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On some modular representations of affine Kac–Moody algebras at the critical level

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Introduction

Let k be a field of arbitrary characteristic, let $\mathfrak{g}$ be an affine Kac–Moody algebra and let U be the Chevalley–Kostant algebra associated to $\mathfrak{g}$. Recall that when k has characteristic zero, U is the enveloping algebra of $\mathfrak{g}$. However in characteristic p, U contains the restricted enveloping algebra together with some divided powers (see Section 1 for a precise definition). For any weight λ, denote by $l(\lambda)$ the simple highest weight U-module with highest weight λ.

The aim of this paper is to prove the following two theorems.

THEOREM 1. *We have* $\mathrm{ch}(l(-\rho)) = e^{-\rho}\Pi_{\alpha\in\Delta_{re}^+} 1/(1 - e^{-\alpha})$.

THEOREM 2. *Assume k is a field of characteristic p. We have* $\mathrm{ch}(l((p-1)\rho)) = e^{(p-1)\rho}\Pi_{\alpha\in\Delta_{re}^+}(1 - e^{-p.\alpha})/(1 - e^{-\alpha})$.

For fields of characteristic 0, Theorem 1 (which has been conjectured by V. G. Kac and D. Kazhdan [KK]) was already known. It has been proved by M. Wakimoto [Wk], N. Wallach [Wl] (in the $\mathfrak{sl}(2)$ case), J. M. Ku [Ku] (in general), T. Hayashi [H] (for classical type affine Lie algebras), B. Feigin and E. Frenkel [FF$_3$], [F] (in general). See also related results by R. Goodman, N. Wallach [GW] and F. Malikov [Ml].

To prove Theorem 1, we need to prove the following two inequalities.
First inequality: $\mathrm{ch}(l(-\rho)) \geqslant e^{-\rho}\Pi_{\alpha\in\Delta_{re}^+} 1/(1 - e^{-\alpha})$.
Second inequality: $\mathrm{ch}(l(-\rho)) \leqslant e^{-\rho}\Pi_{\alpha\in\Delta_{re}^+} 1/(1 - e^{-\alpha})$.

The proof of the first inequality is based on characteristic p methods. When k is a field of characteristic p, we use the isomorphism $l(-\rho) \simeq l((p-1)\rho) \otimes F_* l(-\rho)$ (where F is the Frobenius morphism of U) and the integrability of the module $l((p-1)\rho)$.

To prove the second inequality, we can restrict ourself to the case where k has characteristic zero. Thus this inequality follows from the previously cited works. In particular this part of the proof is not independent of these works. However note that the second inequality follows obviously from the existence of the restricted Wakimoto module $\overline{\mathcal{W}}_\rho$ (as defined in [Wk], [FF1], [FF2]).

It turns out that Theorem 1 and Theorem 2 are actually equivalent. Our approach provides a very simple proof of the first inequality (especially without using Kazdhan–Kac determinant formula [KK]). It gives a new proof of the irreducibility of the Wakimoto module with highest weight $-\rho$ (see [Wk], [FF3]). Moreover we get that the reductions in finite characteristics of the **Z**-form of $l(-\rho)$ are simple modules. For a field of characteristic p, this proves that the Steinberg module $L((p-1)\rho)$ (i.e. the maximal integrable quotient of the Verma module with highest weight $(p-1)\rho$) is not simple what contrasts with the modular theory of finite dimensional Lie algebras. In a forthcoming paper [Mt4], a similar method will be used to study the structure of the endomorphism ring of the Verma module with highest weight $-\rho$.

1. Modular Kac–Moody algebras

(1.1) Let $A = (a_{i,j})_{i,j\in I}$ be a generalized Cartan matrix, let $(\mathfrak{h}_{\mathbf{Q}}, (h_i)_{i\in I}, (\alpha_i)_{i\in I})$ be a realization of A over **Q** (in the sense of Kac, see [K]) and let $\mathfrak{h}_{\mathbf{Z}}$ be an integral realization of A (see [Mt1], ch. 1). Let Q be the root lattice and set $P = \mathfrak{h}_{\mathbf{Z}}^*$. Let $\mathfrak{g}_{\mathbf{Q}}$ be the Kac–Moody algebra over **Q** associated to A. By definition, $\mathfrak{g}_{\mathbf{Q}}$ is generated by $\mathfrak{h}_{\mathbf{Q}}, (e_i)_{i\in I}, (f_i)_{i\in I}$ and defined by some local relations and Serre relations (i.e. it is the maximal choice).

