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The universal Vassiliev-Kontsevich invariant for framed oriented links

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Abstract. We give a generalization of the Reshetikhin-Turaev functor for tangles to get a combinatorial formula for the Kontsevich integral for framed oriented links. The rationality of the Kontsevich integral is established. Many properties of the universal Vassiliev-Kontsevich invariant are discussed. Connections to invariants coming from quantum groups and to multiple zeta functions are explained.

Introduction

The aim of the paper is to give a presentation of the universal Vassiliev-Kontsevich invariant of framed oriented links and to establish some of its properties, including the rationality. We define a mapping from the set of all framed oriented tangles to rather complicated sets. This mapping, when restricted to the set of all framed oriented links in the 3-sphere S^3, is an isotopy invariant called the universal Vassiliev-Kontsevich invariant of framed oriented links. It is as powerful as the set of all invariants of finite type (or Vassiliev invariants) of framed oriented links. Hence it dominates all the invariants coming from quantum groups in which the R-matrix is a deformation of identity, as in [Re-Tu; Tu1]. Similar constructions of the universal Vassiliev-Kontsevich invariants appear in [Car; Piu1].

The values of the universal Vassiliev-Kontsevich invariant of framed knots lie in an algebra, and if we project to an appropriate quotient algebra, we get the Kontsevich integral of knots (Theorem 6).

Actually the universal Vassiliev-Kontsevich invariant is constructed using an object called the Drinfeld associator. This is a solution of a system of equations. Every solution of this system gives rise to a universal Vassiliev-Kontsevich invariant which we prove (Theorem 8) is independent of the solution used. As a corollary we get the rationality of the universal Vassiliev-Kontsevich invariant and Kontsevich integral.

The rationality of the Kontsevich integral was claimed in [Kont1], without proof, citing only Drinfeld's paper [Drin2]. The result of [Drin2] cannot be applied directly to this case because the spaces involved, though related, are in fact different. Here we modify Drinfeld's proof to our situation, using a suggestion of Kontsevich.

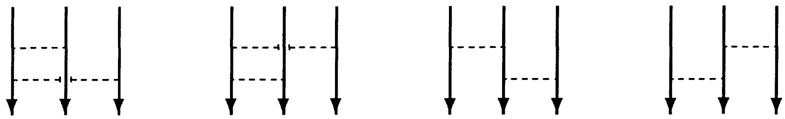


Fig. 1.

Many properties of the universal Vassiliev-Kontsevich invariant are established. By a suitable substitution we can get all the link invariants coming from quantum groups. We explain how to compute the Vassiliev-Kontsevich invariant using braids. Connections to quasi-Hopf algebras and to multiple zeta values are discussed.

For a detailed exposition of the theory of the Kontsevich integral and the universal Vassiliev-Kontsevich invariant for (unframed) knots see [Bar1]. Many arguments in [Bar1] are generalized here.

For technical convenience we use q-tangles instead of tangles. This concept is similar to that of a c-graph introduced in [Al-Co]. Actually the category of q-tangles and the category of tangles are the same, by Maclane's coherence theorem.

1. Chord diagrams

Suppose X is a one-dimensional compact oriented smooth manifold whose components are numbered. *A chord diagram with support* X is a set consisting of a finite number of unordered pairs of distinct non-boundary points on X, regarded up to orientation and component preserving homeomorphisms. We view each pair of points as a chord on X and represent it as a dashed line connecting the two points. The points are called the vertices of chords.

Let $\mathcal{A}(X)$ be the vector space over $\mathbb{Q}$ (rational numbers) spanned by all chord diagrams with support X, subject to the 4-term relation:

$$D_1 - D_2 + D_3 - D_4 = 0,$$

where D_j are four chord diagrams identical outside a ball in which they differ as indicated in Figure 1.

The space $\mathcal{A}(X)$ is graded by the number of chords. We denote the completion with respect to this grading also by $\mathcal{A}(X)$. Every homeomorphism $f\colon X \to Y$ induces an isomorphism between $\mathcal{A}(X)$ and $\mathcal{A}(Y)$.

