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Orthogonal, symplectic and unitary polar spaces sub-weakly embedded in projective space

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Abstract. We show that every sub-weak embedding of any non-singular orthogonal or unitary polar space of rank at least 3 in a projective space $\mathbf{PG}(d, \mathbb{K})$, $\mathbb{K}$ a commutative field, is a full embedding in some subspace $\mathbf{PG}(d, \mathbb{F})$, where $\mathbb{F}$ is a subfield of $\mathbb{K}$; the same theorem is proved for every sub-weak embedding of any non-singular symplectic polar space of rank at least 3 in $\mathbf{PG}(d, \mathbb{K})$, where the field $\mathbb{F}'$ over which the symplectic polarity is defined is perfect in the case that the characteristic of $\mathbb{F}'$ is two and the secant lines of the embedded polar space Γ contain exactly two points of Γ . This generalizes a result announced by LEFÈVRE-PERCSY [5] more than ten years ago, but never published. We also show that every quadric defined over a subfield $\mathbb{F}$ of $\mathbb{K}$ extends uniquely to a quadric over the groundfield $\mathbb{K}$, except in a few well-known cases.

Key words: polar space, weak embedding, sub-weak embedding, projective space

1. Introduction and statement of the results

In this paper we always assume that $\mathbb{K}$ and $\mathbb{F}$ are commutative fields. Any polar space considered in this paper is assumed to be non-degenerate (which means that no point of the polar space is collinear with all points of the polar space), unless explicitly mentioned otherwise.

A *weak embedding* of a point-line geometry Γ with point set $\mathcal{S}$ in a projective space $\mathbf{PG}(d, \mathbb{K})$ is a monomorphism θ of Γ into the geometry of points and lines of $\mathbf{PG}(d, \mathbb{K})$ such that

- (WE1) the set $\mathcal{S}^\theta$ generates $\mathbf{PG}(d, \mathbb{K})$,
- (WE2) for any point x of Γ , the subspace generated by the set $X = \{y^\theta \mid y \in \mathcal{S} \text{ is collinear with } x\}$ meets $\mathcal{S}^\theta$ precisely in X ,
- (WE3) if for two lines L_1 and L_2 of Γ the images L_1^θ and L_2^θ meet in some point x , then x belongs to $\mathcal{S}^\theta$.

In such a case we say that the image Γ^θ of Γ is weakly embedded in $\mathbf{PG}(d, \mathbb{K})$.

A *full embedding in $\mathbf{PG}(d, \mathbb{K})$* is a weak embedding with the additional property that for every line L , all points of $\mathbf{PG}(d, \mathbb{K})$ on the line L^θ have an inverse image under θ .

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Weak embeddings were introduced in [3,5]; in these papers she announced the classification of all weakly embedded finite polar spaces (clearly the polar spaces are considered here as point-line geometries) having the additional property that there exists a line of $\mathbf{PG}(d, \mathbb{K})$ which does not belong to Γ^θ and which meets $\mathcal{S}^\theta$ in at least three points. Only the case $d = 3$, $|\mathbb{K}| < \infty$ and $\text{rank}(\Gamma) = 2$ was published [4]. The question arose again in connection with full embeddings of generalized hexagons (see [7]) and a proof seemed desirable. In the present paper, we will first show that the condition (WE3) is superfluous and then classify all – finite and infinite – weakly embedded non-singular polar spaces of rank at least 3 of orthogonal, symplectic or unitary type, assuming that for the symplectic type the field $\mathbb{F}'$ over which the symplectic polarity is defined is perfect in the case that $\mathbb{F}'$ has characteristic two and no line of $\mathbf{PG}(d, \mathbb{K})$ which does not belong to Γ^θ intersects $\mathcal{S}^\theta$ in at least three points. The classification of all generalized quadrangles weakly embedded in finite projective space can be found in [8].

We call a monomorphism θ from the point-line geometry of a polar space Γ with point set $\mathcal{S}$ to the point-line geometry of a projective space $\mathbf{PG}(d, \mathbb{K})$ a *sub-weak embedding* if it satisfies conditions (WE1) and (WE2). Usually, we simply say that Γ is weakly or sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$ without referring to θ , that is, we identify the points and lines of Γ with their images in $\mathbf{PG}(d, \mathbb{K})$. In such a case the set of all points of Γ on a line L of Γ will be denoted by L^* .

If the polar space Γ arises from a quadric it is called *orthogonal*, if it arises from a hermitian variety it is called *unitary*, and if it arises from a symplectic polarity it is called *symplectic*. In these cases Γ is called *non-singular* either if the hermitian variety is non-singular, or if the symplectic polarity is non-singular, or if the quadric is non-singular (in the sense that the quadric Q , as algebraic hypersurface, contains no singular point over the algebraic closure of the ground field over which Q is defined); in the symplectic and hermitian case this is equivalent to assuming that the corresponding matrix is non-singular. In the orthogonal case with characteristic not 2, in the symplectic case and in the hermitian case, Γ is non-singular if and only if it is non-degenerate; in the orthogonal case with characteristic 2, non-singular implies non-degenerate, but when not every element of $\mathbb{K}$ is a square, and only then, a non-degenerate quadric may be singular.

Our main results read as follows.

THEOREM 1 *Let Γ be a non-singular polar space of rank at least 3 arising from a quadric, a hermitian (unitary) variety or a symplectic polarity, and let Γ be sub-weakly embedded in the projective space $\mathbf{PG}(d, \mathbb{K})$, where for Γ symplectic the polarity is defined over a perfect field $\mathbb{F}'$ in the case that $\mathbb{F}'$ has characteristic two and the secant lines of Γ contain exactly two points of Γ . Then Γ is fully embedded in some subspace $\mathbf{PG}(d, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.*

If Γ is finite, then it is automatically of one of the three types mentioned. Moreover, it is non-degenerate if and only if it is non-singular. Combining this with [8], we have

COROLLARY 1 (i) *Let Γ be a non-degenerate polar space sub-weakly embedded in the finite projective space $\mathbf{PG}(d, q)$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d, q')$ of $\mathbf{PG}(d, q)$, for some subfield $\mathbf{GF}(q')$ of $\mathbf{GF}(q)$, unless Γ is the unique generalized quadrangle of order $(2, 2)$ universally embedded in $\mathbf{PG}(4, q)$ with q odd.*

(ii) *Let Γ be a finite non-degenerate polar space of rank at least 3 sub-weakly embedded in the projective space $\mathbf{PG}(d, \mathbb{K})$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d, q)$ of $\mathbf{PG}(d, \mathbb{K})$, for some subfield $\mathbf{GF}(q)$ of $\mathbb{K}$.*

Our second main result might belong to folklore but we give a full proof here.

THEOREM 2 (i) *Let Q be a non-degenerate non-empty quadric of $\mathbf{PG}(d, \mathbb{F})$, $d \geq 2$, and let $\mathbb{K}$ be a field containing $\mathbb{F}$. Then in the corresponding extension $\mathbf{PG}(d, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{F})$ there exists a unique quadric containing Q , except if $d = 2$ and $\mathbb{F} \in \{\mathbf{GF}(2), \mathbf{GF}(3)\}$, or $d = 3$, $\mathbb{F} = \mathbf{GF}(2)$ and Q is of elliptic type.*

(ii) *Let Γ be a non-singular symplectic polar space defined by a symplectic polarity in $\mathbf{PG}(d, \mathbb{F})$, $d \geq 3$, and let $\mathbb{K}$ be a field extending $\mathbb{F}$. Then in the corresponding extension $\mathbf{PG}(d, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{F})$, there exists a unique symplectic polarity whose corresponding polar space contains Γ .*

(iii) *Let H be a non-singular non-empty hermitian variety of $\mathbf{PG}(d, \mathbb{F})$, $d \geq 2$, with associated $\mathbb{F}$ -involution σ , and let $\mathbb{K}$ be a field containing $\mathbb{F}$ admitting a $\mathbb{K}$ -involution τ the restriction of which to $\mathbb{F}$ is exactly σ . Then in the corresponding extension $\mathbf{PG}(d, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{F})$ there exists a unique hermitian variety with associated field involution τ and containing H .*

Remark It is now easy to extend Theorem 2 to the singular cases with at least one non-singular point over $\mathbb{F}$. Again the extension of the polar space Γ is unique, except for Γ orthogonal and $\mathbb{F} \in \{\mathbf{GF}(2), \mathbf{GF}(3)\}$.

2. Proof of Theorem 1

In the sequel, we adopt the notation $x^\perp$ for the set of all points collinear with the point x in a polar space. After having coordinatized $\mathbf{PG}(d, \mathbb{K})$, we denote by e_i , $1 \leq i \leq d+1$, the point with coordinates $(0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the i th position. By generalizing this, we denote by e_J the point with every coordinate equal to 0 except in each position belonging to the set J , $J \subseteq \{1, 2, \dots, d+1\}$, where the coordinate equals 1. We also remark that polar spaces are *Shult spaces*, i.e. for every point x and every line L , $x^\perp$ contains either all points of L or exactly one point of L (we will call that property the *Buekenhout–Shult axiom*).

We prove Theorem 1 in a sequence of lemmas.

