Composititio Mathematica

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Compositio Mathematica, tome 102, nº 2 (1996), p. 141-145
<http://www.numdam.org/item?id=CM_1996__102_2_141_0>

Compositio Mathematica **102:** 141–145, 1996.

An improved bound for the degree of smooth surfaces in $\mathbf{P}^4$ not of general type

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Received 15 November 1994; accepted in final form 30 March 1995

This is an addendum to the paper of Braun and Fløystad ([BF]) on the bound for the degree of smooth surfaces in $\mathbf{P}^4$ not of general type. (In fact, any statements made here without comment will be found there.) Using their construction and the regularity of curves in $\mathbf{P}^3$, one may lower the bound a little more. We will prove the following:

THEOREM. *Let S be a smooth surface of degree d in $\mathbf{P}^4$ not of general type. Then either $d \leqslant 70$ or S lies on a hypersurface in $\mathbf{P}^4$ of degree 5 and $d \leqslant 80$. Thus $d \leqslant 80$.*

(We will also indicate how one may actually get the bound down to 76.)

The main result of [BF] is to bound $\chi\mathcal{O}_S$ from below using generic initial ideal theory and information coming from a generic hyperplane section of S.

Let C be a generic hyperplane section of S, then C has associated to it two pieces of information relevant to this situation; *connected invariants* $\lambda_0 > \lambda_1 > \cdots > \lambda_{s-1} > 0$ and *sporadic zeros* which are defined as follows:

DEFINITION. Let $\mathbf{C}[x_0, x_1, x_2, x_3]$ be the ring of polynomials of $\mathbf{P}^3$, with reverse lexicographical ordering. Let C be a curve in $\mathbf{P}^3$, then the generic initial ideal of C, $\mathrm{gin}(I_C)$, is generated by elements of the form $x_0^i x_1^j x_2^k$. We say that a monomial $x_0^a x_1^b x_2^c$ is a *sporadic zero* if $x_0^a x_1^b x_2^c \notin \mathrm{gin}(I_C)$, but there exists $c' > c$ such that $x_0^a x_1^b x_2^{c'} \in \mathrm{gin}(I_C)$.

Let α_t is the number of sporadic zeros of degree t and m be the maximal degree of the sporadic zeros.

If π is the genus of C, then

$$\pi = 1 + \sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + (i-1)\lambda_i \right) - \sum_{t=0}^{m} \alpha_t \tag{1}$$

and

$$1+\sum_{i=0}^{s-1}\left(\binom{\lambda_i}{2}+(i-1)\lambda_i\right)\leqslant\frac{d^2}{2s}+(s-4)\frac{d}{2}+1=G(d,s) \tag{2}$$

for $d>(s-1)^2+1$.

(This is due to the work of Gruson and Peskine on the numerical invariants of points (see [GP]).)

In [BF] they show that

$$\chi\mathcal{O}_S\geqslant\sum_{t=0}^{s-1}\left(\binom{\lambda_t+t-1}{3}-\binom{t-1}{3}\right)-\sum_{t=0}^{m}\alpha_t(t-1),$$

and if $s\geqslant 2$ and $d>(s-1)^2+1$,

$$\sum_{t=0}^{s-1}\left(\binom{\lambda_t+t-1}{3}-\binom{t-1}{3}\right)\geqslant s\binom{\frac{d}{s}+\frac{s-3}{2}}{3}+1-\binom{s-1}{4}.$$

They Combine this with the double point formula (see [H, p. 434])

$$d^2-5d-10(\pi-1)+2(6\chi-K^2)=0,$$

and the fact that if the degree of S is $d>5$ then $K^2\leqslant 9$, to get

$$\begin{aligned}18\geqslant 2K^2 &= d^2-5d-10(\pi-1)+12\chi\\
&\geqslant d^2-5d-10\left(\frac{d^2}{2s}+(s-4)\frac{d}{2}-\sum_{t=0}^{m}\alpha_t\right)\\
&\quad +12\left(s\binom{\frac{d}{s}+\frac{s-3}{2}}{3}+1-\binom{s-1}{4}-\sum_{t=0}^{m}\alpha_t(t-1)\right)\\
&= d^2-5d-10\left(\frac{d^2}{2s}+(s-4)\frac{d}{2}\right)\\
&\quad +12\left(s\binom{\frac{d}{s}+\frac{s-3}{2}}{3}+1-\binom{s-1}{4}\right)-\sum_{t=0}^{m}\alpha_t(12t-22).\end{aligned} \tag{3}$$

We will use regularity conditions to find an upper bound for

$$A=\sum_{t=0}^{m}\alpha_t(12t-22).$$

First we should note that if $s = \min\{k|h^0(\mathcal{I}_S(k) \neq 0\}$. Then by the work of Ellingsrud and Peskine ([EP]) if S is a surface, of degree d, not of general type then either $= d \leqslant 90$ or $s \leqslant 5$. Furthermore, if one imitates their argument with $s = 6$, one gets either $d \leqslant 70$ or $s \leqslant 6$. We will assume $s \leqslant 6$. We also know that if $s \leqslant 3$ then $d \leqslant 8$. Hence we only need to bound the degree of surfaces not of general type with $s = 4, 5,$ or 6.