On the plane $\mathbb{R}^2$ with coordinates (x, t) consider the set X_n consisting of n vertical lines $x = a_1$, $x = a_2, \ldots$, $x = a_n$, $a_1 < \cdots < a_n$, lying between two horizontal lines $t = b_1$ and $t = b_2$, $b_1 < b_2$. All the lines are oriented downwards; they are numbered from left to right, i.e. the number of the line $x = a_i$ is i. The space $\mathcal{A}(X_n)$ will be denoted by $\mathcal{P}_n$. A component of X_n is called a string. The vector space $\mathcal{P}_n$ is an algebra with the following multiplication. If D_1 and D_2 are two chord diagrams in $\mathcal{P}_n$, then $D_1 \times D_2$ is the chord diagram gotten by putting D_1 above D_2. The unit is the chord diagram without any chord. Let $\mathcal{P}_0 = \mathbb{Q}$.

PROPOSITION 1. [Kont1]. *The algebra* $\mathcal{P}_1$ *is commutative.*

Suppose that X, X' have distinguished components ℓ, ℓ', and that X consists of loop components only. Let $D \in \mathcal{A}(X)$ and $D' \in \mathcal{A}(X')$ be two chord diagrams. From each of ℓ, ℓ' we remove a small arc which does not contain any vertices. The remaining part of ℓ is an arc which we glue to ℓ' in the place of the removed arc such that the orientations are compatible. The new chord diagram is called *the connected sum of* D, D' *along the distinguished components.* It does not depend on the locations of the removed arcs, which follows from the 4-term relation and the fact that all components of X are loops. The proof is the same as in case $X = X' = S^1$ as in [Bar1].

In case when $X = X' = S^1$, the connected sum defines a multiplication which turns $\mathcal{A}(S^1)$ into an algebra which we will denote simply by $\mathcal{A}$. If D is a chord diagram in $\mathcal{A}$, by removing a small arc of the support S^1 which does not contain any vertices, we get a chord diagram in $\mathcal{P}_1$. This defines an isomorphism between the algebras $\mathcal{A}$ and $\mathcal{P}_1$ (see [Kont1; Bar1]).

Suppose again that X has a distinguished component ℓ. Let X' be the manifold gotten from X by reversing the orientation of ℓ. We define a linear mapping $S_\ell: \mathcal{A}(X) \to \mathcal{A}(X')$ as follows. If $D \in \mathcal{A}(X)$ represents by a diagram with n vertices of chords on ℓ. Reversing the orientation of ℓ, then multiplying by $(-1)^n$, from D we get the chord diagram $S_\ell(D) \in \mathcal{A}(X')$.

Now let us define $\Delta_i: \mathcal{P}_n \to \mathcal{P}_{n+1}$, for $1 \leqslant i \leqslant n$. Suppose D is a chord diagram in $\mathcal{P}_n$ with m vertices on the ith string. Replace the ith string by two strings, the left and the right, very close to the old one. Mark the points on this new set of $n+1$ strings just as in D; if a point of D is on the ith string then it yields two possibilities, marking on the left or on the right string. Summing up all 2^m possible chord diagrams of this type, we get $\Delta_i(D) \in \mathcal{P}_{n+1}$.

Define ε_i by $\varepsilon_i(D) = 0$ if the diagram D has a vertex of chords on the ith string. Otherwise let $\varepsilon_i(D)$ be the diagram in $\mathcal{P}_{n-1}$ gotten by throwing away the ith string. We continue ε_i to a linear mapping from $\mathcal{P}_n$ to $\mathcal{P}_{n-1}$.

Notation: we will write Δ for $\Delta_1: \mathcal{P}_1 \to \mathcal{P}_2$, $\mathrm{id} \otimes \cdots \otimes \Delta \otimes \cdots \otimes \mathrm{id}$ (the Δ is at the ith position) for Δ_i; ε for $\varepsilon_1: \mathcal{P}_1 \to \mathcal{P}_0 = \mathbb{Q}$, $\mathrm{id} \otimes \cdots \otimes \varepsilon \otimes \cdots \otimes \mathrm{id}$ (the ε is at the ith position) for ε_i.

