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ALEXANDRU BUIUM

JOSÉ FELIPE VOLOCH

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# Lang's conjecture in characteristic $p$ : an explicit bound

ALEXANDRU BUIUM and JOSÉ FELIPE VOLOCH

*Institute for Advanced Study, Princeton, NJ 08540, Dept. of Mathematics, Univ. of Texas, Austin, TX 78712, USA, e-mail: voloch@math.utexas.edu*

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Let  $K$  be a function field in one variable with constant field  $k$  and denote by  $K_a, K_s$  its algebraic and separable closures, respectively. Let  $X/K$  be an algebraic curve of genus at least two. The function field analogue of Mordell's conjecture states that  $X(K)$  is finite unless  $X$  is  $K_a$ -isomorphic to a curve defined over  $k$ , in which case  $X$  is called isotrivial. This was first proved by Manin [Man] in characteristic zero and, shortly after, another proof was given by Grauert [Gra] and this proof was then adapted by Samuel [Sa] to positive characteristic. Since then several different proofs were given for Mordell's conjecture over function fields. In particular Szpiro [Sz] was the first to prove an effective version of Mordell's conjecture in characteristic  $p$ .

Mordell's conjecture was generalized by Lang [L] (and proved through work of Raynaud [R] and Faltings [F]). An analogue of Lang's conjecture over function fields of characteristic  $p$  was proved by the second author and Abramovich [V] [AV]. The aim of the present paper is to prove an effective version of Lang's conjecture in characteristic  $p$ . Our approach consists of combining the approaches in [B1] and [V] [AV] which in their turn were motivated by Manin's work [Man]. Here is our main result:

**THEOREM.** *Let  $K$  be a function field in one variable and characteristic  $p > 0$ . Let  $X$  be a smooth projective curve of genus  $g \geq 2$  over  $K$  embedded into its Jacobian  $J$ . Assume  $X$  has non-zero Kodaira-Spencer class (equivalently,  $X$  is not defined over  $K^p$ ). If  $\Gamma$  is a subgroup of  $J(K_s)$  such that  $\Gamma/p\Gamma$  is finite, then:*

$$\#(X \cap \Gamma) \leq \#(\Gamma/p\Gamma) \cdot p^g \cdot 3^g \cdot (8g - 2) \cdot g!$$

The equivalence of the vanishing of the Kodaira-Spencer class of  $X$  and  $X$  being defined over  $K^p$  is proved in [V], Lemma 1.

We stress the fact that we do not assume  $\Gamma$  is finitely generated, which is the main feature in Lang's conjecture that distinguishes it from the Mordell conjecture.

A similar result in characteristic zero was obtained by the first author [B4] but the bound there is huge as compared to the bound here. This is a reflection of the

fact that the characteristic  $p$  case is in some sense ‘easier’ than the characteristic zero case.

The question of the existence of this type of bounds for Lang’s conjecture was raised by Mazur in [Maz] and is quite different from what one understands by ‘effective Mordell’. In particular, even in the special case when  $\Gamma$  in our Theorem is finitely generated, our bound is not a consequence of Szpiro’s [Sz]. Indeed, assuming we are in the hypothesis of the Theorem above with  $\Gamma$  finitely generated, let  $K_1 \subset K_s$  be the field generated over  $K$  by the coordinates of the points in  $\Gamma$ . What Szpiro’s ‘effective Mordell’ yields is a bound for the height of the points in  $X(K_1)$  that depends on  $g = \text{genus of } X, p = \text{characteristic of } k, q_1 = \text{genus of } K_1$  and  $s_1 = \text{number of points of bad reduction of a semistable model of } X \otimes K_1/K_1$ . It follows that  $\#(X \cap \Gamma)$  is bounded by a constant that depends on  $g, p, q_1, s_1$ . But of course  $q_1, s_1$  are not bounded by a constant that depends on  $\#(\Gamma/p\Gamma)$  only; we may always keep  $\#(\Gamma/p\Gamma)$  constant and vary  $\Gamma$  so that both  $q_1$  and  $s_1$  go to infinity.

