i, -1\} = t-i$. Therefore $\operatorname{reg}(\mathcal{R}) \leqslant t$. □

Our next proposition gives a degree bound for the syzygies of $\mathcal{R}$ and G. Conversely, it can also be used to determine the reduction number of the ideal I from the shifts in the resolution of $\mathcal{R}$ (which is often easier to compute than the resolution of G).

PROPOSITION 4.6. *If in addition to the assumptions of Theorem 3.1, I satisfies $AN^-_{\ell-2}$, then*

$$\operatorname{reg}(\mathcal{R}) = \operatorname{reg}(G) = r.$$

In particular, $rt(I) \leqslant r+1$. Furthermore, $r_J(I) = r$ does not depend on the choice of a minimal reduction J of I.

Proof. By Proposition 4.1, $\operatorname{reg}(\mathcal{R}) = \operatorname{reg}(G)$. On the other hand, combining Lemma 2.8 (b) with [36, 3.3], we conclude that $\operatorname{reg}(G) = r$. Furthermore one always has $r \leqslant r_J(I)$, whereas by [36, 3.2], $r_J(I) \leqslant \operatorname{reg}(G)$. Thus $r_J(I) = r$. □

Using the above proposition, we are now going to show that it suffices to check one of the assumptions of Theorem 3.1 locally in codimension ℓ (generalizing results from [20], [21], [41], [38], [33]).

PROPOSITION 4.7. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ and reduction number r, let $k \geqslant 1$ be an integer, assume that I satisfies G_ℓ and $AN^-_{\ell-2}$ locally in codimension ℓ, that I satisfies $AN^-_{\ell-k-1}$, and that depth $R/I^j \geqslant d-\ell+k-j$ for $1 \leqslant j \leqslant k$. The following are equivalent:*

(a) $r(I_p) \leqslant k$ *for every* $p \in V(I)$ *with* $\dim R_p = \ell < \mu(I_p)$.
(b) $r \leqslant k$.

Proof. Let J be a minimal reduction of I with $r_J(I) = r$. By [38, 1.11], J_p is a minimal reduction of I_p for every $p \in V(I)$ with $\dim R_p = \ell < \mu(I_p)$, and therefore by Proposition 4.6, $r_{J_p}(I_p) = r(I_p)$.

(a) $\Rightarrow$ (b): We need to verify that $I_p^{k+1} = J_p I_p^k$ for every $p \in \operatorname{Ass}(R/JI^k)$. Now by Remark 2.7 and Lemma 2.5 (a), $\dim R_p \leqslant \ell$, and by [38, 1.11] we may assume that $\dim R_p = \ell < \mu(I_p)$, since otherwise $I_p = J_p$. But for such prime ideals we have just seen that $r_{J_p}(I_p) = r(I_p) \leqslant k$.

(b) $\Rightarrow$ (a): This follows because $r(I_p) = r_{J_p}(I_p) \leqslant r_J(I) = r \leqslant k$. $\square$

We now wish to further investigate the defining equations of $\mathcal{R}$ and G. To do so it suffices to consider the ideals $\mathcal{A}$ and $\mathcal{B}$ of the symmetric algebras $S(I)$ and $S(I/I^2)$ that fit into the natural exact sequences

$$0 \to \mathcal{A} \to S(I) \to \mathcal{R} \to 0, \tag{4.8}$$

and

$$0 \to \mathcal{B} \to S(I/I^2) \to G \to 0. \tag{4.9}$$

PROPOSITION 4.10. *Let R be a local Gorenstein ring with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, minimal number of generators n, and reduction number r, assume that I satisfies G_ℓ, that the Koszul homology modules $H_j(I)$ are Cohen-Macaulay whenever $0 \leqslant j \leqslant \ell - g$, and that $r \leqslant \ell - g + 1$. Further let $J = (a_1, \ldots, a_\ell)$ be a minimal reduction of I, let $B = R[T_1, \ldots, T_\ell]$ be a polynomial ring, and consider $S(I)$ and $\mathcal{R}$ as B-modules via the R-algebra homomorphisms mapping T_i to $a_i \in I = [S(I)]_1$ and to 0, respectively.*

Then $\mathcal{A} \otimes_B R \cong [S(I/J)]_{\geqslant \ell-g+2}$ and $\mathcal{B} \otimes_B R \cong [S(I/J+I^2)]_{\geqslant \ell-g+2}$. In particular, the ideals $\mathcal{A}$ and $\mathcal{B}$ are minimally generated by $\binom{n-g+1}{n-\ell-1}$ forms of degree $\ell - g + 2$.