REMARK. The reader should not confuse Δ with the co-multiplication introduced in [Bar1] for $\mathcal{A}$.

PROPOSITION 2. *We have* $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.

This follows immediately from the definitions.

Put $\Delta^n = \underbrace{(\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id})}_{n \text{ times}} \underbrace{(\Delta \otimes \mathrm{id} \cdots \otimes \mathrm{id})}_{n-1 \text{ times}} \cdots (\Delta \otimes \mathrm{id})\Delta$. For $n = 0$ let $\Delta^n = \mathrm{id}: \mathcal{P}_1 \to \mathcal{P}_1$.

THEOREM 1. *The image of $\Delta^n: \mathcal{P}_1 \to \mathcal{P}_{n+1}$ lies in the center of $\mathcal{P}_{n+1}$.*

The proof is similar as in case $n = 0$ which is Proposition 1 and is proved in [Bar1].

2. Non-associative words

A non-associative word on some symbols is an element of the free non-associative algebra generated by those symbols. Consider the set of all non-associative words on two symbols $+$ and $-$. If w is such a word, different from $+$ and $-$ and the unit, then w can be presented in a unique way $w = w_1 w_2$, where w_1, w_2 are non-associative non-unit words. Define inductively the length $l(w) = l(w_1) + l(w_2)$ if $w = w_1 w_2$ and $l(+) = l(-) = 1$. A non-associative word can be represented as a sequence of symbols and parentheses which indicate the order of multiplication.

There is a map which transfers each non-associative word into an associative word by simply forgetting the non-associative structure, that is, forgetting the parentheses. An associative word is just a finite sequence of symbols.

If we have a finite sequence of symbols $+, -$, then we can form a non-associative word by performing the multiplication step by step from the left. It will be called the standard non-associative word of the sequence.

Suppose w_1, w_2 are non-associative words. Replacing a symbol in the word w_2 by w_1 one gets another word w. In such a case we will call w_1 a *subword* of w, and write $w_1 < w$.

3. q-tangles

We fix an oriented 3-dimensional Euclidean space $\mathbb{R}^3$ with coordinates (x, y, t). *A tangle* is a smooth one-dimensional compact oriented manifold $L \subset \mathbb{R}^3$ lying between two horizontal planes $\{t = a\}, \{t = b\}, a < b$ such that all the boundary points are lying on two lines $\{t = a, y = 0\}, \{t = b, y = 0\}$, and at every boundary point L is orthogonal to these two planes. These lines are called the top and the bottom lines of the tangle.

A normal vector field on a tangle L is a smooth vector field on L which is nowhere tangent to L (and, in particular, is nowhere zero) and which is given by the vector $(0, -1, 0)$ at every boundary point. *A framed tangle* is a tangle enhanced with a normal vector field. Two framed tangles are isotopic if they can be deformed by a 1-parameter family of diffeomorphisms into one another within the class of framed tangles.

We will consider a *tangle diagram* as the projection onto $\mathbb{R}^2(x,t)$ of tangle in generic position. Every double point is provided with a sign $+$ or $-$ indicating an over or under crossing.

Two tangle diagrams are equivalent if one can be deformed into another by using: (a) isotopy of $\mathbb{R}^2(x,t)$ preserving every horizontal line $\{t = \text{const}\}$, (b) rescaling of $\mathbb{R}^2(x,t)$, (c) translation along the t-axis. We will consider tangle diagrams up to this equivalent relation.

Two isotopic framed tangles may project into two non-equivalent tangle diagrams. But if T is a tangle diagram, then T defines a unique class of isotopic *framed* tangles $L(T)$: let $L(T)$ be a tangle which projects into T and is coincident with T except for a small neighborhood of the double points, the normal vector at every point of $L(T)$ is $(0,-1,0)$.