LEMMA 1 *If L is a line of the sub-weakly embedded polar space Γ , then the only points of Γ on L are the points of L^* .*

Proof Let x be a point of Γ on L with $x \notin L^*$. By the Buekenhout–Shult axiom L^* contains a point y collinear with x . So the lines xy and L of Γ coincide in

$\mathbf{PG}(d, \mathbb{K})$, contradicting the fact that θ is a monomorphism. $\square$

LEMMA 2 *Every sub-weak embedding of a non-degenerate polar space is also a weak embedding.*

Proof Let Γ be a polar space sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$ for some field $\mathbb{K}$. Let L_1 and L_2 be two lines of Γ meeting in a point x of $\mathbf{PG}(d, \mathbb{K})$ which does not belong to $\mathcal{S}$, the point set of Γ . If some point y of Γ is collinear with all points of L_1^* , then $y^\perp$ contains a triangle of the plane $L_1 L_2$ of $\mathbf{PG}(d, \mathbb{K})$ ($y^\perp$ contains some point of L_2^* by the Buekenhout–Shult axiom). Hence (WE2) implies that y is collinear with all points of L_2^* . If we let y vary on L_1^* , then we see that all points of L_1^* are collinear with all points of L_2^* , in other words, L_1^* and L_2^* span a 3-dimensional singular subspace S of Γ . Since Γ is non-degenerate, no point of S is collinear with all other points of Γ , hence there exists a point z of Γ not collinear with all points of S . It is easily seen that $z^\perp$ meets S in the point set of a plane π of Γ . Since any two lines of Γ in π generate the plane $L_1 L_2$, the points of π span the plane $L_1 L_2$ of $\mathbf{PG}(d, \mathbb{K})$. By (WE2), $z^\perp$ must contain all points of S (since they all lie in $L_1 L_2$), a contradiction. $\square$

Let L be any line of $\mathbf{PG}(d, \mathbb{K})$ containing at least two points of Γ which are not collinear in Γ . Then we call L a *secant line*. By Lemma 1, no secant line contains two collinear points. The following result is due to Lefèvre-Percsy [3].

LEMMA 3 *The number of points of Γ on a secant line is a constant.*

We put that number equal to δ (δ is possibly an infinite cardinal) and call it the *degree* of the embedding.

We now prepare the proof of the case $\delta = 2$ by first proving a lemma which certainly belongs to folklore.

A *kernel* of a non-empty non-singular quadric in a projective space is any point belonging to every tangent hyperplane of the quadric. As the quadric is non-singular a kernel does not belong to the quadric. The subspace of all kernels is sometimes called the *radical* of the quadric.

LEMMA 4 *Every non-empty non-singular quadric has at most one kernel.*

Proof Suppose that the non-singular non-empty quadric Γ of $\mathbf{PG}(d, \mathbb{K})$ has a radical V of dimension at least one. Extend Γ over the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ to the non-singular quadric $\overline{\Gamma}$. Then $\overline{\Gamma} \cap \overline{V}$, with $\overline{V}$ the corresponding extension of V , is a non-empty quadric. Let x be a point of it. Every line xp with $p \in \Gamma$, $p \neq x$, is a tangent line of $\overline{\Gamma}$ and all these lines generate the whole projective space $\mathbf{PG}(d, \overline{\mathbb{K}})$. This yields a contradiction as all tangent lines of $\overline{\Gamma}$ at x lie in the tangent hyperplane of $\overline{\Gamma}$ at x . $\square$

LEMMA 5 *Let Γ be a non-singular polar space of rank at least 3 arising from a quadric, a hermitian (unitary) variety or a symplectic polarity, where for Γ sym-*

plectic the polarity is defined over a perfect field $\mathbb{F}'$ in the characteristic two case, and let Γ be sub-weakly embedded of degree 2 in the projective space $\mathbf{PG}(d, \mathbb{K})$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.

Proof We label the steps of the proof for future reference.

(a) Let Γ be a non-singular orthogonal polar space sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$, $d \geq 3$, and suppose that Γ has rank at least 3. We identify the points and lines of Γ with the corresponding points and lines of $\mathbf{PG}(d, \mathbb{K})$. Let π be any plane of Γ . Three non-concurrent lines of π span a unique plane π' of $\mathbf{PG}(d, \mathbb{K})$. Any other line of π meets these three lines in at least two points, hence we see that π' is uniquely determined by π ; moreover, the points and lines of π determine a unique subplane of π' . Hence π is isomorphic to a projective plane over some subfield $\mathbb{F}$ of $\mathbb{K}$. Moreover, since Γ is residually connected (as a polar space or a building, see e.g. [1]), $\mathbb{F}$ is independent from π . Hence, if we coordinatize $\mathbf{PG}(d, \mathbb{K})$, then every re-coordinatization by means of a linear transformation (so without using a field automorphism) which maps the points e_1, e_2, e_3 and $e_{\{1,2,3\}}$ onto points of π , defines a subfield $\mathbb{F}$ of $\mathbb{K}$ which is independent of the choice of π and where $\mathbb{F}$ is equal to the set of quotients of possible coordinates (in the new coordinate system) for points of π . This implies that the set of all points of Γ on any line of Γ is uniquely determined in $\mathbf{PG}(d, \mathbb{K})$ by any three of its points; indeed, re-coordinatize so that these points become e_1, e_2 and $e_{\{1,2\}}$, and then all points of the line are obtained by taking all linear combinations of the vectors $(1, 0, \dots, 0)$ and $(0, 1, 0, \dots, 0)$ over $\mathbb{F}$. All this shows that not only the isomorphism type of $\mathbb{F}$ is fixed, but also the subfield $\mathbb{F}$ itself.

(b) Now consider a line L_1 of Γ and a point x_1 of Γ on it. Through x_1 there is a line M_1 of Γ with the property that L_1 and M_1 are not in a common plane of Γ . Now we take a point y_1 of Γ not collinear with x_1 and we consider the unique line L_2 of Γ passing through y_1 and meeting M_1 in a point of Γ . Now we show that in Γ no point on L_2 is collinear with all points of L_1 . The point x_1 is not collinear with y_1 , and as L_1 and M_1 are not in a common plane of Γ the point $M_1 \cap L_2$ is not collinear with all points of L_1 . As x_1 is not collinear with y_1 , it is not collinear with two distinct points of L_2 ; hence no point of L_2 different from y_1 and $M_1 \cap L_2$ is collinear with all points of L_1 . Similarly, in Γ no point on L_1 is collinear with all points on L_2 . If L_1 and L_2 would span a plane $L_1 L_2$, then every point of L_2 is in the space spanned by $x^\perp$ for every $x \in L_1^*$, since there is at least one point of $x^\perp$ on L_2^* . So by (WE2) the point $x \in L_1^*$ is collinear with every point of L_2^* , a contradiction. Hence L_1 and L_2 generate a 3-space U of $\mathbf{PG}(d, \mathbb{K})$. In Γ the lines L_1, L_2 and their points generate a polar space Ω ; Ω corresponds to a hyperbolic quadric Q_3^+ (of a 3-space) on the non-singular quadric from which Γ arises. The point set of Ω will also be denoted by Q_3^+ , and the sets of lines of Ω corresponding to the reguli of Q_3^+ will also be called the reguli of Ω . Since all points of Ω lie on lines meeting both L_1 and L_2 , we see that Ω is entirely contained in U . Let $M_2 \neq M_1$ belong to the regulus of Ω defined by M_1 . Put $x_2 = L_1 \cap M_2$, $x_3 = L_2 \cap M_1$ and $x_4 = L_2 \cap M_2$.

Let x_5 be one further point of Ω not on one of the lines L_1, L_2, M_1, M_2 and let L_3 , respectively M_3 , be the line of Ω through x_5 and belonging to the regulus defined by L_1 , respectively M_1 . No four of the points $\{x_1, x_2, x_3, x_4, x_5\}$ are coplanar, so they determine a unique subspace V of U over $\mathbb{F}$.

(c) We claim that Ω is fully embedded in V , that is, we claim that all points of Ω are contained in V . Indeed, the points on L_1 in V are uniquely determined by the three points x_1, x_2 and $M_3 \cap L_1$. But as remarked above, these points are precisely all points of Γ on L_1 . Similarly for L_2, M_1 and M_2 . Let M_4 be a line of Ω meeting L_1, L_2 in points of Ω , so of V , with $M_1 \neq M_4 \neq M_2$; then M_4 is a line of V . As L_3 is a line of V , also $L_3 \cap M_4$ is a point of V . It follows that the points of M_4 in V are exactly the points of M_4 in Ω . Similarly, for any line L_4 of Ω meeting M_1, M_2 in points of Ω , the points of L_4 in V are exactly the points of L_4 in Ω . If y is any point of Ω , then the line of Γ through y meeting L_1, L_2 , respectively M_1, M_2 , contains at least two points of V , and hence the intersection y of these two lines also belongs to V . This shows our claim.