Proof. We first show that $\mathcal{A} \otimes_B R \cong [S(I/J)]_{\geqslant \ell-g+2}$.

From our assumption on the Koszul homology modules we know that the graded pieces $[\mathcal{M}]_j$ of the $\mathcal{M}$-complex are acyclic for $0 \leqslant j \leqslant \ell - g$ ([17, the proof of 5.1]). By the acyclicity of these complexes, $S_j(I) \cong I^j$ ([17, the proof of 4.6]) and depth $R/I^j \geqslant d - g - j + 1$ in the range $1 \leqslant j \leq \ell - g + 1$. Thus $[\mathcal{A}]_j = 0$ whenever $j \leqslant \ell - g + 1$, and I satisfies $AN^-_{\ell-2}$ (Theorem 2.2 (a)). Furthermore by Proposition 4.6, $r_J(I) = r$, and hence Lemma 2.8 applies to the ideal J.

After changing the generators of $a_1, \ldots, a_\ell$ of J if needed, we conclude from that lemma that in G, $[(a'_1, \ldots, a'_i) : (a'_{i+1})]_{\geqslant \ell-g+1} = [(a'_1, \ldots, a'_i)]_{\geqslant \ell-g+1}$ whenever $0 \leqslant i \leqslant \ell-1$. Thus in $\mathcal{R}$, $[(a_1t, \ldots, a_it) : (a_{i+1}t)]_{\geqslant \ell-g+1} = [(a_1t, \ldots, a_it)]_{\geqslant \ell-g+1}$ whenever $0 \leqslant i \leqslant \ell - 1$, as can be seen from [4, the proof of 6.5]. Now let H denote Koszul homology with values in $\mathcal{R}$. Using the long exact sequence

$$\begin{aligned} &H_1(a_1t, \ldots, a_it) \to H_1(a_1t, \ldots, a_{i+1}t) \\ &\quad \to H_0(a_1t, \ldots, a_it)(-1) \xrightarrow{a_{i+1}t} H_0(a_1t, \ldots, a_it) \end{aligned}$$

and induction on i, one concludes that $[H_1(a_1t, \ldots a_it)]_{\geqslant \ell-g+2} = 0$ whenever $0 \leqslant i \leqslant \ell$ (cf. also [4, 4.4]). In particular, $[\mathrm{Tor}_1^B(\mathcal{R}, R)]_{\geqslant \ell-g+2} = [H_1(a_1t, \ldots, a_\ell t)]_{\geqslant \ell-g+2} = 0$.

On the other hand, we had seen that $[\mathcal{A}]_j = 0$ whenever $j \leqslant \ell - g + 1$. Thus applying $- \otimes_B R$ to (4.8) yields an exact sequence

$$0 \to \mathcal{A} \otimes_B R \to S(I/J) \to \mathcal{R} \otimes_B R \to 0.$$

Since $[\mathcal{R} \otimes_B B/B_+]_{\geqslant \ell-g+2} = 0$, we deduce that $[\mathcal{A} \otimes_B R]_{\geqslant \ell-g+2} \cong [S(I/J)]_{\geqslant \ell-g+2}$, and therefore $\mathcal{A} \otimes_B R \cong [S(I/J)]_{\geqslant \ell-g+2}$.