One can assign a symbol $+$ or $-$ to all the boundary points of a tangle diagram according to whether the tangent vector at this point directs downwards or upwards. Then on the top boundary line of a tangle diagram we have a sequence of symbols consisting of $+$ and $-$. Similarly on the bottom boundary line there is also a sequence of symbols $+$ and $-$.

A *q-tangle diagram* T is a tangle diagram enhanced with two non-associative words $w_b(T)$ and $w_t(T)$ such that when forgetting about the non-associative structure from $w_t(T)$ (resp. $w_b(T)$) we get the sequence of symbols on the top (resp. bottom) boundary line. Similarly, *a framed q-tangle* L is a framed tangle enhanced with two non-associative words $w_b(L)$ and $w_t(L)$ such that when forgetting about the non-associative structure from $w_t(L)$ (resp. $w_b(L)$) we get the sequence of symbols on the top (resp. bottom) boundary line.

If T_1, T_2 are q-tangle diagrams such that $w_b(T_1) = w_t(T_2)$ we can define the product $T = T_1 \times T_2$ by putting T_1 above T_2. The product is a q-tangle diagram with $w_t(T) = w_t(T_1)$, $w_b(T) = w_b(T_2)$.

Every q-tangle diagram can be decomposed (non-uniquely) as the product of the following *elementary* q-tangle diagrams:

(1a) $X^+_{w,v}$, where $v < w$ are two non-associative words on one one symbol $+$, $v = {++}$, the underlying tangle diagram is in Figure 2a, with $w_t = w_b = w$, the two strings of the crossing correspond to two symbols of the word v.

(1b) $X^-_{w,v}$: the same as $X^+_{w,v}$, only the overcrossing is replaced by the undercrossing (Figure 2b).

(2a) $Y_{w,v}$ with $v = (+-) < w$, all the symbols in w outside v are $+$. The underlying tangle is in Figure 2c. Here $w_t = w$, w_b is obtained from w by removing v.

(2b) $Y^{-1}_{w,v}$ with $v = (-+) < w$, all the symbols in w outside v are $+$. The underlying tangle is in Figure 2d. Here $w_b = w$, w_t is obtained from w by removing v.

(3a) $T^+_{w_1w_2w_3,w}$ where w_1, w_2, w_3, w are non-associative words on one symbol $+$, and $((w_1w_2)w_3)$ is a subword of w. The underlying tangle diagram is trivial, consisting of $l(w)$ parallel lines, all are directed downwards, and

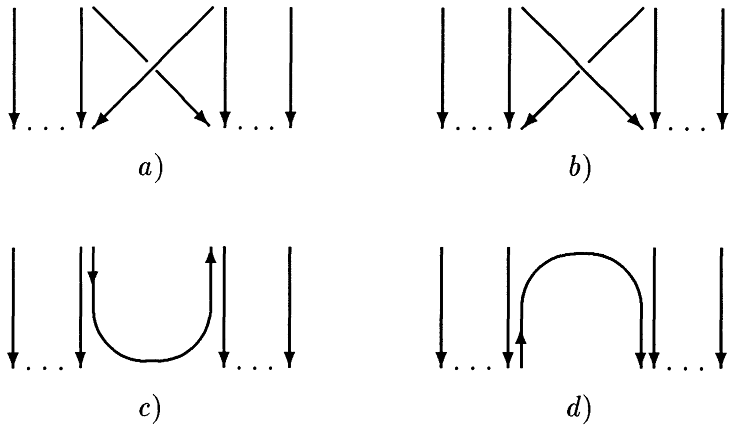


Fig. 2.

$w_b(T^+_{w_1w_2w_3,w}) = w$ while $w_t(T^+_{w_1w_2w_3,w})$ is obtained from w by substituting $((w_1w_2)w_3)$ by $(w_1(w_2w_3))$.

(3b) $T^-_{w_1w_2w_3,w}$ where w_1, w_2, w_3, w are non-associative words on one symbol $+$, and $((w_1w_2)w_3)$ is a subword of w. The underlying tangle diagram is trivial, consisting of $l(w)$ parallel lines, all are directed downwards, and $w_t(T^-_{w_1w_2w_3,w}) = w$ while $w_b(T^-_{w_1w_2w_3,w})$ is obtained from w by substituting $((w_1w_2)w_3)$ by $(w_1(w_2w_3))$.