In order to prove the Theorem let us start by recalling a construction from [B1]. Assume we have fixed a derivation  $\delta = \partial/\partial t$  of  $K$  where  $t \in K$  is a separable transcendence basis of  $K/k$ . Then for any  $K$ -scheme  $X$  one defines the ‘first jet scheme along  $\delta$ ’ by the formula

$$X^1 := \text{Spec}(S(\Omega_{X/k})/I),$$

where  $I$  is the ideal generated by sections of the form  $df - \delta f$  ( $f \in \mathcal{O}_X$ ). This object was analysed in [B2], [B3] where the characteristic zero case only was considered. But many of the facts proved there extend, with identical proofs, in positive characteristic. In particular the following hold. Assume  $X$  above is a smooth variety over  $K$ . Then exactly as in [B1], p. 1396,  $X^1$  identifies with the torsor for the tangent bundle  $TX := \text{Spec}(S(\Omega_{X/K}))$  corresponding to the Kodaira-Spencer class

$$\rho(\delta) \in H^1(X, T_{X/K})$$

(where  $\rho: \text{Der}_k K \rightarrow H^1(X, T_{X/K})$  is the Kodaira-Spencer map; this map played various roles in virtually all approaches to the Mordell and Lang conjectures over function fields.) So exactly as in [B2], section 1, we may write  $X^1$  as the complement of a divisor in a projective bundle:

$$X^1 = \mathbf{P}(E) \setminus \mathbf{P}(\Omega_{X/K}),$$

where  $E$  is the vector bundle defined by the extension

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \Omega_{X/K} \rightarrow 0$$

corresponding to  $\rho(\delta) \in H^1(X, T_{X/K}) \simeq \text{Ext}^1(\Omega_{X/K}, \mathcal{O}_X)$ .

If  $X/K$  is a smooth group scheme then so is  $X^1/K$ .

Also, since  $\delta$  lifts to a derivation of  $K_s$ , there is an obvious ‘lifting map’

$$\nabla: X(K_s) \rightarrow X^1(K_s)$$

which in case  $X/K$  is a group is a homomorphism.

The following is the characteristic  $p$  analogue of a fact from [B3], (2.2):

LEMMA. *If  $X/K$  is a smooth projective curve of genus  $\geq 2$  with non zero-Kodaira-Spencer class then  $X^1$  is an affine surface.*

*Proof.* By the discussion preceding the Lemma it is enough to check that the divisor  $\mathbf{P}(\Omega_{X/K})$  is ample in  $\mathbf{P}(E)$ , equivalently that  $E$  is ample, which is the same as  $E_a$  being ample (where  $E_a$  is the pull back of  $E$  on  $X_a := X \otimes_K K_a$ ). Let  $F: X_a \rightarrow X_a$  be the absolute Frobenius (viewed as a scheme morphism over the integers). Assume  $E_a$  is not ample and seek a contradiction. By the characteristic  $p$  analogue of 'Gieseker's Theorem' [Gie] due to Martin-Deschamps [MD] it follows that there exists a power  $F^m: X_a \rightarrow X_a$  of  $F$  such that the pull back of the sequence

$$(*) \quad 0 \rightarrow \mathcal{O}_{X_a} \rightarrow E \rightarrow \Omega_{X_a/K_a} \rightarrow 0$$

via  $F^m$  splits. Now, since  $\Omega_{X_a/K_a}$  has degree  $2g - 2 > (2g - 2)/p$ , a result of Tango [T] Theorem 15 p. 73 implies that the sequence  $(*)$  itself must be split, which contradicts the fact that the Kodaira-Spencer class of  $X/K$  is non zero. This completes the proof of the Lemma.