There is a triangular decomposition $\mathfrak{g}_{\mathbf{Q}} = \mathfrak{n}_{\mathbf{Q}}^+ \oplus \mathfrak{h}_{\mathbf{Q}} \oplus \mathfrak{n}_{\mathbf{Q}}^-$ where $\mathfrak{n}_{\mathbf{Q}}^+$ (respectively $\mathfrak{n}_{\mathbf{Q}}^-$) is the subalgebra generated by $(e_i)_{i\in I}$ (respectively by $(f_i)_{i\in I}$). Denote by $U_{\mathbf{Q}}, U_{\mathbf{Q}}^+, U_{\mathbf{Q}}^0, U_{\mathbf{Q}}^-$ the enveloping algebras of $\mathfrak{g}_{\mathbf{Q}}, \mathfrak{n}_{\mathbf{Q}}^+, \mathfrak{h}_{\mathbf{Q}}$ and $\mathfrak{n}_{\mathbf{Q}}^-$.

Following Tits [T1], [T2] (see also [M1]), we denote by $U_{\mathbf{Z}}$ (respectively $U_{\mathbf{Z}}^-, U_{\mathbf{Z}}^+, U_{\mathbf{Z}}^0$) the subring of $U_{\mathbf{Q}}$ generated hy the elements $e_i^{(n)}, f_i^{(n)}, \binom{h}{n}$ (respectively by $f_i^{(n)}$, by $e_i^{(n)}$, by $\binom{h}{n}$) where i, n, h runs over $I, \mathbf{Z}_+, \mathfrak{h}_{\mathbf{Z}}$. We set $\mathfrak{g}_{\mathbf{Z}} = \mathfrak{g}_{\mathbf{Q}} \cap U_{\mathbf{Z}}$. Similarly there is a triangular decomposition $\mathfrak{g}_{\mathbf{Z}} = \mathfrak{n}_{\mathbf{Z}}^+ \oplus \mathfrak{h}_{\mathbf{Z}} \oplus \mathfrak{n}_{\mathbf{Z}}^-$, where $\mathfrak{n}_{\mathbf{Z}}^\pm = \mathfrak{n}_{\mathbf{Q}}^\pm \cap U_{\mathbf{Z}}^\pm$.

Let W be the Wely group of $\mathfrak{g}_{\mathbf{Q}}$, and let Δ (respectively $\Delta^-, \Delta^+, \Delta_{re}^+$) be the set of roots (respectively of negative roots, of positive roots, of real positive roots). For any $\lambda \in P$ we will denote by $l_{\mathbf{Q}}(\lambda)$ the unique irreducible highest weight module over **Q** with highest weight λ, see [K].

For any simple root $\alpha = \alpha_i$, denote by $U_{\mathbf{Z}}^\alpha$ be subring generated by $U_{\mathbf{Z}}^+, U_{\mathbf{Z}}^0$ and the elements $f_i^{(n)}, n \geqslant 0$. Some natural duals of the Hopf **Z**-algebras $U_{\mathbf{Z}}^\pm, U_{\mathbf{Z}}^0$ and $U_{\mathbf{Z}}^\alpha$ are defined in [Mt1] (see Ch. 1). They are denoted respectively by $\mathbf{Z}[\mathbf{U}^\pm]$, $\mathbf{Z}[\mathbf{H}]$ and $\mathbf{Z}[\mathbf{P}_\alpha]$. Recall that $\mathbf{Z}[\mathbf{U}^\pm]$ are polynomial algebras over **Z** ([Mt2], see Lemma 2 and its proof). Roughly speaking, these Hopf algebras are the $\mathbf{Z}$-forms of the space of regular functions over, respectively, the unipotent radicals $\mathbf{U}^{\pm}$ of the two opposite Borel subgroups, the Cartan subgroups $\mathbf{H}$ and the minimal parabolic subgroups $\mathbf{P}_{\alpha}$.