(d) Next we prove that no other point of Γ belongs to U . Indeed, suppose the point z of Γ lies in U , but is not contained in Ω . Then z does not belong to V since the unique line M in V through z meeting both L_1 and L_2 contains three points of Γ , say z, x_1, x_4 , hence belongs to Γ , contradicting the fact that z does not belong to Ω . In Γ the points of Ω collinear with z either are all the points of Ω , or are the points of a point set $\mathcal{C}$ of Ω corresponding to a non-singular conic of the hyperbolic quadric Q_3^+ , or are the points of Ω on two lines of Ω , say L_1 and M_1 . Noticing that for every point y of Ω , the space generated by $y^\perp$ in $\mathbf{PG}(d, \mathbb{K})$ meets U in a plane (by axiom (WE2)), we see that in the first case z must lie in every plane containing two lines of Ω . This yields a contradiction since these planes have no intersection point in V , hence neither in U . In the second case z must lie in the planes tangent to Q_3^+ at points of $\mathcal{C}$. These planes meet in at most one point, which lies in V , a contradiction. In the third case z must lie in all planes of V containing L_1 or M_1 , hence $z = x_1$, a contradiction. This proves our claim.

(e) An orthogonal subspace of Γ containing lines is called s -dimensional if the corresponding subquadric on the quadric from which Γ arises generates an $(s+1)$ -dimensional space. Now suppose that any $(c-1)$ -dimensional non-singular orthogonal subspace Ω' of Γ containing lines is fully embedded in a c -dimensional projective subspace over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$, $3 \leq c \leq d-1$. We show that, if Ω is a c -dimensional non-singular orthogonal subspace of Γ containing lines, then Ω is fully embedded in some $(c+1)$ -dimensional projective subspace $\mathbf{PG}(c+1, \mathbb{F})$ of $\mathbf{PG}(c+1, \mathbb{K})$. Since Ω is non-singular, it contains some $(c-1)$ -dimensional non-singular orthogonal subspace Ω' containing lines. By assumption Ω' is contained in a c -dimensional projective space V' over $\mathbb{F}$. Let U' be the extension of V' over $\mathbb{K}$. We first show that U' does not contain any point of $\Omega \setminus \Omega'$. Let the point x of $\Omega \setminus \Omega'$ belong to U' . Then $x^\perp$ and the point set of Ω' intersect in a point set Q'' which corresponds to a non-singular subquadric of the quadric from which Γ arises. By (WE2) Q'' is contained in a $(c-1)$ -dimensional subspace V'' of V' .

Assume that Q'' does not generate V'' . Then Ω' contains a point u of V'' not on Q'' . Every line of Ω' through u contains a point of $x^\perp$, so every line of Ω' through u contains a point of Q'' . Hence V'' contains all lines of Ω' through u . Analogously, V'' contains all lines of Ω' through u' , with $u' \neq u$ a second point of Ω' in $V'' \setminus Q''$. So the tangent hyperplanes of the point set of Ω' at u and u' coincide with V'' , a contradiction. We conclude that Q'' generates V'' . The extension of V'' over $\mathbb{K}$ will be denoted by U'' . If $x \notin U''$, then $x^\perp \cap U'$ spans U' , hence by (WE2) all points of Ω' are collinear with x , a contradiction. So $x \in U''$. Let y be a point of Q'' and let V'_y be the tangent hyperplane of Ω' at y ; the extension of V'_y to $\mathbb{K}$ is denoted by U'_y . If $x \notin U'_y$, then the space generated by x and U'_y is U' , so by (WE2) $y^\perp$ contains all points of Ω' , a contradiction. Hence $x \in U'_y$. Let V''_y be the tangent hyperplane of Q'' at y , and let U''_y be the extension of V''_y to $\mathbb{K}$; then $V''_y = V'_y \cap V''$ and $U''_y = U'_y \cap U''$. As $x \in U''$, we have $x \in U''_y$ for every point y of Q'' . This implies that $x \in V''$ and that x is the unique kernel of Q'' in V'' . Since Q'' has a unique kernel, the dimension $c - 1$ of the space generated by Q'' is even and the matrix defined by Q'' has rank equal to $c - 1$. If x is also a kernel of Ω' , then as $c + 1$ is even Ω' admits at least a line L of kernels. Over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$ the extension $\overline{L}$ of L contains a point r of the extension $\overline{\Omega}'$ of Ω' . The point r is singular for $\overline{\Omega}'$, hence Ω' is singular, a contradiction. Consequently x is not a kernel for Ω' . Hence there is a line N of V' containing x and two distinct points y_1, y_2 of Ω' . Since the degree of the weak embedding is equal to 2, N is a line of Γ , so $y_1 = y_2 \in Q''$, a contradiction. It follows that U' does not contain any point of $\Omega \setminus \Omega'$.

(f) Let x_1 be any point of $\Omega \setminus \Omega'$ and let L_1 be any line of Ω through x_1 . Evidently, L_1 meets Ω' in a unique point y . Let L_2 be any line of Ω' such that L_1, L_2 and their points in Ω' generate a polar space in Ω with as point set a hyperbolic quadric $Q = Q_3^+$. Take any point $x_2 \neq x_1$ on L_1^* with $x_2 \neq y$. The space V' together with the two points x_1, x_2 defines a unique $(c + 1)$ -dimensional subspace V over $\mathbb{F}$, which contains x_1, x_2 and y and hence all points of Ω on L_1 . Also, V contains all points of Ω on L_2 and all points of the line of Ω' containing y and concurrent with L_2 . Similarly as in (c), one now shows that Q_3^+ is completely contained in a 3-dimensional subspace over $\mathbb{F}$ which clearly belongs to V .

(g) We now show that all points of Ω belong to V . Let z be any point of $\Omega \setminus \Omega'$. First suppose that z is not collinear with y . Consider a line M_1 on Ω' through y and such that L_1 and M_1 are not contained in a plane of Ω . Let L_3 be the unique line of Ω through z meeting M_1 in a point of Ω . Then clearly L_1 and L_3 define a hyperbolic quadric Q' over $\mathbb{F}$ on Ω . We show that the polar subspace of Ω with point set Q' has two different lines M_1 and L'_2 in common with Ω' . If we identify the point set of Ω with a quadric in some $\text{PG}(c + 1, \mathbb{F})$, then the 3-space of Q' and the hyperplane defined by Ω have a plane ζ in common, which intersects Q' in two distinct lines. Hence Q' has two different lines M_1 and L'_2 in common with Ω' . Interchanging roles of L_2 and L'_2 , we now see that z also belongs to the space

V . Now suppose that the point z of $\Omega \setminus \Omega'$, $z \neq y$, is collinear with y . Let L_3 and L_4 , with $L_3 \neq yz \neq L_4$, be two distinct lines of Ω through z for which yL_3 and yL_4 are not planes of Ω . By the foregoing all points of $L_3^* \setminus \{z\}$ and $L_4^* \setminus \{z\}$ belong to V . Hence also the intersection of L_3 and L_4 , that is z , belongs to V . So we conclude that each of the points of Ω belongs to V , and consequently Ω is fully embedded in the space V over $\mathbb{F}$.

(h) Applying consecutively the previous paragraphs for $c = 3, 4, \dots, d-1$, we finally obtain that Γ is fully embedded in some $\mathbf{PG}(d, \mathbb{F})$.

(i) Now let Γ be a non-singular hermitian polar space sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$, $d \geq 3$, and suppose that the degree is 2. On the non-singular hermitian variety $\bar{\mathcal{H}}$ from which Γ arises we consider a non-singular hermitian variety $\mathcal{H}'$, where $\mathcal{H}'$ generates a 3-dimensional space. The corresponding point set on Γ will be denoted by $\mathcal{H}$ and the corresponding polar subspace of Γ by Ω . Let L, M be two non-intersecting lines of Ω . In $\mathbf{PG}(d, \mathbb{K})$, the lines L and M generate a 3-dimensional subspace $U = \mathbf{PG}(3, \mathbb{K})$, which contains all points of $\mathcal{H}$ (Ω is generated by L, M and their points in Ω). Now consider two points x and y in $\mathcal{H}$ which are not collinear in $\mathcal{H}$. Let $\mathcal{H}_x$ and $\mathcal{H}_y$ be the set of points of $\mathcal{H}$ collinear in $\mathcal{H}$ with x and y respectively. Clearly neither $\mathcal{H}_x$ nor $\mathcal{H}_y$ can be contained in a line of U . Also, by condition (WE2), neither $\mathcal{H}_x$ nor $\mathcal{H}_y$ generates U . Hence $\mathcal{H}_x$ and $\mathcal{H}_y$ define unique planes U_x and U_y respectively. These planes meet in a unique line N of U . Clearly N contains all points of $\mathcal{H}$ collinear in Ω with both x and y . Assume that z is any point of Γ on N . Further, let $u, v \in N \cap \mathcal{H}$, $u \neq v$. Then z is collinear in Γ with all points of $u^\perp \cap v^\perp$. Let u', v', z' be the points of $\bar{\mathcal{H}}$ which correspond to u, v, z respectively. As z' is collinear in $\bar{\mathcal{H}}$ with all points of $u'^\perp \cap v'^\perp$, it belongs to $\bar{\mathcal{H}} \cap u'v' = \mathcal{H}' \cap u'v'$. Hence z belongs to $\mathcal{H} \cap uv$. It follows that the set of all points of Γ on N corresponds to the point set $\bar{\mathcal{H}} \cap u'v' = \mathcal{H}' \cap u'v'$. As N meets Γ in more than 2 points, we are in contradiction with $\delta = 2$.