(4) All the above listed q-tangle diagrams with reversed orientations on some strings and the corresponding change of signs of the boundary points.

4. The Drinfeld associator

Let M be the algebra over $\mathbb{C}$ of all formal series on two non-commutative, associative symbols A, B. Consider a function G: $(0,1) \to M$ satisfying the following Knizhnik-Zamolodchikov equation

$$\frac{d}{dt}G = \left(\frac{A}{t} + \frac{B}{t-1}\right)G.$$

Let $G_\lambda(A, B)$ be the value at $t = 1-\lambda$ of the solution to this equation which takes the value 1 at $t = \lambda$. The following limit exists

$$\varphi(A, B) = \lim_{\lambda\to 0} \lambda^{-B} G_\lambda \lambda^A,$$

and can be written in the form

$$\varphi(A, B) = 1 + \sum_W f_W W, \tag{1}$$

where the summation is over all the associative words on two symbols A and B. The coefficients f_W are highly transcendent and are computed in [Le-Mu2]. Each f_W is the sum of a finite number of numbers of type

$$\zeta(i_1, \ldots, i_k) = \sum_{n_1 < \cdots < n_k \in \mathbb{N}} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}} \tag{2}$$

with natural numbers $i_1, \ldots, i_k, i_k \geqslant 2$. These numbers, called multiple zeta values, have recently gained much attention among number theorists (see [Za]).

More precisely, let bold letters $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ stand for non-negative multi-indices. For a multi-index $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_k)$, k is the length of $\mathbf{p}$. Let $\mathbf{1}_k$ be the multi-index consisting of k letters 1. Let $|\mathbf{p}| = \sum \mathbf{p}_i$.

For two multi-indices $\mathbf{p}, \mathbf{q}$ of the same length k, define: $\eta(\mathbf{p}; \mathbf{q}) = \mathbf{0}$ if one of p_i, q_i is 0, otherwise

$$\eta(\mathbf{p}; \mathbf{q}) = \zeta(\mathbf{1}_{p_1 - 1}, q_1 + 1, \mathbf{1}_{p_2 - 1}, q_2 + 1, \ldots, \mathbf{1}_{p_k - 1}, q_k + 1).$$

Set $(A, B)^{(\mathbf{p}, \mathbf{q})} = A^{p_1} B^{q_1} A^{p_2} B^{q_2} \ldots A^{p_k} B^{q_k}$,

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} \cdots \begin{pmatrix} p_k \\ q_k \end{pmatrix}.$$

Then

$$\varphi(A, B) = 1 + \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \geqslant 0} (-1)^{|\mathbf{r}| + |\mathbf{q}|} \eta(\mathbf{p} + \mathbf{r}; \mathbf{q} + \mathbf{s})$$

$$\times \begin{pmatrix} \mathbf{p} + \mathbf{r} \\ \mathbf{r} \end{pmatrix} \begin{pmatrix} \mathbf{q} + \mathbf{s} \\ \mathbf{s} \end{pmatrix} \mathbf{B}^{|\mathbf{s}|} (\mathbf{A}, \mathbf{B})^{(\mathbf{p}, \mathbf{q})} \mathbf{A}^{|\mathbf{r}|}.$$

Here the sum is over all multi-indices $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ of the same length k, $k = 1, 2, 3, \ldots$. This formula follows from the result of [Le-Mu2].