(1.2) From now we denote by k a perfect field of finite characteristic p. The tensor product of k with $\mathfrak{g}_{\mathbf{Z}}, \mathfrak{n}^{\pm}_{\mathbf{Z}}, \mathfrak{h}_{\mathbf{Z}}, U_{\mathbf{Z}}, U^{\pm}_{\mathbf{Z}}, U^{0}_{\mathbf{Z}}, U^{\alpha}_{\mathbf{Z}}, \mathbf{Z}[\mathbf{U}^{\pm}], \mathbf{Z}[\mathbf{H}]$ and $\mathbf{Z}[\mathbf{P}_{\alpha}]$ will be denoted by $\mathfrak{g}, \mathfrak{n}^{\pm}, \mathfrak{h}, U, U^{\pm}, U^{0}, U^{\alpha}, k[\mathbf{U}^{\pm}], k[\mathbf{H}]$ and $k[\mathbf{P}_{\alpha}]$. For any $\lambda \in P$, let $l(\lambda)$ be the unique irreducible highest weight module with highest weight λ, and let v_{λ} be its highest weight vector. For any dominant weight $\lambda \in P$, let $L(\lambda)$ be the maximal integrable quotient of the Verma module with highest weight λ. Also denote by v_{λ} its highest weight vector.

LEMMA 1.3. *There is a k-linear morphism of algebras $F: U \to U$ uniquely defined on the generators of U by $F(e_i^{(n)}) = F(f_i^{(n)}) = F\binom{h}{n} = 0$ if p does not divide n and $F(e_i^{(pn)}) = e_i^{(n)}, F(f_i^{(pn)}) = f_i^{(n)}, F(\binom{h}{pn}) = \binom{h}{n}$ for any $i \in I$, $n \in \mathbf{Z}_{+}$ and $h \in \mathfrak{h}_{\mathbf{Z}}$.*

Proof. We can identify $U^{\pm}$ with the space of right invariant, left $\mathrm{Ad}(U^{\pm})$ locally nilpotent differential operators of the k-algebra $k[\mathbf{U}^{\pm}]$. As $k[\mathbf{U}^{\pm}]$ has no zero divisors and is naturally defined over $\mathbf{F}_p$, there is a natural k-linear isomorphism $\sigma_{\pm} : (k[\mathbf{U}^{\pm}])^{p} \simeq k[\mathbf{U}^{\pm}]$. For any $u \in U^{\pm}$, we have $u.(k[\mathbf{U}^{\pm}])^{p} \subset (k[\mathbf{U}^{\pm}])^{p}$. Define k-algebra morphisms $F_{\pm} : U^{\pm} \to U^{\pm}$ by the formula $F_{\pm}(u) = \sigma_{\pm} \circ v \circ \sigma_{\pm}^{-1}$, where v is the restriction of u to $(k[\mathbf{U}^{\pm}])^{p}$. Define in a similar way the morphisms F_0, F_{α} of the k-algebras U^{0}, U_{α}.

As we have $U \simeq U^{+} \otimes U^{0} \otimes U^{-}$, we can define a k-linear map $F : U \to U$ by $F = F_{+} \otimes F_0 \otimes F_{-}$. To check that F is a morphism of algebras, it suffices to prove that $F(xy) = F(x)F(y)$ at least when x is one of the generators of U. There is a simple root α such that x belongs to U_{α} and we can assume that $y = z.w$ where $z \in U^{+} \otimes U^{0}$ and $w \in U^{-}$. It is clear that the restriction of F to U_{α} is precisely F_{α}. In particular one gets $F(x.z) = F(x).F(z)$. Using that F_{-} is a morphism of algebras, one gets $F(a.w) = F(a).F(w)$ for any $a \in U$. So we get $F(x.y) = F(x.z.w) = F(x.z).F(w) = F(x).F(z).F(w) = F(x).F(z.w) = F(x).F(y)$. □