(j) Finally let Γ be a non-singular symplectic polar space sub-weakly embedded in $\mathbf{PG}(d, \mathbb{K})$, $d \geq 3$. Let $\mathbb{F}'$ be the ground field over which the symplectic polarity ζ from which Γ arises is defined.

If the characteristic of $\mathbb{F}'$ is not two, then a similar proof as for the hermitian case leads to a contradiction; here the secant line N will contain $|\mathbb{F}'| + 1$ points (note that the secant lines of Γ correspond (bijectively) to the non-isotropic lines of the symplectic polarity ζ).

If the characteristic of $\mathbb{F}'$ is two, then $\mathbb{F}'$ is perfect, hence Γ is also orthogonal. Now it follows from (a)–(h) that Γ is fully embedded in some $\mathbf{PG}(d, \mathbb{F})$. $\square$

The next lemma is a result similar to Theorem 1 for projective spaces. A *sub-space* of a projective space $\mathbf{PG}(n, \mathbb{K})$ is any space $\mathbf{PG}(n, \mathbb{F})$, $\mathbb{F}$ a subfield of $\mathbb{K}$, obtained from $\mathbf{PG}(n, \mathbb{K})$ by restricting coordinates to $\mathbb{F}$ (with respect to some coordinatization). Note that, for many fields $\mathbb{K}$ and positive integers n , there exist subsets $\mathcal{S}$ of the point set of $\mathbf{PG}(n, \mathbb{K})$ such that the linear space induced in $\mathcal{S}$ by the lines of $\mathbf{PG}(n, \mathbb{K})$ is the point-line space of a $\mathbf{PG}(m, \mathbb{F})$ with $m > n$. The

following result gives a necessary and sufficient condition for such a structure to be a sub- n -space. These conditions are basically (WE1) and some analogue of (WE2).

LEMMA 6 *Let $\mathcal{S}$ be a generating set of points in the projective space $\mathbf{PG}(n, \mathbb{K})$, $\mathbb{K}$ a skewfield and let $\mathcal{L}$ be the collection of all intersections of size > 1 of $\mathcal{S}$ with lines of $\mathbf{PG}(n, \mathbb{K})$. Suppose $(\mathcal{S}, \mathcal{L})$ is the point-line space of some projective space $\mathbf{PG}(m, \mathbb{F})$, for some skewfield $\mathbb{F}$ and some positive integer m . Then $\mathbb{F}$ is a subfield of $\mathbb{K}$, $m = n$ and $\mathcal{S}$ and $\mathcal{L}$ are the point set and line set respectively of some sub- n -space $\mathbf{PG}(n, \mathbb{F})$ of $\mathbf{PG}(n, \mathbb{K})$ if and only if there exists a dual basis of hyperplanes in $\mathbf{PG}(m, \mathbb{F})$ such that each element H of that basis is contained in a hyperplane H' of $\mathbf{PG}(n, \mathbb{K})$ with $H' \cap \mathcal{S} = H$.*

Proof It is clear that the given condition is necessary. Now we show that it is also sufficient. If $m + 1$ points of $\mathcal{S}$ generate $\mathbf{PG}(m, \mathbb{F})$, then by the condition that lines of $\mathbf{PG}(m, \mathbb{F})$ are line intersections of $\mathbf{PG}(n, \mathbb{K})$ with $\mathcal{S}$, these $m + 1$ points must also span $\mathbf{PG}(n, \mathbb{K})$ (otherwise $\mathcal{S}$ is contained in some proper subspace of $\mathbf{PG}(n, \mathbb{K})$). Hence $m \geq n$. Now let $\{H_i : i = 0, 1, \dots, m-1, m\}$ be a collection of hyperplanes of $\mathbf{PG}(m, \mathbb{F})$ meeting the requirements of the lemma. Put $S_i = H_0 \cap H_1 \cap \dots \cap H_i$, $i = 0, 1, \dots, m$. Suppose that S_j generates the same space as S_{j+1} in $\mathbf{PG}(n, \mathbb{K})$ for some j , $0 \leq j \leq m-1$. Let H_i be contained in the hyperplane H'_i (not necessarily unique at this point) of $\mathbf{PG}(n, \mathbb{K})$, $i = 0, 1, \dots, m$. If x is a point of S_j not lying in S_{j+1} (x exists by the assumptions on H_i), then in $\mathbf{PG}(n, \mathbb{K})$ x is not generated by the points of H_{j+1} , since H'_{j+1} meets $\mathcal{S}$ precisely in H_{j+1} . But $S_{j+1} \subseteq H_{j+1}$, hence in $\mathbf{PG}(n, \mathbb{K})$ x is not in the space generated by S_{j+1} , a contradiction. So S_j generates a space in $\mathbf{PG}(n, \mathbb{K})$ which is strictly larger than S_{j+1} . That means that we have a chain of $m + 1$ subspaces of $\mathbf{PG}(n, \mathbb{K})$ consecutively properly contained in each other and all contained in H'_0 ; hence $n \geq m$. We conclude that $n = m$.

Now if we choose a basis of $\mathbf{PG}(n, \mathbb{F})$ (this is also a basis of $\mathbf{PG}(n, \mathbb{K})$), then it is clear that the corresponding coordinatization of $\mathbf{PG}(n, \mathbb{F})$ is the restriction of the coordinatization of $\mathbf{PG}(n, \mathbb{K})$ to the field $\mathbb{F}$. The result follows. $\square$

LEMMA 7 *Let Γ be a non-singular polar space of rank at least 3 arising from a quadric, a symplectic polarity or a hermitian variety, and let Γ be sub-weakly embedded of degree $\delta > 2$ in the projective space $\mathbf{PG}(d, \mathbb{K})$. Then Γ is fully embedded in some subspace $\mathbf{PG}(d, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$, for some subfield $\mathbb{F}$ of $\mathbb{K}$.*

Proof Let $\mathbb{F}'$ be the field underlying Γ .

(1) First, let the characteristic of $\mathbb{F}'$ be odd and let Γ be a non-singular symplectic polar space. By (j) in the proof of Lemma 5, secant lines of Γ correspond (bijectively) with non-isotropic lines of the symplectic polarity ζ from which Γ arises. Now the space Ω with point set $\mathcal{S}$, the point set of Γ , and line set $\{L^* : L \text{ is a line of } \Gamma\} \cup \{S \cap \mathcal{S} : S \text{ is the point set in } \mathbf{PG}(d, \mathbb{K}) \text{ of a secant line of } \Gamma\}$ is a projective space. Every hyperplane H in that projective space Ω is the set of points of $\mathcal{S}$ collinear in Γ with some fixed point x of $\mathcal{S}$. It is easy to see that, as $\mathcal{S}$ is a generating set of $\mathbf{PG}(d, \mathbb{K})$, the hyperplane H of Ω generates a hyperplane

H' of $\mathbf{PG}(d, \mathbb{K})$. Now by (WE2) the assumptions of Lemma 6 are satisfied and the result follows.

Next, assume that the characteristic of $\mathbb{F}'$ is two and let Γ be a non-singular symplectic polar space. Let ζ be again the symplectic polarity from which Γ arises. If ζ is defined in $\mathbf{PG}(d', \mathbb{F}')$, then we consider a subspace $\mathbf{PG}(3, \mathbb{F}')$ of $\mathbf{PG}(d', \mathbb{F}')$ in which ζ induces a non-singular symplectic polarity η . The polar space defined by ζ is Γ' , and the polar space defined by η is Ω' . With Ω' corresponds the polar subspace Ω of Γ . Let L, M be two non-intersecting lines of Ω and let L', M' be the corresponding lines of Ω' . Let x be a point of Ω on L and y a point of Ω on M , where x and y are not collinear in Ω . The points of $\mathbf{PG}(3, \mathbb{F}')$ which correspond to x, y are denoted by x', y' respectively. As $\delta > 2$ the line xy contains a third point z of Γ . As, by (WE2), z is collinear in Γ to all points of $x^\perp \cap y^\perp$, the corresponding point z' of $\mathbf{PG}(d', \mathbb{F}')$ is collinear in Γ' to all points of $x'^\perp \cap y'^\perp$. Hence z' belongs to the line $x'y'$, so belongs to Ω' . It follows that z belongs to Ω . As Ω' is generated by z', L', M' and all points of L' and M' , also Ω is generated by z, L, M and all points of L and M . Hence Ω is contained in a subspace $\mathbf{PG}(3, \mathbb{K})$ of $\mathbf{PG}(d, \mathbb{K})$. Then a similar argument as in (i) of Lemma 5 shows that the secant lines of Γ correspond (bijectively) to the non-isotropic lines of ζ . Now, analogously as in the odd characteristic case, the result follows.