By [BS] the regularity of an ideal I is equal to the regularity of gin(I), which is the highest degree of a minimal generator of gin(I). Furthermore, by [GLP], the regularity of a smooth curve of degree d in $\mathbf{P}^3$ is $\leqslant d-1$. Hence the largest degree of the minimal generators of gin(I_C) is $\leqslant d-1$ and so all the sporadic zeros of C are in degree $\leqslant d-2$.

We will now bound the number of sporadic zeros. Let $\gamma = G(d,s) - \pi$. Any bound on γ will also bound the number of sporadic zeros (see equations (1) and (2)). By [EP], $\gamma \leqslant \frac{d(s-1)^2}{2s}$. Furthermore, by the double point formula and the fact that if S is a surface not of general type $K^2 < 6\chi$, we get $\pi \geqslant \frac{d^2-5d+10}{10}$ and thus $\gamma \leqslant \frac{d^2}{2s} + (s-4)\frac{d}{2} + 1 - \frac{d^2-5d+10}{10}$. Taking the minimum of these bounds for γ, we get, for $s = 4, \gamma \leqslant \frac{9d}{8}$, for $s = 5, \gamma \leqslant d$ and for $s = 6, \gamma \leqslant \frac{d(90-d)}{60}$.

Furthermore, by connectedness of the invariants $\lambda_0 \leqslant \frac{d}{s} + s - 1$ and $\lambda_1 \leqslant \frac{d}{s} + s - 2$.

Putting all this together, we have

$$A \leqslant \sum_{t=\frac{d}{s}+s-1}^{d-2} (12t-22) + \sum_{t=\frac{d}{s}+s-1}^{\gamma-d+\frac{2d}{s}+2s-2} (12t-22)$$

and using the bounds on γ, we get

$$\text{for } s = 4,\ A \leqslant \tfrac{243}{32}d^2 - 9d + 192,$$
$$\text{for } s = 5,\ A \leqslant \tfrac{162}{25}d^2 - 16d + 300.$$

Substituting back into the original equation (3) above, we get

$$\text{for } s = 4,\ 0 \geqslant \tfrac{1}{8}d^3 - \tfrac{275}{32}d^2 + \tfrac{7}{2}d - 195 \quad \text{and hence } d \leqslant 68,$$
$$\text{for } s = 5,\ 0 \geqslant \tfrac{2}{25}d^3 - \tfrac{162}{25}d^2 + 4d - 318 \quad \text{and hence } d \leqslant 80.$$

Now suppose S does not lie on a hypersurface of degree 5. Then the degree of $S = d \leqslant 90$. By [EP], we know that either $d \leqslant 70$ or S is contained in a hypersurface of degree 6. Therefore it remains to show that there are no smooth surfaces not of general type with $s = 6$ and $71 \leqslant d \leqslant 90$.

Suppose such a surface existed. Then $\gamma \leqslant \frac{71(90-71)}{60}$, i.e. $\gamma \leqslant 22$ and so

$$A \leqslant \sum_{t=\frac{d}{6}+5}^{\frac{d}{6}+5+21} (12t-22) = 44d + 3608.$$

Substituting as above we get

$$0 \geqslant \frac{d^3}{18} + \frac{2}{3}d^2 - \frac{119}{2}d - \frac{7321}{2}.$$

But this is a contradiction for $71 \leqslant d \leqslant 90$. □

By more careful consideration of the equations involved, it is possible to lower the bound a little more:

Making use of the equation

$$0 \leqslant \sum_{t=0}^{m} \alpha_t \leqslant \sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + i(\lambda_i) \right) - \frac{d^2 + 5d}{10}$$

arising from the double point formula, and

$$0 \geqslant d^2 - 5d - 18 - 10 \left(\sum_{i=0}^{s-1} \left(\binom{\lambda_i}{2} + (i-1)(\lambda_i) \right) \right) + 12 \left(\sum_{i=0}^{s-1} \binom{\lambda_i + i - 1}{3} - \binom{s-1}{4} \right) - \sum_{t=0}^{m} \alpha_t (12t - 22)$$

it is possible to show that $d \leqslant 76$ and $s \leqslant 8$. This is done be writing out all the possible connected invariants of length s and degree d and checking the inequalities on "Mathematica". (I would like to thank Rich Liebling for sending me the program which calculates the invariants.)

The temptation at this point is to try to lower the bound by considering each degree individually and using ad hoc methods. For example, we know for certain configurations of the invariants the curve is Arithmetically Cohen-Macaulay and hence has no sporadic zeros. But, to make any significant improvements using these methods, one needs to understand which configurations of the sporadic zeros can occur.

Acknowledgements

I would like to thank Mei-Chu Chang for asking me to talk on the paper of Braun and Fløystad, which made me read the details of the paper. I would like to thank Robert Braun and Gunnar Fløystad for answering all my questions and Robert for pointing out the better bound for γ for $s = 6$. Finally, I would like to thank Bruce Crauder and Sheldon Katz for making me feel welcome during my stay here at OSU.

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