(1.4) For a restricted Lie algebra $\mathfrak{a}$ over k, denote by $\mathfrak{u}(\mathfrak{a})$ its restricted enveloping algebra, see [J]. The Lie algebra $\mathfrak{g}$ is a restricted Lie algebra. As $\mathfrak{g}$ is the sum of the three Lie algebras $\mathfrak{n}^{\pm}$ and $\mathfrak{h}$, it suffices to describe the p-structure on each of these three Lie sub-algebras. The p-structure $x \to x^{[p]}$ on $\mathfrak{n}^{\pm}$ is simply given by the requirement that $x^{[p]}$ is the derivation x^{p} on $k[\mathbf{U}^{\pm}]$. Over $\mathfrak{h}$ the requirement is $h^{[p]} = h$ for any $h \in \mathfrak{h} \otimes \mathbf{Z}/p\mathbf{Z}$. As U is an Hopf algebra, the natural map $\mathfrak{u}(g) \to U$ is one-to-one.

A weight $\lambda \in P$ is called restricted if we have $0 \leqslant \lambda(h_i) \leqslant p - 1$ for any $i \in I$. However the use of the same terminology 'restricted' for weights and for representations is meaningless. E.g. as $\mathfrak{g}$-module, any U-module is restricted.

(1.5) Let $\mathcal{O}$ be the category of all U-modules M which are locally nilpotent as U^+-module and such that $M = \oplus_{\lambda\in P} M_\lambda$ as U^0-module, where all weight spaces M_λ are finite dimensional. There is an involution of U exchanging the generators $e_i^{(n)}$ and $f_i^{(n)}$. Thus one defines the $\mathcal{O}$ dual $M^\#$ of M as the vector space $\oplus_{\lambda\in P}(M_\lambda^*)$ where the action is twisted by this involution.

For a formal expression $A = \Sigma_{\lambda\in P} m_\lambda e^\lambda$, set $FA = \Sigma_{\lambda\in P} m_\lambda e^{p.\lambda}$. Thus for any module $M \in \mathcal{O}$, we have $F\,\mathrm{ch}(M) = \mathrm{ch}(F_* M)$ and $\mathrm{ch}(M^\#) = \mathrm{ch}(M)$.

(1.6) In order to give another description of the modular Kac–Moody algebras (Proposition 1.7), we will introduce a few other notations. We will use only part of Proposition 1.7, which is stated here for clarity and for a future reference [Mt4]. Set $\mathfrak{n}'^- = \Sigma_{i\in I}\mathrm{Ad}(U^-)f_i$. When k is infinite, it is clear that $\mathfrak{n}'^-$ is the span of $\mathrm{Ad}(V)(f_i)$, where V is the subgroup of the adjoint group generated by the elements $\exp(t.f_j)$ $(t \in k, j \in I)$. Thus it is clear that $\mathfrak{n}'^-$ is a restricted ideal of $\mathfrak{n}^-$.

LEMMA 1.7. *Let λ be a restricted weight, and let L be a nonzero quotient of $L(\lambda)$. Then we have $L = \mathfrak{u}(\mathfrak{n}'^-).v_\lambda$. Assume now that L is simple. Then L is simple as $\mathfrak{g}$-module and we have $H^0(\mathfrak{n}^+, L) = k.v_\lambda$.*

Set $L' = \mathfrak{u}(\mathfrak{n}'^-).v_\lambda$ and let $i \in I$ and $u \in \mathfrak{u}(\mathfrak{n}'^-)$. As we have $f_i^{(m)}.v_\lambda = 0$ for any $m \geqslant p$, we get

$$f_i^{(m)}.u.v_\lambda = \sum_{0\leqslant s<p} \mathrm{Ad}(f_i^{(m-s)})(u).f_i^{(s)}.v_\lambda \quad \text{for any } m \geqslant 0.$$

Hence L' is a U^--submodule of L. Thus we get $L = L'$. Assume now that L is simple. Then L is isomorphic to its $\mathcal{O}$-dual $L^\#$. Thus by duality $H^0(\mathfrak{n}^+, L)$ is one dimensional. It follows that L is simple as $\mathfrak{g}$-module.