(2) Now suppose that Γ is of orthogonal type. Let Γ' be the image of a natural full embedding of Γ in a projective space $\mathbf{PG}(d', \mathbb{F}')$ where the point set of Γ' is a non-degenerate quadric Q' of $\mathbf{PG}(d', \mathbb{F}')$. Denote by x' the element of Γ' corresponding to any element x of Γ . Let M be a secant line in $\mathbf{PG}(d, \mathbb{K})$. Let p_1, p_2, p_3 be three points of Γ on M . Consider a point r of Γ collinear with both p_1 and p_2 . By (WE2) all points of Γ on M are collinear with r . If the lines $r'p'_1, r'p'_2, r'p'_3$ lie in a plane of $\mathbf{PG}(d', \mathbb{F}')$, then this must be a plane of Γ' and hence M is a line of Γ , a contradiction. Consequently r', p'_1, p'_2, p'_3 generate a 3-dimensional subspace $\mathbf{PG}(3, \mathbb{F}')$ of $\mathbf{PG}(d', \mathbb{F}')$. Let $\mathbf{PG}(4, \mathbb{F}') \supseteq \mathbf{PG}(3, \mathbb{F}')$ intersect Q' in a non-singular quadric Q'_1 . Suppose the characteristic of $\mathbb{F}'$ is not 2. Then there is a unique second point s' of Q'_1 collinear with p'_1, p'_2, p'_3 . So s is collinear with p_1, p_2, p_3 . Since s and r are not collinear in Γ , s is not in the plane $rp_1p_2p_3$ by (WE2). Let N be a line of Γ concurrent with rp_1 and sp_2 in Γ , but not incident with r or s . The line R of Γ through p_3 meeting N^* lies in the 3-dimensional space $srp_1p_2p_3$. By (WE2) R is in the plane p_3rs . Let w be the unique point of R^* collinear with p_1 ; then w is also collinear with p_2 (by (WE2)). Clearly $w' \in Q'_1$, a contradiction. Hence the characteristic of $\mathbb{F}'$ is equal to 2.

Let p'_1, p'_2, p'_3 and r' be as above, and let $p'_1p'_2p'_3 \cap Q' = C'$; further let Q'_1 be as above. Let $s' \neq r'$ be a point of Q'_1 collinear with p'_1, p'_2 (s' exists since Q'_1 defines itself a polar space). By (WE2), s' is also collinear with p'_3 . As in the previous paragraph, we construct the line R and the point w . Let V' be a line on Q'_1 through w' , not containing p'_1, p'_2 . There is a line L' meeting $r'p'_1, s'p'_2$ and V' , thus implying that V belongs to the space $rsp_1p_2 = rsp_1p_2$. By (WE2), V is contained in the plane wp_1p_2 . Let W be a line of Γ containing r and meeting V^* . Then W is in the

plane $rp_1p_2 \neq wp_1p_2$, hence $V \cap W$ is on M . So M contains all the points x such that x' is on the conic C' . Note that the kernel k' of C' coincides with the kernel of Q'_1 (as all tangents $k'r'$, $k's'$ and $k'p'$ with $p' \in C'$ generate the 4-space of Q'_1). We now show that for any point x of Γ on M , the point x' belongs to C' . By (WE2), each point of Γ on M lies in $(\{p_1, p_2\}^\perp)^\perp$. But $(\{p'_1, p'_2\}^\perp)^\perp$ is the intersection of Q' with either a line (and this happens if and only if d' is odd) or a plane π (and this happens if and only if d' is even) containing the kernel k' of Q' . The first case contradicts $\delta > 2$, hence only the latter case occurs. But clearly π must meet Q' in C' and our claim follows.

Note that the argument of the previous paragraph also shows that all points of every conic on Q' lying in a plane which contains the kernel k' of Q' correspond to the points of intersection of Γ with some secant line M . Also, every two non-collinear points of Q' lie in such a unique plane. Projecting Γ' from the kernel k' onto some hyperplane $\mathbf{PG}(d' - 1, \mathbb{F}')$ not containing k' , we obtain an embedding of Γ' into $\mathbf{PG}(d' - 1, \mathbb{F}')$ such that secant lines of Γ correspond with secant lines of the image Γ'' of Γ' in $\mathbf{PG}(d' - 1, \mathbb{F}')$. Note that if $\mathbb{F}'$ is perfect, in particular when $\mathbb{F}'$ is finite, then Γ'' is a non-singular symplectic space and the result follows from the first part of the proof.

(3) Remark that in (1) and (2) the proof does not depend on the rank of Γ , as long as it is at least 2.

From now on we use the fact that the rank of the orthogonal polar space Γ is at least 3. By the last part of (2) we may assume that the field $\mathbb{F}'$ is not perfect. As in paragraph (a) of the proof of Lemma 5, one shows that any set L^* , with L a line of Γ , is a subline of L over a subfield $\mathbb{F}$ of $\mathbb{K}$ which is independent of L (and clearly $\mathbb{F}$ is isomorphic to $\mathbb{F}'$). We now proceed in the same style as in the proof of Lemma 5, adapting the arguments to our present case $\delta > 2$.

We denote by x'' the element of Γ'' in $\mathbf{PG}(d' - 1, \mathbb{F}')$ corresponding to any element x of Γ in $\mathbf{PG}(d, \mathbb{K})$. Let L_1 and L_2 be two lines of Γ such that in $\mathbf{PG}(d', \mathbb{F}')$ L'_1 and L'_2 span a 3-space which intersects Q' in a non-singular quadric Q^+ . Let Q'_1 be the intersection of Q' with the 4-dimensional subspace of $\mathbf{PG}(d', \mathbb{F}')$ generated by L'_1, L'_2 and the kernel k' of Q' ; note that Q'_1 is non-singular. Let Ω be the polar subspace of Γ which corresponds with the quadric Q^+ . As in paragraphs (b) and (c) of the proof of Lemma 5, one shows that Ω is fully embedded in a unique 3-dimensional subspace V over $\mathbb{F}$ of the 3-dimensional subspace U (over $\mathbb{K}$) of $\mathbf{PG}(d, \mathbb{K})$ generated by L_1 and L_2 . Let V'' be the 3-dimensional subspace of $\mathbf{PG}(d' - 1, \mathbb{F}')$ generated by L''_1 and L''_2 (where L''_1 and L''_2 are the respective projections of L'_1 and L'_2). Let x'' be any point of Γ'' in V'' . Then $x' \in Q'_1$ and since Q'_1 is non-singular, x is not collinear with all points of L^*_i , $i = 1, 2$. Suppose x' does not lie on Q^+ and let y be the unique point on L_1 collinear with x in Γ . Let x_1, x_2 be two other points of Γ on L_1 . Let L be the line of Γ containing y and concurrent with L_2 . The lines $x'y'$, L' and L'_1 define a cone on Q'_1 and consequently there is a unique conic C'_i on that cone with kernel k' and containing x' and x'_i , $i = 1, 2$. These conics correspond with the respective secant lines M_1 and M_2 of

Γ . Hence M_i , $i = 1, 2$, contains x_i and another point y_i of Γ on L . But $x_i, y_i \in V$, hence M_i defines a line of V , $i = 1, 2$. Since x is the intersection of M_1 and M_2 , it belongs to V . So we obtain a full embedding of the polar subspace of Γ determined by Q'_1 .

Now let z be any other point of Γ contained in U . If z belongs to V then there is a unique line M in V meeting both L_1 and L_2 and containing z . The extension of M to $\mathbb{K}$ is a secant line of Γ and hence it corresponds with a conic on Q'_1 ; hence z' belongs to Q'_1 , a contradiction.

Suppose now $z \in U \setminus V$. Considering the polar subspace of Γ generated by L_1 , L_2 and their points in Γ , one shows as in paragraph (d) of the proof of Lemma 5 that $z \in V$, a contradiction. Hence the only points x of Γ in U satisfy $x' \in Q'_1$.

As in paragraphs (e), (f), (g) and (h) of the proof of Lemma 5 we use an inductive argument. The assumption is that any $(2c-1)$ -dimensional non-singular orthogonal subspace Γ_1 of Γ , whose corresponding subspace V'_1 in $\mathbf{PG}(d', \mathbb{F}')$ contains k' , is fully embedded in a $(2c-1)$ -dimensional projective subspace V_1 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$, $2 \leq c < \frac{d}{2}$. We want to show that every $(2c+1)$ -dimensional non-singular orthogonal subspace Γ_2 of Γ , whose corresponding subspace of $\mathbf{PG}(d', \mathbb{F}')$ contains k' , is fully embedded in a $(2c+1)$ -dimensional projective subspace over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$.

Let Γ_2 be a $(2c+1)$ -dimensional non-singular subspace of Γ , whose corresponding subspace V'_2 of $\mathbf{PG}(d', \mathbb{F}')$ contains k' , $2 \leq c < \frac{d}{2}$. Further, let Γ_1 be a $(2c-1)$ -dimensional non-singular subspace of Γ_2 , whose point set corresponds to the set of all points of Γ'_2 collinear with two given non-collinear points u' and v' of Γ'_2 . Then the subspace V'_1 of $\mathbf{PG}(d', \mathbb{F}')$ containing Γ'_1 , also contains the kernel k' . Hence Γ_1 is fully embedded in a $(2c-1)$ -dimensional projective subspace V_1 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$.