To prove the remaining assertions of the proposition, notice that upon applying $- \otimes_R R/I$ to the latter isomorphism, one obtains a commutative diagram

$$\begin{tikzcd} \mathcal{A} \otimes_B R/I \arrow[rr, "\sim"] \arrow[dr, "\varphi"'] & & {[S(I/J+I^2)]_{\geqslant \ell-g+2}} \\ & \mathcal{B} \otimes_B R \arrow[ur, "\psi"'] & \end{tikzcd}$$

where φ is surjective. Thus ψ is an isomorphism as well. Finally, note that both $S(I)$-modules $[S(I/J)]_{\geqslant \ell-g+2}$ and $[S(I/J+I^2)]_{\geqslant \ell-g+2}$,are minimally generated by $\binom{n-g+1}{n-\ell-1}$ homogeneous elements of degree $\ell - g + 2$. $\square$

5. The Gorensteinness of the Associated Graded Ring

Let R be a local Gorenstein ring, let I be a proper R-ideal, and let G stand for the associated graded ring of I.

The Gorensteinness of G has been investigated under suitable assumptions in [17, 9.1], [18, 1.3] (in conjunction with [19, 3.6]), [11, 1.3], [13, 1.1], [33, 3.7 and 3.8], [35, 6.1], and [16, 2.1 and 2.5]. These results (except for [16, 2.5]) are contained in the next two theorems (and their corollaries). After this paper was written, it came to our attention that other generalizations have been independently obtained in [14].

THEOREM 5.1. *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, let $k \geqslant 0$ be an integer with $r \leqslant k$, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\ell-\max\{2,k\}}$, that depth $R/I^j \geqslant d - \ell + k - j + 1$ for $1 \leqslant j \leqslant k$ and that depth $R/I^j \geqslant d - g - j + 1$ for $1 \leqslant j \leqslant \ell - g - k$.*

Then G is Gorenstein.

THEOREM 5.2. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, let k be an integer with $0 \leqslant k \leqslant \ell - g$, assume that I satisfies G_ℓ, that I is generically a complete intersection, and that G is Gorenstein. The following are equivalent:*

(a) depth $R/I^j \geqslant d-g-j+1$ *for* $1 \leqslant j \leqslant \ell-g-k$
(b) $r \leqslant k$ *and* I *satisfies* $AN^-_{\ell-k-1}$.

Combining Theorems 5.1 and 5.2, one immediately obtains the following characterization:

COROLLARY 5.3. *Let* R *be a local Gorenstein ring of dimension* d *with infinite residue field, let* I *be an* R*-ideal with grade* g*, analytic spread* ℓ*, and reduction number* r*, let* $k \geqslant 1$ *be an integer, assume that* I *has no embedded associated primes, that* I *satisfies* G_ℓ *and* $AN^-_{\ell-3}$ *locally in codimension* $\ell-1$*, that* I *satisfies* $AN^-_{\ell-\max\{2,k\}}$*, and that depth* $R/I^j \geqslant d-\ell+k-j+1$ *for* $1 \leqslant j \leqslant k$*. Any two of the following conditions implies the third one:*

(a) $r \leqslant k$.
(b) depth $R/I^j \geqslant d-g-j+1$ *for* $1 \leqslant j \leqslant \ell-g-k$.
(c) G *Gorenstein.*

Before proving Theorems 5.1 and 5.2 we derive some further consequences. The next corollary deals with the Rees algebra $\mathcal{R} = R[It] \subset R[t]$.

COROLLARY 5.4. *If in addition to the assumptions of Theorem 5.1,* $g \geqslant 1$*, then*

$$\omega_{\mathcal{R}} \cong (1,t)^{g-2}\mathcal{R}(-1).$$

Proof. The assertion follows from Theorem 5.1, Corollary 3.4, and [18, 2.5]. □

COROLLARY 5.5. *Let* R *be a local Gorenstein ring of dimension* d *with infinite residue field, let* I *be an* R*-ideal with grade* g*, analytic spread* ℓ*, and reduction number* r*, assume that* I *satisfies* G_ℓ*, that depth* $R/I^j \geqslant d-g-j+1$ *for* $1 \leqslant j \leqslant \ell-g$*, and that* $r \leqslant \ell-g$.