PROPOSITION 1.8. (1.7.1) *The module $L((p-1).\rho)$ is $\mathfrak{u}(n^-)$-free with generator $v_{(p-1).\rho}$.*

(1.7.2) *We have $\mathfrak{n}'^- = \mathfrak{n}^-$. In particular $\mathfrak{g}$ is generated by $\mathfrak{h}$, $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ as U-module.*

(1.7.3) *Assume that $p > -a_{i,j}$, for any $i \neq j$. As Lie algebra, $\mathfrak{g}$ is generated by $\mathfrak{h}$, $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$.*

Proof. Recall that $\mathrm{ch}(L(p-1)\rho)$ is given by Weyl–Kac formula [Mt1]. By using the denominator formula, we get

$$\mathrm{ch}(L(p-1)\rho)) = e^{(p-1)\rho} \prod_{\alpha\in\Delta^+} (1 - e^{-p.\alpha})^{m_\alpha}/(1 - e^{-\alpha})^{m_\alpha},$$

where m_α is the multiplicity of the root α. So we get $\mathrm{ch}(L((p-1)\rho) = e^{(p-1).\rho}\,\mathrm{ch}(\mathfrak{u}(\mathfrak{n}^-))$. By Lemma 1.6, the natural map $\Theta : \mathfrak{u}(\mathfrak{n}'^-) \to L((p-1)\rho)$,

$u \mapsto u.v_{(p-1)\rho}$ is surjective. Hence the $\mathfrak{u}(n^-)$-module $L((p-1).\rho)$ is free and we have $\mathfrak{n}'^- = \mathfrak{n}^-$.

Moreover it follows that $U^- = \mathfrak{u}(n)^- \oplus I$, where I is the left ideal generated by the $f_i^{(n)}$ where $i \in I$ and $n \geqslant p$. Under hypothesis (1.7.3), we have $\mathrm{Ad}(I)(f_i) = 0$, hence $\mathfrak{n}^-$ is generated as a $\mathfrak{u}(n)^-$ by the f_i. It follows easily that $\mathfrak{n}^-$ is generated by the $f_i, i \in I$. As a similar result holds for $\mathfrak{n}^+$, assertion (1.7.3) is proved. □

LEMMA 1.9 (Steinberg tensor product theorem). *Let $\lambda, \mu, \nu \in P$. Assume $\lambda = \mu + p.\nu$ and μ is restricted. Then we have $l(\lambda) = l(\mu) \otimes F_* l(\nu)$ and* $\mathrm{ch}(l(\lambda)) = \mathrm{ch}(l(\mu)).F\,\mathrm{ch}(l(\nu))$.

Proof. Let x, y be nonzero highest weight vectors of $l(\mu), F_* l(\nu)$ and let Z be the space of U^+ invariant vectors of $l(\mu) \otimes F_* l(\nu)$. Note that $\mathfrak{n}^+$ acts trivially on $F_* l(\nu)$. So by Lemma 1.6, we get $H^0(\mathfrak{n}^+, l(\mu) \otimes F_* l(\nu)) = k.x \otimes F_* l(\nu)$. Hence Z is the space of U^+-invariant vectors of $k.x \otimes F_* l(\nu)$, i.e. $Z = k.x \otimes y$. By using the duality in category $\mathcal{O}$, we also get that $l(\mu) \otimes F_* l(\nu)$ is generated by $x \otimes y$ as U^--module. Hence $l(\mu) \otimes F_* l(\nu)$ is simple. As its highest weight is λ, this module is isomorphic to $l(\lambda)$. □

LEMMA 1.10. *We have* $\mathrm{ch}(l(-\rho)) = \mathrm{ch}(l((p-1)\rho)).F\,\mathrm{ch}(l(-\rho)))$.