First, suppose there is a point x of $\Gamma_2 \setminus \Gamma_1$ with the property that the subspace V'_3 of $\mathbf{PG}(d', \mathbb{F}')$ generated by V'_1 and x' meets the point set of Γ'_2 in a non-degenerate quadric Q'_3 , i.e. the singular point of Q'_3 lies in a proper extension of V'_3 over some extension field $\mathbb{F}_1$ of $\mathbb{F}$, but not in V'_3 itself. Let U_1 be the extension of V_1 over $\mathbb{K}$. We first show that U_1 does not contain any point of $\Gamma_3 \setminus \Gamma_1$, where Γ_3 is the polar subspace of Γ which corresponds to Q'_3 . Let the point z of $\Gamma_3 \setminus \Gamma_1$ belong to U_1 . Since Γ_3 is generated by Γ_1 and z , all points of Γ_3 belong to U_1 . All points of Γ_1 are collinear with u . Since the point set of Γ_1 generates U_1 , by (WE2) all points of Γ_3 are collinear with u . As Γ_3 is non-degenerate the point u does not belong to Γ_3 , and so the set of all points of Γ_3 collinear with u is just the point set of Γ_1 . This yields a contradiction. Consequently no point of $\Gamma_3 \setminus \Gamma_1$ is contained in U_1 . Similarly to parts (f) and (g) of the proof of Lemma 5 we can now show that Γ_3 is fully embedded in a subspace $\mathbf{PG}(2c, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$. Let $\mathbf{PG}(2c, \mathbb{K})$ be the extension of $\mathbf{PG}(2c, \mathbb{F})$ over $\mathbb{K}$. Assume, by way of contradiction, that $\mathbf{PG}(2c, \mathbb{K})$ contains a point r of $\Gamma_2 \setminus \Gamma_3$. Since Γ_2 is generated by Γ_3 and r , all points of Γ_2 belong to $\mathbf{PG}(2c, \mathbb{K})$. Hence u belongs to $\mathbf{PG}(2c, \mathbb{K})$. By (WE2) the points u and v belong to the $(2c-1)$ -dimensional space U_1 . Since Γ_2 is generated by Γ_1 , u

and v , the polar space Γ_2 belongs to U_1 . Hence Γ_3 belongs to U_1 , a contradiction. Consequently no point of $\Gamma_2 \setminus \Gamma_3$ is contained in $\mathbf{PG}(2c, \mathbb{K})$. Similarly to parts (f) and (g) of the proof of Lemma 5 we now show that Γ_2 is fully embedded in a subspace $\mathbf{PG}(2c + 1, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$.

Next, suppose that for each point x of $\Gamma_2 \setminus \Gamma_1$ the subspace V'_3 of $\mathbf{PG}(d', \mathbb{F}')$ generated by V'_1 and x' meets the point set of Γ'_2 in a degenerate quadric Q'_3 , that is, the singular point y' of Q'_3 belongs to V'_3 . The set of all singular points y' is a non-singular conic C' with kernel k' . Let L' be any line through k' in the plane π' of C' . Then the $(2c + 1)$ -dimensional space generated by V'_1 and L' intersects the point set of Γ'_2 in a degenerate quadric with singular point on C' and L' . It follows that each line L' in π' through k' contains a point of C' . Consequently the field $\mathbb{F}'$ is perfect, a contradiction.

As in (h) of the proof of Lemma 5, induction now shows that $d = d' - 1$ and that Γ is fully embedded in a subspace $\mathbf{PG}(d, \mathbb{F})$ of $\mathbf{PG}(d, \mathbb{K})$.

(4) Finally suppose that Γ is a non-singular unitary polar space of rank at least 3 arising from some hermitian variety $\mathcal{H}' = H(d', \mathbb{F}', \sigma)$ in $\mathbf{PG}(d', \mathbb{F}')$ with σ an involutory field automorphism of $\mathbb{F}'$. Again we can copy part (a) of the proof of Lemma 5. As in (b) of that proof we can choose two lines L_1 and L_2 of Γ generating a 3-space U of $\mathbf{PG}(d, \mathbb{K})$. In Γ the lines L_1 and L_2 and their points generate a non-singular polar space Ω which corresponds to a hermitian surface $\mathcal{H}'_3$ (of a 3-space) on $\mathcal{H}'$. Now L_1 and L_2 (but not all their points) are contained in a polar subspace Ω_0 corresponding to a symplectic space $W(3, \mathbb{F}'_\sigma)$ in a 3-dimensional subspace $\mathbf{PG}(3, \mathbb{F}'_\sigma)$ of $\mathbf{PG}(d', \mathbb{F}')$ over the field $\mathbb{F}'_\sigma$ which consists of all elements of $\mathbb{F}'$ fixed by σ . By part (1) of this proof we know that there exists a subfield $\mathbb{F}_\sigma$ of $\mathbb{K}$ isomorphic to $\mathbb{F}'_\sigma$ and a 3-dimensional subspace V_σ of $\mathbf{PG}(d, \mathbb{K})$ over $\mathbb{F}_\sigma$ such that Ω_0 is fully embedded in V_σ . We also know that for any line L of Γ the set L^* is a projective subline of L in $\mathbf{PG}(d, \mathbb{K})$ over some field $\mathbb{F}$, which is independent of L . Evidently $\mathbb{F}$ contains $\mathbb{F}_\sigma$. Let V be the extension of V_σ over $\mathbb{F}$. Let L be a line of Ω_0 and let x be a point on L belonging to $\Omega \setminus \Omega_0$. Then clearly x lies in V . We will show that every point x of Ω lies on a line of Ω_0 .

Let x be an arbitrary point of $\Omega \setminus \Omega_0$ and let x' be the corresponding point of $\mathcal{H}'_3$. Since $\mathbf{PG}(3, \mathbb{F}'_\sigma)$ is a Baer subspace of $\mathbf{PG}(3, \mathbb{F}')$, there is a unique line L' of $\mathbf{PG}(3, \mathbb{F}'_\sigma)$ containing x' . If L' were not a line of $W(3, \mathbb{F}'_\sigma)$, then it would meet $\mathcal{H}'_3$ in a subline of L' over $\mathbb{F}'_\sigma$, hence x' would be a point of $\mathbf{PG}(3, \mathbb{F}'_\sigma)$, a contradiction. So L' is a line of $\mathcal{H}'_3$ (alternatively, this can be easily seen by considering the dual generalized quadrangle). The corresponding line L of Ω is incident with x and belongs to Ω_0 . Hence Ω is fully embedded in V and U is the extension of V over $\mathbb{K}$.

Now we show that no other point of Γ belongs to U . Suppose, by way of contradiction, that the point z of Γ lies in U but is not contained in Ω . Let z' be the corresponding point of $\mathcal{H}'$. If T' is the set of all points of $\mathcal{H}'_3$ collinear with z' , then either $\mathcal{H}'_3 = T'$, or T' is a non-singular hermitian curve, or T' is a singular hermitian curve. Let T be the corresponding point set of Ω . First, let $\mathcal{H}'_3 = T'$.

Noticing that for every point y of Ω , the space generated by $y^\perp$ in $\mathbf{PG}(d, \mathbb{K})$ meets U in a plane (by axiom (WE2)), we see that z must lie in every plane containing two intersecting lines of Ω . Hence the extensions over $\mathbb{K}$ of all tangent planes of the unitary polar space Ω (the point set of Ω is a hermitian variety of V) have a common point, clearly a contradiction. Hence $\mathcal{H}'_3 \neq T'$. Then, by (WE2), T and z are contained in a common plane $\mathbf{PG}(2, \mathbb{K})$. Assume that T' is a singular hermitian curve, with singular point u' . Let $r' \in T' \setminus \{u'\}$. As r is collinear with u and z in Γ , by (WE2) it is collinear in Γ with all points of T , clearly a contradiction. Finally, let T' be a non-singular hermitian curve. Let s be any point of T , and let M_1, M_2 be any two distinct lines of Ω through s . By (WE2) the lines M_1, M_2, zs are contained in a common plane, which is the extension over $\mathbb{K}$ of the tangent plane of the unitary polar space Ω at s . Hence z belongs to the extensions of all tangent planes of Ω at points of T , so z belongs to V . It follows that all tangent lines of the hermitian curve T concur at z , a contradiction. We conclude that the only points of Γ in U are the points of Ω .