Then $r=0$*, and* I *satisfies* G_∞ *and is strongly Cohen-Macaulay.*

Proof. We may assume that $\ell \geqslant g+1$, since otherwise I is a complete intersection. By using Theorem 2.2 (a) and Theorem 5.1 with $k=\ell-g$ we see that G is Gorenstein, but then Theorem 5.2 with $k=0$ implies $r=0$. The rest follows from [38, 2.13]. □

COROLLARY 5.6. *Let* R *be a local Gorenstein ring of dimension* d *with infinite residue field, let* I *be an* R*-ideal with grade* g*, analytic spread* ℓ*, and reduction number* r*, assume that* I *is generically a complete intersection, that* R/I *is Cohen-Macaulay, and that depth* $R/I^j \geqslant d-g-j+1$ *for* $1 \leqslant j \leqslant \ell-g-1$.

Then G *is Gorenstein if and only if* $r \leqslant 1$ *and* I *satisfies* G_ℓ.

Proof. By Theorems 2.2 (a), 5.1, and 5.2, it suffices to prove that if G is Gorenstein then I satisfies G_ℓ. Suppose this implication does not hold, and choose $p \in V(I)$ minimal with the property that $\mu(I_p) > \dim R_p \leqslant \ell-1$. Since

$\ell(I_p)$ − grade $I_p \leqslant \ell - g - 1$, applying Theorem 5.2 with $k = 0$ to the ideal I_p shows that $r(I_p) = 0$. But then $\mu(I_p) = \ell(I_p) \leqslant \dim R_p$, which yields a contradiction. □

COROLLARY 5.7. *Let R be a local Gorenstein ring, let I be an R-ideal with grade g and analytic spread ℓ, assume that I is generically a complete intersection and the Koszul homology modules H_j of I are Cohen-Macaulay for $0 \leqslant j \leqslant \frac{\ell-g}{2}$.*

Then G is Gorenstein if and only if I satisfies G_∞ and is strongly Cohen-Macaulay.

Proof. We may suppose that the residue field of R is infinite. By [17, 9.1] we are reduced to showing that if G is Gorenstein then I is G_∞ and strongly Cohen-Macaulay. Proceeding by induction on d, we may assume that $\ell \geqslant g+1$ and that the assertion holds locally on the punctured spectrum of R. Now using the Approximation Complex, one sees that depth $R/I^j \geqslant d - g - j + 1$ for $1 \leqslant j \leqslant \frac{\ell-g+2}{2}$, and hence by Theorem 5.2, $r \leqslant \frac{\ell-g-1}{2}$. On the other hand, the Approximation Complex and our induction hypothesis also imply that either $r = 0$, or else $r \geqslant \frac{\ell-g+1}{2}$. Therefore $r = 0$, and hence $\mu(I) = \ell \leqslant d$. The result now follows by the duality of Koszul homology ([27, 2.13 and 2.22]). □

To prove the theorems we need the following result:

PROPOSITION 5.8. *Let R be a local Cohen-Macaulay ring of dimension d with infinite residue field and canonical module ω, let I be an R-ideal with grade g, analytic spread ℓ, and reduction number r, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimension $\ell - 1$, that I satisfies $AN^-_{\min\{\ell-2,g\}}$, that depth $R/I^j \geqslant d - g - j + 1$ for $1 \leqslant j \leqslant \ell - g$, and that $r \leqslant \ell - g$.*

Then $\omega_G \cong gr_I(\omega)(-g)$.

Proof. Theorem 3.1 already shows that G is Cohen-Macaulay (which is the only place where the condition $AN^-_{\min\{l-2,g\}}$ is used). But then by [18, 2.4], our assertion is equivalent to saying that ω_G is generated in degree g, which we may check after reducing modulo a G-regular sequence in R. Now since G is Cohen-Macaulay, a sequence of general elements $x_1, \ldots, x_{d-\ell}$ in R is regular on R and on G. Furthermore, factoring out these elements does not change our assumptions (in the setting of the proposition, the local property $AN^-_{\ell-3}$ is equivalent to I being strongly Cohen-Macaulay locally in codimension $\ell - 1$, cf. [38, the proof of 2.13], and the latter condition is preserved under specialization, cf. [27, 2.15]). But after specializing, $d = \ell$, hence depth $R/I^j \geqslant d - g - j + 1$ for $1 \leqslant j \leq \ell - g + 1$. In this case our assertion follows from Proposition 3.13. □