Denote by Ω_{ij} the chord diagram in $\mathcal{P}_3$ with one chord connecting the ith and the jth strings. The element $\Phi_{KZ} = \varphi(\frac{1}{2\pi\sqrt{-1}} \Omega_{12}, \frac{1}{2\pi\sqrt{-1}} \Omega_{23}) \in \mathcal{P}_3 \otimes \mathbb{C}$ is called the KZ Drinfeld associator. It is a solution of the following equations (for a proof see [Drin1; Drin2]):

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\Phi) \times (\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\Phi)$$

$$= (1 \otimes \Phi) \times (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\Phi) \times (\Phi \otimes 1), \tag{A1}$$

$$(\Delta \otimes \mathrm{id})(R) = \Phi^{312} \times R^{13} \times (\Phi^{132})^{-1} \times R^{23} \times \Phi, \tag{A2a}$$

$$(\mathrm{id} \otimes \Delta)(R) = (\Phi^{231})^{-1} \times R^{13} \times \Phi^{213} \times R^{12} \times \Phi^{-1}, \tag{A2b}$$

$$\Phi^{-1} = \Phi^{321}, \tag{A3}$$

$$\varepsilon_1(\Phi) = \varepsilon_2(\Phi) = \varepsilon_3(\Phi) = 1. \tag{A4}$$

Here Φ^{ijk} is the element of $\mathcal{P}_3 \otimes \mathbb{C}$ obtained from Φ by permuting the strings: the first to the ith, the second to the jth, the third to the kth and $R^{ij} = \exp(\Omega_{ij}/2)$. Equation (A1) holds in $\mathcal{P}_4 \otimes \mathbb{C}$, equations (A2a, A2b, A3) in $\mathcal{P}_3 \otimes \mathbb{C}$, and equation (A4) in $\mathcal{P}_2 \otimes \mathbb{C}$.

REMARK. (A2b) follows from the other identities in (A1–A4).

Besides, Δ, Φ_{KZ} and $R = \exp(\Omega_{12}/2) \in \mathcal{P}_2 \otimes \mathbb{C}$ satisfy:

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \Phi \times (\Delta \otimes \mathrm{id})(\Delta(a)) \times \Phi^{-1}, \tag{B1}$$

$$(\varepsilon \otimes \mathrm{id})_o\Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)_o\Delta, \tag{B2}$$

$$\Delta(a) = R \times \Delta(a) \times R^{-1}. \tag{B3}$$

The first follows from (1) and theorem 1 for any Φ, the second is trivial, the third follows from theorem 1. The equations (A1–A4, B1–B3) mimic the axioms of a quasi-triangular quasi-Hopf algebra in [Drin1].

Every solution Φ of (A1–A4) is called an associator. Theorem A'' of [Drin2] asserts that there is an associator $\Phi_{\mathbb{Q}}$ with rational coefficients, i.e. $\Phi_{\mathbb{Q}} \in \mathcal{P}_3$.

5. The representation of framed q-tangles

Every tangle diagram T defines a framed tangle $L(T)$, and every framed tangle K is $L(T)$ for some tangle diagram.

Suppose T is a q-tangle diagram. Then $L(T)$ is a framed q-tangle. Regarding $L(T)$ as a 1-dimensional manifold, we can define the vector space $\mathcal{A}(L(T))$, which we will abbreviate as $\mathcal{A}(T)$. This vector space depends only on the underlying tangle diagram of T but not on the words w_b and w_t.

If $D_i \in \mathcal{A}(T_i), i = 1, 2$ and $T = T_1 \times T_2$, then we can define $D_1 \times D_2 \in \mathcal{A}(T)$ in the obvious way, just putting D_1 above D_2.

We will define a mapping $T \to Z_f(T) \in \mathcal{A}(T) \otimes \mathbb{C}$ for any q-tangle diagram such that if $T = T_1 \times T_2$ then $Z_f(T) = Z_f(T_1) \times Z_f(T_2)$. Such a mapping is uniquely defined by the values of the elementary q-tangle diagrams listed in the previous section. Let

$$Z_f(X^+_{w,v}) = \exp(\Omega/2) := 1 + \Omega/2 + \frac{1}{2!}(\Omega/2)^2 + \cdots, \tag{D1a}$$

where Ω^n is the chord diagram in $\mathcal{A}(X^+_{w,v})$ with n chords, each is parallel to the horizontal line and connects the two strings that form the double point of $X^+_{w,v}$.

$$Z_f(X^-_{w,v}) = \exp(-\Omega/2). \tag{D1b}$$

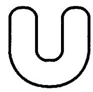


Fig. 3.

$$Z_f(Y_{w,v}) = 1, \tag{D2a}$$

i.e. $Z_f(Y_{w,v})$ is the chord diagram in $\mathcal{A}(Y_{w,v})$ without any chord.

$$Z_f(Y_{w,v}^{-1}) = 1, \tag{D2b}$$

i.e. $Z_f(Y_{w,v}^{-1})$ is the chord diagram in $\mathcal{A}(Y_{w,v}^{-1})$ without any chord.