As in paragraphs (e), (f), (g) and (h) of the proof of Lemma 5 (and as in (3) of the present proof) we use an inductive argument. Let Γ_1 be the polar subspace of Γ arising from a non-degenerate hermitian subvariety $\mathcal{H}'_1$ of $\mathcal{H}'$ containing lines, and obtained from $\mathcal{H}'$ by intersecting it with a c -dimensional subspace W'_1 of $\mathbf{PG}(d', \mathbb{F}')$, $3 \leq c < d'$. Suppose that Γ_1 is fully embedded in a c -dimensional subspace V_1 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$. Let Γ_2 be the polar subspace of Γ arising from a non-degenerate hermitian subvariety $\mathcal{H}'_2$ of $\mathcal{H}'$ obtained from $\mathcal{H}'$ by intersecting it with a $(c+1)$ -dimensional subspace W'_2 of $\mathbf{PG}(d', \mathbb{F}')$ containing W'_1 . Then we will show that Γ_2 is fully embedded in some $(c+1)$ -dimensional subspace V_2 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$. Let x be a point of $\Gamma_2 \setminus \Gamma_1$. Let U_1 be the extension of V_1 over $\mathbb{K}$. Suppose by way of contradiction that x belongs to U_1 . The points of Γ_1 collinear with x in Γ_2 form a point set $\mathcal{H}_3$ corresponding to a non-singular hermitian subvariety $\mathcal{H}'_3$ of $\mathcal{H}'_1$ obtained by intersecting $\mathcal{H}'_1$ with a hyperplane of W'_1 . By (WE2), x must belong to the extension over $\mathbb{K}$ of every hyperplane of V_1 tangent to Γ_1 at a point of $\mathcal{H}_3$. Also by (WE2), x and $\mathcal{H}_3$ are contained in a common hyperplane W_3 of U_1 . As the polar space with point set $\mathcal{H}'_1$ is generated by $\mathcal{H}'_3$ and any point of $\mathcal{H}'_1 \setminus \mathcal{H}'_3$, also Γ_1 is generated by $\mathcal{H}_3$ and any point of Γ_1 not in $\mathcal{H}_3$. Hence $\mathcal{H}_3$ generates a hyperplane R_3 of V_1 . Clearly W_3 is the extension over $\mathbb{K}$ of the hyperplane R_3 . It follows that the extensions over $\mathbb{K}$ of the tangent hyperplanes of Γ_1 at points of $\mathcal{H}_3$ intersect in a unique point which belongs to $V_1 \setminus R_3$. Hence $x \notin W_3$, a contradiction. Consequently no point of $\Gamma_2 \setminus \Gamma_1$ belongs to U_1 . Let L be any line of $\Gamma_2 \setminus \Gamma_1$; then L^* defines a projective subline over $\mathbb{F}$ and hence there is a unique $(c+1)$ -dimensional subspace V_2 over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$ containing V_1 and all elements of L^* . We now show that all points of Γ_2 are contained in V_2 . Let x be any point of Γ_2 . Clearly we may assume that x does not belong to Γ_1 nor to L^* .

In the sequel, we again denote the corresponding element in $\mathbf{PG}(d', \mathbb{F}')$ of an element e of Γ by e' .

First suppose that x is collinear in Γ_2 with a point $y \in L^*$ which does not belong to Γ_1 . All points of the line $x'y'$ belong to $\mathcal{H}'_2$ and hence there is a unique point z' of $x'y'$ in $\mathcal{H}'_1$. Let w be the unique point of Γ_1 on L^* . The line wz is either a line of Γ_1 or a secant line. In the first case the points of Γ_2 in the plane xwz of $\mathbf{PG}(d, \mathbb{K})$ form a projective subplane over $\mathbb{F}$ sharing all points of at least two lines with V_2 . Hence all points of that subplane belong to V_2 and so does x . In the second case let u be any point of Γ_1 on wz , $w \neq u \neq z$ (this is possible by the assumption $\delta > 2$). By Proposition 4 of [5] the line xu meets L in a point of Γ . Hence both xu and xz are lines of V_2 and the result follows.

Now suppose that x is not collinear in Γ_2 with an element of L^* not belonging to Γ_1 . By the Buekenhout–Shult axiom x is collinear in Γ_2 with the unique point w of L^* in Γ_1 . Let $y \in L^*$, $y \neq w$. It is easy to see that there is at most one point on the line $y'w'$ collinear in $\mathcal{H}'_2$ to all points of $\mathcal{H}'_1$ which are collinear to x' (since all such points belong to a secant line of $\mathcal{H}'_2$). So there is a point $y_1 \neq w$ on L^* and a point r of Γ_1 collinear with y_1 in Γ_2 , but not collinear with x in Γ_2 . By the Buekenhout–Shult axiom, there exists a unique line M of Γ_2 incident with x and containing a point s of Γ_2 on the line ry_1 . By assumption $s \neq r$, so s does not belong to Γ_1 . By the previous paragraph, all points of Γ on ry_1 belong to V_2 . Interchanging the roles of ry_1 and L , we now see that x belongs to V_2 . We conclude that Γ_2 is fully embedded in a $(c + 1)$ -dimensional subspace over $\mathbb{F}$ of $\mathbf{PG}(d, \mathbb{K})$. Applying this for $c = 3, 4, \dots, d' - 1$, we finally obtain that Γ is fully embedded in some $\mathbf{PG}(d', \mathbb{F})$ from which immediately follows that $d' = d$.

This completes the proof of the lemma. $\square$

The previous lemmas prove Theorem 1. $\square$

Remarks 1. When Γ arises from a non-degenerate but singular quadric (and that can only happen if the characteristic of the ground field $\mathbb{F}'$ is equal to 2), Theorem 1 is not valid. For example consider in $\mathbf{PG}(7, \mathbb{F}')$, where $\mathbb{F}'$ is a non-perfect field with characteristic 2, the quadric Q with equation

$$X_0^2 + X_1^2 + X_0X_1 + X_2^2 + aX_3^2 + X_4^2 + X_5^2 + X_4X_5 + X_6X_7 = 0,$$

where $a \in \mathbb{F}'$ is a non-square. Let $\mathbb{K}$ be the algebraic closure of $\mathbb{F}'$ and let $\mathbf{PG}(7, \mathbb{K})$ be the corresponding extension of $\mathbf{PG}(7, \mathbb{F}')$. The point $x(0, 0, \sqrt{a}, 1, 0, 0, 0, 0)$ is the unique singular point of Q . If we project Q from x onto a hyperplane $\mathbf{PG}(6, \mathbb{K})$ of $\mathbf{PG}(7, \mathbb{K})$ which does not contain x , then we obtain a weakly embedded polar space which is not fully embedded in any subspace $\mathbf{PG}(6, \mathbb{F})$, for any subfield $\mathbb{F}$ of $\mathbb{K}$. In a forthcoming paper, we will classify sub-weakly embedded singular polar spaces, degenerate or not, arising from quadrics, symplectic polarities or hermitian varieties.

2. When Γ has $\delta = 2$ and arises from a non-singular symplectic polar space of rank at least three over a non-perfect field of characteristic two, then Theorem 1 is not valid. We give an example. Let $\mathbb{K}$ be a field of characteristic two for which the subfield $\mathbb{F}$ of squares is not perfect. Then also $\mathbb{K}$ is not perfect. Now consider in

$\mathbf{PG}(6, \mathbb{K})$ the set $\mathcal{S}$ of points $(x_0, x_1, \dots, x_6)$ with $x_0, x_1, \dots, x_5 \in \mathbb{F}$, $x_6 \in \mathbb{K}$, and lying on the quadric Q with equation

$$X_0X_3 + X_1X_4 + X_2X_5 = X_6^2.$$

Then $\mathcal{S}$, provided with lines and planes induced by Q , is a polar space Γ isomorphic to the non-singular symplectic polar space $W(5, \mathbb{F})$ in $\mathbf{PG}(5, \mathbb{F})$ by projecting $\mathcal{S}$ from $(0, 0, 0, 0, 0, 0, 1)$ into the subspace U with equation $X_6 = 0$ over $\mathbb{F}$. Clearly Γ is sub-weakly embedded in $\mathbf{PG}(6, \mathbb{K})$. Let e_i , $0 \leq i \leq 5$, be the point of $\mathbf{PG}(6, \mathbb{K})$ with all coordinates 0 except the $(i + 1)$ th coordinate, which is equal to 1. Let e be the point all coordinates of which are equal to 1 and let e_{01} be the point with coordinates $(1, 1, 0, 0, 0, 0, 0)$. Then it is easy to see that the set V of points of $\mathcal{S}$ on the lines e_ie_{i+1} , $i \in \{0, 1, \dots, 4\}$, on e_0e_5 and on ee_{01} generates the subspace $\mathbf{PG}(6, \mathbb{F})$ of $\mathbf{PG}(6, \mathbb{K})$ consisting of all points with coordinates in $\mathbb{F}$. Hence, if $\mathcal{S}$ were fully embedded in a subspace of $\mathbf{PG}(6, \mathbb{K})$ over a subfield of $\mathbb{K}$, then this subspace would be $\mathbf{PG}(6, \mathbb{F})$. As $\mathcal{S}$ contains the point $(0, 0, 1, 0, 0, a^2, a)$, $a \in \mathbb{K} \setminus \mathbb{F}$, which does not belong to $\mathbf{PG}(6, \mathbb{F})$, the polar space Γ is not fully embedded in a subspace of $\mathbf{PG}(6, \mathbb{K})$.