*Proof of the Theorem.* The closed immersion  $X \subset J$  induces a closed immersion  $X^1 \subset J^1$ . For any point  $P \in X(K_s) \cap pJ(K_s)$  we have  $\nabla(P) \in X^1(K_s) \cap pJ^1(K_s)$ . Since  $J^1$  is an extension of  $J$  by a vector group (same argument as in [B2] (2.2)) the algebraic group  $B = pJ^1$  coincides with the maximum abelian subvariety of  $J^1$  and the projection  $B \rightarrow J$  is an isogeny. Moreover by [Ro], p. 704, Lemma 2, the natural isogeny (the Verschiebung)  $J^{(p)} \rightarrow J$  factors through  $B \rightarrow J$ . Since Verschiebung is of degree  $p^g$ ,  $B \rightarrow J$  has degree at most  $p^g$ .

In order to prove the Theorem it is obviously enough to prove that, over  $K_a$ ,  $X^1 \cap B$  is finite, of cardinality at most  $p^g \cdot 3^g \cdot (8g - 2) \cdot g!$ . Finiteness follows trivially from our Lemma above:  $X^1$  is affine and  $B$  is complete and both are closed in  $J^1$  so their intersection is closed in both  $X^1$  and  $B$ , so  $X^1 \cap B$  is both affine and complete, hence it is finite over  $K$ . To estimate its cardinality we use Bézout's theorem in Fulton's form, along the lines of [B4] (except that here we do not need any 'iteration' and we do not have to take multiplicities into account!).

Recall that  $X^1$  and  $J^1$  are Zariski locally trivial principally homogeneous spaces for the tangent bundles of  $X$  and  $J$  respectively, corresponding to the Kodaira-Spencer class. Let

$$0 \rightarrow \mathcal{O}_X \rightarrow E_X \rightarrow \Omega_{X/K} \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_J \rightarrow E_J \rightarrow \Omega_{J/K} \rightarrow 0$$

be the corresponding extensions. Consider the divisors  $D_X = \mathbf{P}(\Omega_{X/K}) \subset \mathbf{P}(E_X)$  and  $D_J = \mathbf{P}(\Omega_{J/K}) \subset \mathbf{P}(E_J)$ . Since  $\Omega_{J/K} \simeq \mathcal{O}_J^g$  we have  $D_J \simeq J \times \mathbf{P}^{g-1}$ .

Recall also that these divisors belong to the linear systems associated to  $\mathcal{O}_{\mathbf{P}(E_X)}(1)$  and  $\mathcal{O}_{\mathbf{P}(E_J)}(1)$  respectively and that we have identifications  $X^1 \simeq \mathbf{P}(E_X) \setminus D_X$  and  $J^1 \simeq \mathbf{P}(E_J) \setminus D_J$ . Let  $\alpha: X \rightarrow J$  be the inclusion. There is a natural restriction homomorphism  $\alpha^*E_J \rightarrow E_X$  prolonging the natural homomorphism  $\alpha^*\Omega_{J/K} \rightarrow \Omega_{X/K}$  since  $E_X = \Omega_{X/K^p} = \Omega_{X/k}$  and similarly for  $E_J$ . The homomorphism  $\alpha^*E_J \rightarrow E_X$  is clearly surjective so it induces a closed embedding  $\mathbf{P}(E_X) \subset \mathbf{P}(E_J)$  prolonging the embedding  $X^1 \subset J^1$ . By abuse we shall still denote by  $\pi_X, \pi_J$  the projections  $\mathbf{P}(E_X) \rightarrow X, \mathbf{P}(E_J) \rightarrow J$ .

*Claim.* The line bundle  $\mathcal{H} := \pi_J^*\mathcal{O}_J(3\Theta) \otimes \mathcal{O}_{\mathbf{P}(E_J)}(1)$  is very ample on  $\mathbf{P}(E_J)$ . (Here  $\Theta$  is the theta divisor on  $J$ .)