The Proof of Theorem 5.1. Adopt the notation of Lemma 2.8, write $K_i = \mathfrak{a}_i : I$ for $0 \leqslant i \leqslant \ell$, and $K_{-1} = 0$. Our assumptions together with Theorem 2.2 (b), Lemma 2.3 ((c)(iv)), and Lemma 2.8 (a) imply that there are natural isomorphisms

$$\omega_{R/K_i} \cong I\omega_{R/(K_{i-1},a_i)} \quad \text{for } g \leqslant i \leqslant \ell - k - 1. \tag{5.9}$$

We now replace the assumption of R being Gorenstein by the weaker condition that R is Cohen-Macaulay and that (5.9) holds. With this new assumption, we are going to show that $\omega_G \cong \mathrm{gr}_I(\omega_R)(-g)$. We use increasing induction on $\delta = \delta(I) = \ell - g - k \geqslant 0$, the cases $\delta = 0$ and $\delta = \ell - g$ being covered by Proposition 5.8.

So let $0 < \delta < \ell - g$. As in the proof of Theorem 3.1, we replace R by $R/(a_1, \dots, a_g) = R/(K_{g-1}, a_g)$, thus reducing to the case $g = 0$. Write $\omega = \omega_R$, $K = 0 : I$, and let '$^-$' denote images in $\bar{R} = R/K$. Since $\ell - g > k > 0$, we may use the property $AN^-_{\ell-\max\{2,k\}}$ and (5.9), to conclude that $\bar{R}$ is Cohen-Macaulay with $\omega_{\bar{R}} \cong I\omega$. Furthermore, by the same arguments as in the proof of Theorem 3.1, all our assumptions are preserved as we pass from I and R to $\bar{I}$ and $\bar{R}$ (including (5.9), whereas $\bar{R}$ need no longer be Gorenstein), grade $\bar{I} = 1$, and $\delta(\bar{I}) < \delta(I)$. Thus our induction hypothesis implies that the canonical module of $\mathrm{gr}_{\bar{I}}(\bar{R})$ is $\mathrm{gr}_{\bar{I}}(\omega_{\bar{R}})(-1)$, where $\mathrm{gr}_{\bar{I}}(\omega_{\bar{R}})(-1) \cong \mathrm{gr}_I(I\omega)(-1)$.

Furthermore, by Theorem 3.1, $G = \mathrm{gr}_I(R)$ is Cohen-Macaulay, and from (3.16) we have an exact sequence

$$0 \to K \to \mathrm{gr}_I(R) \to \mathrm{gr}_{\bar{I}}(\bar{R}) \to 0. \tag{5.10}$$

Since the canonical module of $\mathrm{gr}_{\bar{I}}(\bar{R})$ is $\mathrm{gr}_I(I\omega)(-1)$ and since $\mathrm{Hom}_G(K, \omega_G) \cong \mathrm{Hom}_R(K, \omega)$, we may dualize (5.10) into ω_G to obtain an exact sequence of graded G-modules,

$$0 \to \mathrm{gr}_I(I\omega)(-1) \to \omega_G \to \mathrm{Hom}_R(K, \omega). \tag{5.11}$$

Now (5.11) shows that ω_G is concentrated in nonnegative degrees and that $[\omega_G]_{\geqslant 1}$ can be identified with $\mathrm{gr}_I(I\omega)(-1)$.

On the other hand, there is the natural exact sequence

$$0 \to \mathrm{gr}_I(I\omega)(-1) \to \mathrm{gr}_I(\omega) \to \omega \otimes_R R/I \to 0. \tag{5.12}$$

Since $\bar{R} \cong R/K$ is Cohen-Macaulay and since $\omega_{\bar{R}} \cong I\omega$, it follows that $\omega \otimes_R R/I \cong \mathrm{Hom}_R(K, \omega)$, which is a maximal Cohen-Macaulay G-module. Thus dualizing (5.12) into ω_G, one sees that there exists a homogeneous G-linear map $\varphi : \mathrm{gr}_I(\omega) \to \omega_G$ lifting the identity on $\mathrm{gr}_I(I\omega)(-1)$. Now $\mathrm{gr}_I(\omega)$ is generated in degree zero, ω_G is concentrated in nonnegative degrees, and φ coincides with the identity map in positive degrees. Therefore ω_G is also generated in degree zero, which by [18, 2.4] means that $\omega_G \cong \mathrm{gr}_I(\omega)$. □