3. Proof of Theorem 2

(i) First suppose that the non-degenerate quadric Q does not contain lines. Since by assumption the points of Q span $\mathbf{PG}(d, \mathbb{F})$, we may assume that $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the i th position, lies on Q for every i . The plane $e_ie_je_k$, $1 \leq i < j < k \leq d + 1$, meets Q in a non-singular non-empty conic. Assume that the coefficient of $X_\ell X_m$ in a fixed equation for Q over $\mathbb{F}$ is $a_{\ell m} = a_{m\ell}$. Let the quadric Q' of $\mathbf{PG}(d, \mathbb{K})$, with $\mathbb{K}$ an extension of $\mathbb{F}$ and $\mathbf{PG}(d, \mathbb{K})$ the corresponding extension of $\mathbf{PG}(d, \mathbb{F})$, contain Q . The coefficient of $X_\ell X_m$ in a fixed equation for Q' over $\mathbb{K}$ is denoted by $a'_{\ell m} = a'_{m\ell}$. If $|\mathbb{F}| \geq 4$, then, either $e_ie_je_k \cap Q'$ is a non-singular non-empty conic or the plane $e_ie_je_k$ itself. As a non-singular non-empty conic is uniquely defined by any five of its points, we have $a'_{\ell m} = c_{\{i,j,k\}} a_{\ell m}$ with $\ell, m \in \{i, j, k\}$ and $c_{\{i,j,k\}} \in \mathbb{K}$ (as $e_ie_je_k \cap Q$ is non-singular we have $a_{\ell m} \neq 0$). By fixing i and j we see that $c_{\{i,j,k\}} = c_{\{i,j,k'\}}$, for every k, k' and now it is easy to see that $c_{\{i,j,k\}}$ is a constant c ; it is clear that $c \neq 0$, whence the result for $|\mathbb{F}| \geq 4$. Suppose now $|\mathbb{F}| = 3$. As Q does not contain lines we have $d \in \{2, 3\}$. For $d = 2$, there are indeed distinct conics in $\mathbf{PG}(2, \mathbb{K})$, where $\mathbb{K}$ is a field of characteristic 3 with $|\mathbb{K}| > 3$, containing the four points of a conic in a subplane isomorphic with $\mathbf{PG}(2, 3)$, and the same remark holds for $|\mathbb{F}| = 2$ and $d = 2$. If $d = 3$ and $|\mathbb{F}| = 3$, then a direct and straightforward computation shows that the ten points of Q are on a unique quadric in every extension $\mathbf{PG}(3, \mathbb{K})$. For $|\mathbb{F}| = 2$ and $d = 3$, the five points of Q are contained in several non-singular quadrics over every proper extension of $\mathbb{F}$. This completes the case where Q does not contain lines.

Now suppose that Q contains lines. Let Q' be a quadric in $\mathbf{PG}(d, \mathbb{K})$ containing Q , with $\mathbb{K}$ an extension of $\mathbb{F}$ and $\mathbf{PG}(d, \mathbb{K})$ the corresponding extension of $\mathbf{PG}(d, \mathbb{F})$.

Again we can assume that $e_i \in Q$ for all i . Let $a_{ij} = a_{ji}$ respectively $a'_{ij} = a'_{ji}$ be the coefficient of $X_i X_j$ in the equation of Q respectively Q' . The tangent hyperplane U_i of Q at e_i is spanned by all lines through e_i contained in Q . If e_i is not singular for Q' , then also the tangent hyperplane U'_i of Q' at e_i is spanned by all lines through e_i contained in Q' ; in such a case the hyperplane U_i is necessarily a subhyperplane of U'_i . The equation of U_i is $\sum_j a_{ij} X_j = 0$ (note that $a_{ii} = a'_{ii} = 0$ for all i). If e_i is not singular for Q' , then the equation of U'_i is $\sum_j a'_{ij} X_j = 0$; if e_i is singular for Q' , then $a'_{ij} = 0$ for all j . From the foregoing it follows that $a'_{ij} = c_i a_{ij}$ for all j , with $c_i \in \mathbb{K}$. Hence if $a_{ij} = 0$, then also $a'_{ij} = 0$. Now consider $1 \leq i < j \leq d+1$ and $1 \leq k < \ell \leq d+1$ with $\{i, j\} \cap \{k, \ell\} = \emptyset$ and suppose that $a_{ij} \neq 0 \neq a_{k\ell}$. From the preceding it immediately follows that if a_{ik} , $a_{i\ell}$, a_{jk} and $a_{j\ell}$ are not all zero, then

$$\frac{a'_{ij}}{a_{ij}} = \frac{a'_{k\ell}}{a_{k\ell}}.$$

On the other hand, if $a_{ik} = a_{i\ell} = a_{jk} = a_{j\ell} = 0$, then the same equality follows from considering the tangent hyperplane of Q at the point $e_{ik} = (0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$, with the 1 in the i th and the k th position, from considering the tangent hyperplane of Q' at e_{ik} if this point is not singular for Q' (if this point is singular for Q' , then $a'_{ij} = a'_{k\ell} = 0$), and from considering the coefficients of X_j and X_ℓ in the equations of these hyperplanes. Now it immediately follows that Q' is uniquely determined by Q .

(ii) The proof is similar to the last part of (i) and in fact it can be simplified a great deal because we can immediately use standard equations.

(iii) First suppose that the non-singular non-empty hermitian variety H does not contain lines. Since the points of H span $\mathbf{PG}(d, \mathbb{F})$, $d \geq 2$, we may assume that $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 is in the i th position, lies on H for every i . The plane $e_i e_j e_k$, $1 \leq i < j < k \leq d+1$, meets H in a non-singular non-empty hermitian curve C . Assume that the coefficient of $X_\ell X_m^\sigma$ in a fixed equation for H over $\mathbb{F}$ is $a_{\ell m}$. Let $\mathbb{K}$ be a field containing $\mathbb{F}$ admitting a $\mathbb{K}$ -involution τ the restriction of which to $\mathbb{F}$ is σ , let $\mathbf{PG}(d, \mathbb{K})$ be the corresponding extension of $\mathbf{PG}(d, \mathbb{F})$, and let the hermitian variety H' of $\mathbf{PG}(d, \mathbb{K})$ contain H . The coefficient of $X_\ell X_m^\tau$ in a fixed equation for H' over $\mathbb{K}$ is denoted by $a'_{\ell m}$. The intersection of C with the line $e_i e_j$ is determined by the equation $a_{ij} X_i X_j^\sigma + a_{ji} X_j X_i^\sigma = 0$ (as C is non-singular we have $a_{ij} \neq 0$). For each point of that intersection also the equation $a'_{ij} X_i X_j^\sigma + a'_{ji} X_j X_i^\sigma = 0$ is satisfied. Let $(0, \dots, 0, 1, 0, \dots, 0, u, 0, \dots, 0)$ be a point of $C \cap e_i e_j$ with $u \neq 0$. Then $a_{ij} u^\sigma + a_{ji} u = a'_{ij} u^\sigma + a'_{ji} u = 0$. Hence

$$\frac{a'_{ij}}{a_{ij}} = \frac{a'_{ji}}{a_{ji}}.$$

Let us now consider a point $(0, \dots, 0, 1, 0, \dots, 0, u, 0, \dots, 0, v, 0, \dots, 0)$ of $C \cap e_i e_j e_k$ with the u as above and $v \neq 0$. Then $a_{ik}v^\sigma + a_{ki}v + a_{jk}uv^\sigma + a_{kj}vu^\sigma = a'_{ik}v^\sigma + a'_{ki}v + a'_{jk}uv^\sigma + a'_{kj}vu^\sigma = 0$. As

$$\frac{a'_{ik}}{a_{ik}} = \frac{a'_{ki}}{a_{ki}} \quad \text{and} \quad \frac{a'_{jk}}{a_{jk}} = \frac{a'_{kj}}{a_{kj}},$$

we have

$$\begin{aligned} & a_{ik}v^\sigma + a_{ki}v + a_{jk}uv^\sigma + a_{kj}vu^\sigma \\ &= b(a_{ik}v^\sigma + a_{ki}v) + c(a_{jk}uv^\sigma + a_{kj}vu^\sigma) \\ &= 0, \end{aligned}$$

with $b, c \in \mathbb{K}$. Assume, by way of contradiction, that

$$\begin{cases} a_{ij}u^\sigma + a_{ji}u &= 0, \\ a_{ik}v^\sigma + a_{ki}v &= 0, \\ a_{jk}uv^\sigma + a_{kj}vu^\sigma &= 0. \end{cases}$$

Then it readily follows that $a_{ij}a_{jk}a_{ki} + a_{ji}a_{ik}a_{kj} = 0$. As C is non-singular, we have $a_{ij}a_{jk}a_{ki} + a_{ji}a_{ik}a_{kj} \neq 0$, a contradiction. Hence $a_{ik}v^\sigma + a_{ki}v$ and $a_{jk}uv^\sigma + a_{kj}vu^\sigma$ are not both zero, so that $b = c$. Hence

$$\frac{a'_{ik}}{a_{ik}} = \frac{a'_{ki}}{a_{ki}} = \frac{a'_{jk}}{a_{jk}} = \frac{a'_{kj}}{a_{kj}}.$$

Now it readily follows that H' is uniquely determined by H .

Now suppose that H contains lines. If the line $e_i e_j$, $i \neq j$, does not belong to H , then as in the first part of (iii) we obtain

$$\frac{a'_{ij}}{a_{ij}} = \frac{a'_{ji}}{a_{ji}}.$$

If the line $e_i e_j$, $i \neq j$, belongs to H , then $a_{ij} = a_{ji} = a'_{ij} = a'_{ji} = 0$. Now we proceed as in the second part of the proof of (i). $\square$

Remark In the finite case, any $\mathbf{GF}(q^2)$ contains a unique involution. But in the infinite case, examples arise where distinct choices for τ can be made. For instance, one can extend the unique involution $x \mapsto x^q$ of $\mathbf{GF}(q^2)$, q odd, to the involutions $\sum a_i t^i \mapsto \sum a_i^q t^i$ and $\sum a_i t^i \mapsto \sum a_i^q (-t)^i$ of $\mathbf{GF}(q^2)(t)$.

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