To check the Claim, note first that the trace of the linear system  $|\mathcal{H}|$  on  $D_J$  is very ample. Indeed

$$\mathcal{H} \otimes \mathcal{O}_{D_J} = \mathcal{H} \otimes \mathcal{O}_{\mathbf{P}(\Omega_{J/K})} = p_1^*\mathcal{O}_J(3\Theta) \otimes p_2^*\mathcal{O}_{\mathbf{P}^{g-1}}(1),$$

where  $p_1, p_2$  are the two projections of  $D_J = J \times \mathbf{P}^{g-1}$  onto the factors. So  $\mathcal{H} \otimes \mathcal{O}_{D_J}$  is very ample on  $D_J$ , cf. [Mum] p. 163. Furthermore we have an exact sequence

$$H^0(\mathbf{P}(E_J), \mathcal{H}) \rightarrow H^0(D_J, \mathcal{H} \otimes \mathcal{O}_{D_J}) \rightarrow H^1(\mathbf{P}(E_J), \pi_J^*\mathcal{O}_J(3\Theta)).$$

But the  $H^1$  above is zero (use the Leray spectral sequence and the vanishing theorem in [Mum] p. 150) so the trace of  $|\mathcal{H}|$  on  $D_J$  is a complete linear system and hence is very ample. In particular  $|\mathcal{H}|$  separates points of  $D_J$  and ‘vectors tangent to  $D_J$ ’. Since  $|\mathcal{H}|$  has no base points outside  $D_J$  either, it follows that  $|\mathcal{H}|$  is base point free on  $\mathbf{P}(E_J)$ . Hence  $|\mathcal{H}|$  restricted to the fibres of  $\pi_J$  is base point free. Since any base point free linear subsystem of  $|\mathcal{O}_{\mathbf{P}^g}(1)|$  equals actually the whole of  $|\mathcal{O}_{\mathbf{P}^g}(1)|$  it follows that  $|\mathcal{H}|$  separates points in each fibre of  $\pi_J$  and separates ‘vectors tangent to each fibre’. All these imply that  $|\mathcal{H}|$  separate points and tangent vectors on the whole of  $\mathbf{P}(E_J)$  and our Claim is proved.

Our last step is to compute the degrees  $\deg_{\mathcal{H}}\mathbf{P}(E_X)$  and  $\deg_{\mathcal{H}}B$  of  $\mathbf{P}(E_X)$  and  $B$  respectively, as subvarieties of  $\mathbf{P}(E_J)$  with respect to the embedding defined by  $\mathcal{H}$ . Note that

$$\mathcal{H} \otimes \mathcal{O}_{\mathbf{P}(E_X)} = \pi_X^*\mathcal{O}_X(3\Theta) \otimes \mathcal{O}_{\mathbf{P}(E_X)}(1).$$

We may compute the selfintersection

$$(\mathcal{O}_{\mathbf{P}(E_X)}(1) \cdot \mathcal{O}_{\mathbf{P}(E_X)}(1))_{\mathbf{P}(E_X)} = \deg \Omega_{X/K} = 2g - 2$$

and since  $(\Theta \cdot X)_J = g$  we get

$$\deg_{\mathcal{H}}\mathbf{P}(E_X) = 2g - 2 + 6g = 8g - 2.$$

On the other hand we have

$$\mathcal{H} \otimes \mathcal{O}_B \simeq \pi^* \mathcal{O}_J(3\Theta),$$

where  $\pi: B \rightarrow J$  is the projection which we already know has degree at most  $p^g$ . So we get, using  $(\Theta^g)_J = g!$ , that

$$\deg_{\mathcal{H}} B = p^g \cdot 3^g \cdot g!$$

Now Bezout's theorem in Fulton's form [Fu] p. 148, says that the number of irreducible components in the intersection of two projective varieties of degrees  $d_1, d_2$  cannot exceed  $d_1 d_2$ . In particular

$$\#(X^1 \cap B) \leq \deg_{\mathcal{H}} \mathbf{P}(E_X) \cdot \deg_{\mathcal{H}} B \leq (8g - 2) \cdot p^g \cdot 3^g \cdot g!$$

and our Theorem is proved.

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