The Proof of Theorem 5.2. By [15, the proof of (11.16) (b)] R is Gorenstein, and by [33, 2.1] I is unmixed. Furthermore by way of Theorem 2.2 (a), we may suppose that I satisfies $AN^-_{\ell-k-1}$. With this additional assumption, we are going to show that (a) holds if and only if $r \leqslant k$.

Since I satisfies G_ℓ, there exists a minimal reduction J of I such that $\mathrm{ht}\, J : I \geqslant \ell$, and $r_J(I) = r$ (this follows, for instance, from [28, the proof of 3.2] and [33, the proof of 3.4]). Now let $a_1, \ldots, a_\ell$ be a generating set of J as in Lemma 2.3 (a), write $\mathfrak{a}_i = (a_1, \ldots, a_i)$ and $K_i = \mathfrak{a}_i : I$ for $0 \leqslant i \leqslant \ell$, $K_{-1} = 0$, and let S_i be the associated graded ring of $I + K_i/K_i$ in R/K_i. We may choose $a_1, \ldots, a_\ell$ so that for $1 \leqslant i \leqslant \ell$, the image of a_i in $[S_{i-1}]_1$ does not lie in any associated prime of S_{i-1} not containing S_{i-1+}.

According to [38, 2.9 and 2.18], (a) is equivalent to

$$\omega_{R/K_i} \cong I^{i-g+1} + K_i/K_i \quad \text{for} \quad g \leqslant i \leqslant \ell - k - 1. \tag{5.13}$$

We first prove by induction on $\ell - k - 1$, that (5.13) holds if and only if

$$\omega_{R/K_i} \cong I\omega_{R/(K_{i-1}, a_i)} \quad \text{for} \quad g \leqslant i \leqslant \ell - k - 1. \tag{5.14}$$

This equivalence being clear for $\ell - k - 1 = g - 1$, we may assume that $\ell - k - 1 \geqslant g$. By induction hypothesis, we know that (5.13) holds for $g \leqslant i \leqslant \ell - k - 2$. Thus by [38, 2.18], depth $R/I^j \geqslant d - g - j + 1$ for $1 \leqslant j \leqslant \ell - g - k - 1$, and hence by Lemma 2.5 (c), $K_{\ell-k-1} \cap I^{\ell-g-k} = \mathfrak{a}_{\ell-k-1} I^{\ell-g-k-1}$. This equality together with our induction hypothesis and Lemma 2.3 ((c)(iv)) implies that $I^{\ell-g-k} + K_{\ell-k-1}/K_{\ell-k-1} \cong I^{\ell-g-k}/K_{\ell-k-1} \cap I^{\ell-g-k} \cong I^{\ell-g-k}/K_{\ell-k-2} \cap I^{\ell-g-k} + a_{\ell-k-1} I^{\ell-g-k-1} \cong I\omega_{R/(K_{\ell-k-2}, a_{\ell-k-1})}$, which establishes the equivalence of (5.13) and (5.14).

We are now going to prove by induction on $\delta = \ell - g - k \geqslant 0$ that (5.14) holds if and only if $r \leqslant k$. To do so, we replace the assumption of I being generically a complete intersection and of R and G being Gorenstein, by the weaker condition that R is Cohen-Macaulay with canonical module ω and G is Cohen-Macaulay with $\omega_G \cong \mathrm{gr}_I(\omega)(-g)$ (that this isomorphism holds if G is Gorenstein, follows from [33, the proof of 2.1]). Write $a(G) = -\min\{j | [\omega_G]_j \neq 0\}$. Now for $\delta = 0$, (5.14) is trivially satisfied, and on the other hand $r \leqslant \ell + a(G) = \ell - g$ ([33, 3.5]).