For a q-tangle diagram $T^{\pm}_{w_1 w_2 w_3, w}$ let $\#l$ and $\#r$ be respectively the number of strings (in the underlying tangle diagram) left and right of the block of strings which form the word $((w_1 w_2) w_3)$. Define

$$Z_f(T^{+}_{w_1 w_2 w_3, w}) = 1^{\otimes \#l} \otimes [(\Delta^{l(w_1)-1} \otimes \Delta^{l(w_2)-1} \otimes \Delta^{l(w_3)-1})\Phi_{KZ}] \otimes 1^{\otimes \#r}, \tag{D3a}$$

$$Z_f(T^{-}_{w_1 w_2 w_3, w}) = 1^{\otimes \#l} \otimes [(\Delta^{l(w_1)-1} \otimes \Delta^{l(w_2)-1} \otimes \Delta^{l(w_3)-1})(\Phi_{KZ})^{-1}] \otimes 1^{\otimes \#r}. \tag{D3b}$$

Here $1^{\otimes n_1} \otimes D \otimes 1^{\otimes n_2}$, for $D \in \mathcal{P}_n$, is the element of $\mathcal{P}_{n_1+n+n_2}$ which has no chords on the first n_1 strings and on the last n_2 strings, and on the middle n strings it looks like D.

Finally, if T' is a q-tangle diagram obtained from T by reversing the orientation of some components $\ell_1, \dots, \ell_k$, then $Z_f(T')$ is obtained from $Z_f(T)$ by successively applying the mappings $S_{\ell_1}, \dots, S_{\ell_k}$. The result does not depend on the order of these mappings.

THEOREM 2. *1. The mapping $T \to Z_f(T)$ is well-defined: it does not depend on the decomposition of a q-tangle diagram into elementary q-tangle diagrams.*

2. Let $\phi = Z_f(U) \in \mathcal{A} \otimes \mathbb{C}$, where U is tangle diagram in Figure 3. Then

$$Z_f(T') = \phi . Z_f(T),$$

where the RHS is the connected sum of ϕ and Z_f along the indicated component, and T, T' are two q-tangles identical outside a ball in which they differ as indicated in Figure 4.

3. Suppose the coordinate function t on the ith component of T has s_i maximal points. Define

$$\hat{Z}_f(T) = (\phi^{-s_1} \otimes \cdots \otimes \phi^{-s_k}).Z_f(T), \tag{3}$$

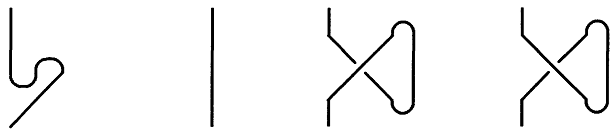


Fig. 4.

where the right hand side is the element obtained by successively taking the connected sum of ϕ^{-s_i} *and* $Z_f(T)$ *along the ith component. If two q-tangle diagrams* T_1, T_2 *define the same isotopic class of framed q-tangles, then* $\hat{Z}_f(T_1) = \hat{Z}_f(T_2)$, *hence* $\hat{Z}_f$ *is an isotopy invariant of framed q-tangles.*

In particular, $\hat{Z}_f$ is an isotopy invariant of *framed oriented links* regarded as framed q-tangles without boundary points.