Proof. Apply the previous lemma to $\lambda = -\rho, \mu = (p-1)\rho, \nu = -\rho$. Note as corollary that $\mathrm{ch}(l(-\rho))$ is uniquely determined by $\mathrm{ch}(l((p-1)\rho))$ and conversely. □

2. The first inequality

For any $\alpha \in \Delta_{re}^+$, let f_α be a basis of $\mathfrak{g}_\alpha$. For $w \in W$, set $\Delta_w = \{\alpha \in \Delta^+ | w.\alpha \in \Delta^-\}$ and set $\mathfrak{n}_w = \oplus_{\alpha \in \Delta_w} \mathfrak{g}_{-\alpha}$. Let $s_{i_l}.s_{i_{l-1}} \dots s_{i_1}$ be a reduced decomposition of w where $l = l(w)$. Set $\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}\alpha_{i_2}, \dots, \beta_l = s_{i_1} \dots s_{i_{l-1}} \alpha_{i_l}$, where $l = l(w)$. Recall that we have $\Delta_w = \{\beta_1, \dots, \beta_l\}$. Set $F_w = f_{\beta_l}^{(p-1)} \dots f_{\beta_2}^{(p-1)}.f_{\beta 1}^{(p-1)}$.

LEMMA 2.1. (2.1.1) *We have* $\mathrm{ch}\,\mathfrak{u}(\mathfrak{n}_w) = \Pi_{0 \leqslant j < l}(\Sigma_{0 \leqslant s < p} e^{-s.\beta_j})$.

(2.1.2) *The dimension of* $\mathfrak{u}(\mathfrak{n}_w)_{(p-1)(w^{-1}.\rho-p)}$ *is exactly one, and F_w is a basis of this vector space.*

(2.1.3) *For any nonzero element $u \in \mathfrak{u}(\mathfrak{n}_w)$, there exists $u' \in \mathfrak{u}(\mathfrak{n}_w)$ such that $u'.u = F_w$.*

Proof. For a n dimensional vector space V, let $\bar{S}V$ be the quotient of SV by the ideal generated by all the elements $x^p, x \in V$. In other words $\bar{S}V$ is the restricted enveloping algebra of the commutative Lie algebra V endowed with a trivial p-structure. Note that $\mathfrak{u}(\mathfrak{n}_w)$ has a natural filtration $(\mathcal{F}_s)_{s \geqslant 0}$ defined by $\mathcal{F}_0 = k$ and $\mathcal{F}_{s+1} = \mathcal{F}_s + \mathfrak{n}_w.\mathcal{F}_s$ for $s \geqslant 0$. The associated graded vector space is canonically isomorphic with $\bar{S}\mathfrak{n}_w$. The assertion on the character follows. Then an easy calculation shows that $H^0(V, \bar{S}V)$ is one dimensional. This space is exactly the space of elements of degree $n(p-1)$ in $\bar{S}V$ and it is canonically isomorphic with $(\wedge^n V)^{\otimes(p-1)}$. Thus the isomorphism $\mathcal{F}_{l(p-1)}/\mathcal{F}_{l(p-1)-1} \simeq \wedge^l \mathfrak{n}_w$ induces a map $\sigma : \mathfrak{u}(\mathfrak{n}_w) \to (\wedge^l \mathfrak{n}_w)^{\otimes p-1}$. We have $\sigma(F_w) = (f_{\beta_1} \wedge \dots f_{\beta_l})^{\otimes(p-1)}$. Hence F_w is not zero (see [Mt3], proof of Lemma 6.3.2 for more details).

Note that $(p-1)(w^{-1}.\rho - \rho) - \beta_j$ is not a weight of $\mathfrak{u}(\mathfrak{n}_w)$ (for any j, $1 \leqslant j \leqslant l$). Hence F_w is $\mathfrak{n}_w$-invariant for the left action. Thus F_w is the only (up to scalar) left invariant vector in $\mathfrak{u}(\mathfrak{n}_w)$. The Lie algebra $\mathfrak{n}_w$ is nilpotent and the left multiplication by any basis element f_{β_i} is a nilpotent endomorphism. Hence for the left action, $\mathfrak{u}(\mathfrak{n}_w)$ is a nilpotent $\mathfrak{n}_w$-module and any nonzero submodule of $\mathfrak{u}(\mathfrak{n}_w)$ contains a nonzero invariant vector. Thus for any nonzero $u \in \mathfrak{u}(\mathfrak{n}_w)$, we can find u' such that $u'.u = F_w$. □

For now on, we will denote the highest weight vector of $l((p-1)\rho)$ by v.