So let $\delta > 0$. By our choice of $a_1, \ldots, a_\ell$, the images $a'_1, \ldots, a'_g$ in $[G]_1$ form a G-regular sequence, and we do not change our assumptions and conclusions by factoring out $a_1, \ldots, a_g$, thus reducing to the case $g = 0$. Write $K = 0 : I$ and let '$\bar{\ }$' denote images in $\bar{R} = R/K$. Since $\delta > 0$, I satisfies AN^-_0, and thus $\bar{R}$ is Cohen-Macaulay. Furthermore, G_ℓ and $AN^-_{\ell-k-1}$ still hold for $\bar{I}$ (Lemma 2.4 (b)), and $\ell(\bar{I}) = \ell$, $r(\bar{I}) = r$, grade $\bar{I} = 1$ (Lemma 2.3 (c)), which shows that $\delta(\bar{I}) < \delta$.

As in the proof of Theorem 5.1, we have an exact sequence

$$0 \to K \to \mathrm{gr}_I(R) \to \mathrm{gr}_{\bar{I}}(\bar{R}) \to 0, \tag{5.15}$$

which upon dualizing into ω_G yields

$$0 \to \omega_{\mathrm{gr}_{\bar{I}}(\bar{R})} \to \omega_G \xrightarrow{\varphi} \mathrm{Hom}_R(K,\omega) \to \mathrm{Ext}^1_G(\mathrm{gr}_{\bar{I}}(\bar{R}),\omega_G) \to 0. \tag{5.16}$$

On the other hand, the exact sequence of maximal Cohen-Macaulay R-modules,

$$0 \to K \to R \to \bar{R} \to 0$$

gives rise to

$$0 \to \omega_{\bar{R}} \to \omega_R \xrightarrow{\psi} \mathrm{Hom}_R(K,\omega) \to 0. \tag{5.17}$$

Since $[\omega_G]_0 \cong \omega/I\omega$, comparing (5.16) and (5.17) shows that $[\varphi]_0$ is surjective. Thus φ is surjective, hence $\mathrm{Ext}^1_G(\mathrm{gr}_{\bar{I}}(\bar{R}),\omega_G) = 0$, and therefore by (5.15), $\mathrm{gr}_{\bar{I}}(\bar{R})$ is Cohen-Macaulay. Now if (5.14) holds, then $\omega_{\bar{R}} \cong I\omega$, and thus again by comparing (5.16) and (5.17) we see that $[\varphi]_0$ is an isomorphism. Therefore $\omega_{\mathrm{gr}_{\bar{I}}(\bar{R})}$ has all its generators in degree one, and hence by [18, 2.4], $\omega_{\mathrm{gr}_{\bar{I}}(\bar{R})} \cong \mathrm{gr}_{\bar{I}}(\omega_{\bar{R}})(-1)$. Conversely, if $r \leqslant k$, then [33, 3.5] implies that $a(\mathrm{gr}_{\bar{I}}(\bar{R})) = \max\{-\mathrm{grade}\ \bar{I}, r(\bar{I}) - \ell(\bar{I})\} = \max\{-1, r-\ell\} = -1$. Thus also in this case $[\varphi]_0$ is an isomorphism. Hence by (5.16) and [18, 2.4], $\omega_{\mathrm{gr}_{\bar{I}}(\bar{R})} \cong \mathrm{gr}_{\bar{I}}(\omega_{\bar{R}})(-1)$, and by (5.17), $\omega_{\bar{R}} \cong I\omega$. Therefore in either case, $\omega_{\bar{R}} \cong I\omega$ and $\mathrm{gr}_{\bar{I}}(\bar{R})$ is Cohen-Macaulay with canonical module $\mathrm{gr}_{\bar{I}}(\omega_{\bar{R}})(-1)$.

Thus all our assumptions are preserved as we pass from I to $\bar{I}$, except that now $\delta(\bar{I}) < \delta$. Moreover since $\omega_{\bar{R}} \cong I\omega$, (5.14) holds for I if and only if it holds for $\bar{I}$, and by induction hypothesis, the latter condition is equivalent to $r = r(\bar{I}) \leqslant k$. □

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