LEMMA 2.2. *The* $\mathfrak{u}(\mathfrak{n}_w)$*-module* $\mathfrak{u}(\mathfrak{n}_w).v$ *is free.*

Proof. By an easy induction on j we prove that $f_{\beta_j}^{(p-1)}.f_{\beta_{j-1}}^{(p-1)} \dots f_{\beta_1}^{(p-1)}.v$ is a nonzero extremal weight vector of $l((p-1)\rho)$. In particular we get $F_w.v \neq 0$. Using Lemma 2.1.3 we deduce that the map $\mathfrak{u}(\mathfrak{n}_w) \to l((p-1)\rho), u \mapsto u.v$ is injective. Thus $\mathfrak{u}(\mathfrak{n}_w).v$ is free. □

A sequence $\omega = (w_n)_{n \in \mathbf{Z}_+}$ of elements of W such $\Delta_{w_n} \subset \Delta_{w_{n+1}}$ is called an end of W. For an end ω of W, we set $\Delta_\omega = \cup_{n \in \mathbf{Z}_+} \Delta_{w_n}$ and $\mathfrak{n}_\omega = \cup_{n \in \mathbf{Z}_+} \mathfrak{n}_{w_n}$. Thus we get

LEMMA 2.3. *For any end* ω *of* W*, the* $\mathfrak{u}(\mathfrak{n}_\omega)$*-module* $\mathfrak{u}(\mathfrak{n}_\omega).v$ *is free*

From now on we will assume that $\mathfrak{g}$ is affine. Write $I = I_0 \cup \{0\}$ as in [K], and let δ be any positive imaginary root. Moreover, choose an element $t \in W$ and $h \in \mathfrak{h}$ with $\alpha_i(h) > 0$ for any $i \in I_0$, such that $t(\mu) = \mu + \mu(h)\delta$ for any $\mu \in Q$. It is clear that the sequence $\omega = (t^n)_{n \in \mathbf{Z}_+}$ is an end of W. For $n \geqslant 0$, set $\Delta(n) = \Delta_\omega \cup \Delta_{t^{-n}}$. Roughly speaking, Δ_ω contains half of the set of positive real roots. Indeed, we have $\cup_{n \in \mathbf{Z}_+} \Delta(n) = \Delta_{re}^+$ and $t^n \Delta_\omega = \Delta_\omega \cup -\Delta_{t^{-n}}$.

LEMMA 2.4. *(First inequality)*

(2.4.1) *We have* $\mathrm{ch}(l((p-1)\rho)) \geqslant e^{(p-1)\rho} \Pi_{\alpha \in \Delta_{re}^+} (1 - e^{-p.\alpha})/(1 - e^{-\alpha})$.

(2.4.2) *We have* $\mathrm{ch}(l(-\rho)) \geqslant e^{-\rho} \Pi_{\alpha \in \Delta_{re}^+} 1/(1 - e^{-\alpha})$.

Proof. By Lemma 2.3 we have $\mathrm{ch}(l((p-1)\rho)) \geqslant e^{(p-1)\rho} \Pi_{\alpha \in \Delta_\omega} (1 - e^{-p.\alpha})/(1 - e^{-\alpha})$. Let $n \geqslant 0$. Recall that we have $\Sigma_{\alpha \in \Delta_{t^{-n}}} \alpha = \rho - t^n.\rho$. Thus we deduce

$$e^{(p-1)\rho} \prod_{\alpha \in \Delta_{t^{-n}}} \frac{(1 - e^{-p.\alpha})}{(1 - e^{-\alpha})} = e^{(p-1).t^n \rho} \prod_{\alpha \in \Delta_{t^{-n}}} \frac{(1 - e^{p.\alpha})}{(1 - e^{\alpha})}.$$

As we have $t^n\Delta_\omega = \Delta_\omega \cup -\Delta_{t^{-n}}$, we get

$$t^n.\left(e^{(p-1)\rho}\prod_{\alpha\in\Delta_\omega}\frac{(1-e^{-p.\alpha})}{(1-e^{-\alpha})}\right)$$

$$= e^{(p-1).t^n\rho}\prod_{\alpha\in\Delta_{t^{-n}}}\frac{(1-e^{p.\alpha})}{(1-e^{\alpha})}\prod_{\alpha\in\Delta_\omega}\frac{(1-e^{-p.\alpha})}{(1-e^{-\alpha})}$$

$$= e^{(p-1)\rho}\prod_{\alpha\in\Delta(n)}\frac{(1-e^{-p.\alpha})}{(1-e^{-\alpha})}.$$

Using the W-invariance of $\mathrm{ch}(l(p-1)\rho)$, we get

$$\mathrm{ch}(l((p-1)\rho)) \geqslant e^{(p-1)\rho}\prod_{\alpha\in\Delta(n)}\frac{(1-e^{-p.\alpha})}{(1-e^{-\alpha})}.$$

By passing to the limit we finally get

$$\mathrm{ch}(l((p-1)\rho)) \geqslant e^{(p-1)\rho}\prod_{\alpha\in\Delta^+_{re}}\frac{(1-e^{-p.\alpha})}{(1-e^{-\alpha})}.$$

Thus Assertion (2.4.1) is proved. Moreover (2.4.2) follows from (2.4.1) and Lemma 1.9.

COROLLARY 2.5. (2.5.1) *We have* $\mathrm{ch}(l_{\mathbf{Q}}(-\rho)) \geqslant e^{-\rho}\Pi_{\alpha\in\Delta^+_{re}}1/(1-e^{-\alpha})$.

(2.5.2) *The restricted Wakimoto module* $\overline{\mathcal{W}}_{-\rho}$ *of highest weight* $-\rho$ *(as defined in general by Feigin and Frenkel [FF1], [FF2]) is irreducible.*

Proof. Inequality (2.5.1) follows from Lemma 2.4.2 and the semicontinuity principle. By construction $\mathcal{W}_{-\rho}$ contains $l(-\rho)$ as subquotient and its character is $e^{-\rho}\Pi_{\alpha\in\Delta^+_{re}}1/(1-e^{-\alpha})$ (see [FF1], [FF2]). Thus inequality (2.5.1) implies that $\overline{\mathcal{W}}_{-\rho}$ is simple. □

3. The second inequality

LEMMA 3.1. *Let* $\lambda \in P$. *The we have* $\mathrm{ch}(l_{\mathbf{Q}}(\lambda)) \geqslant \mathrm{ch}(l(\lambda))$.

Proof. Set $l_{\mathbf{Z}}(\lambda) = U^-_{\mathbf{Z}}.v_\lambda$. It is clear that $l_{\mathbf{Z}}(\lambda)$ is a $U_{\mathbf{Z}}$-invariant lattice of $l_{\mathbf{Q}}(\lambda)$. It is clear that $l(\lambda)$ is a quotient of $k \otimes l_{\mathbf{Z}}(\lambda)$. Thus we get the inequality (3.1). □

References concerning next lemma are given in the introduction. In particular it follows from the construction of the restricted Wakimoto module $\overline{\mathcal{W}}_{-\rho}$.

LEMMA 3.2. [Wk], [GW], [Ku], [H], [FF3].

We have $\mathrm{ch}(l_{\mathbf{Q}}(-\rho)) \leqslant e^{-\rho}\Pi_{\alpha\in\Delta^+_{re}}1/(1-e^{-\alpha})$.

LEMMA 3.3. *(Second inequality)*

(3.3.1) *We have* $\mathrm{ch}(l(-\rho)) \leqslant e^{-\rho}\Pi_{\alpha\in\Delta_{re}^{+}} 1/(1-e^{-\alpha})$.

(3.3.2) *We have* $\mathrm{ch}(l((p-1)\rho)) \leqslant e^{(p-1)\rho}\Pi_{\alpha\in\Delta_{re}^{+}}(1-e^{-p.\alpha})/(1-e^{-\alpha})$.

Proof. Inequality (3.3.1) follows from Lemmas 3.1 and 3.2. Inequality (3.3.2) follows from inequality (3.3.1) and Lemma 1.8. □

Both Theorems follow from Lemmas 2.4 and 3.3.

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