There are at least two ways to prove Theorem 2. In the first which is more algebraic, we use MacLane's coherence theorem to reduce the category of q-tangles to the category of tangles and then verify that $\hat{Z}_f$ does not change under certain local moves (see the definition of the local moves in [Re-Tu; Al-Co]). Similar proofs are sketched in [Car; Piu1]. In the second which is more analytical (see [Le-Mu3]), we first define the regularized Kontsevich integral for framed oriented tangles, then we prove that the value Z_f of a q-tangle is the limit (in some sense) of the regularized Kontsevich integrals. In this approach we can avoid using MacLane's coherence theorem and verifying the invariance under local moves. The second proof also shows the relation between $\hat{Z}_f$ and the original Kontsevich integral (see Theorem 6 below).

REMARK. We have chosen the normalization in which $\hat{Z}_f$ of the unframed trivial knot is ϕ^{-1}, of the empty knot is 1.

THEOREM 3. *Let* ω *be the unique element of* $\mathcal{A}$ *with one chord. Then a change in framing results on* $\hat{Z}_f$ *by multiplying by* $\exp(\omega/2)$:

$$e^{-\omega/2} . \hat{Z}_f(T_+) = \hat{Z}_f(T) = e^{\omega/2} . \hat{Z}_f(T_-),$$

where T_+, T_-, T *are identical outside a ball in which they differ as indicated in Figure 4.*

This can be proved easily by moving the minimum point to the left then using the representations of q-tangles.

The invariant $\hat{Z}_f$ is called a universal Vassiliev-Kontsevich invariant of framed oriented links. As in [Bar1], it is easy to prove that $\hat{Z}_f(K_1) = \hat{Z}_f(K_2)$ if and only if all the invariants of finite type are the same for framed links K_1 and K_2. This means $\hat{Z}_f$ is as powerful as the set of all invariants of finite type, in particular it dominates all invariants coming from R-matrices which are deformations of identity, as in [Tu1; Re-Tu].

Let ℓ be a component of a one-dimensional compact manifold X and X' be obtained from X by replacing ℓ by two copies of ℓ. In a similar manner as in Section 1 one can define the operator $\Delta_\ell\colon \mathcal{A}(X) \to \mathcal{A}(X')$.

THEOREM 4. *Suppose L is a framed oriented link with a distinguished component ℓ, L' is obtained from L by replacing ℓ by two its parallels, close to ℓ, L'' is obtained from L by reversing the orientation of ℓ. Then*

$$\hat{Z}_f(L') = \Delta_\ell(\hat{Z}_f(L)). \tag{4}$$

$$\hat{Z}_f(L'') = S_\ell(\hat{Z}_f(L)). \tag{5}$$

The second identity follows trivially from the definition of $\hat{Z}_f$. The first is more difficult and can be proved by analyzing the parallel of the elementary q-tangles. Note that the chosen normalization of $\hat{Z}_f$ plays important role in the identity. The formula for the parallel version of Z_f (not $\hat{Z}_f$) would require an additional factor. Applying this identity to the unknot we get a beautiful formula

$$\Delta(\phi) = \phi \otimes \phi. \tag{6}$$

THEOREM 5. *Suppose L_1, L_2 are framed links with distinguished components and L is the connected link along the distinguished components. Then*

$$\hat{Z}_f(L) = \phi\,.\,(\hat{Z}_f(L_1))\,.\,(\hat{Z}_f(L_2)).$$

The right hand side is the connected sum of ϕ, $\hat{Z}_f(L_1)$ and $\hat{Z}_f(L_2)$ along the distinguished components. Theorem 5 follows easily from the construction of $\hat{Z}_f$.

6. The Kontsevich integral

Let $\mathcal{A}_0 = \mathcal{A}/(\omega\mathcal{A})$, where w is the only chord diagram in $\mathcal{A}$ with one chord. With respect to connected sum, $\mathcal{A}_0$ is a commutative algebra. There is a natural projection $p\colon \mathcal{A} \to \mathcal{A}_0$.

Let K be the image of an embedding of the oriented circle into $\mathbb{R}^3$ lying between two horizontal planes $\{t = t_{\min}\}, \{t = t_{\max}\}$. We will consider the 2-dimensional plane (x, y) as the complex plane with coordinate $z = x + y\sqrt{-1}$. The Kontsevich integral of K is defined as an element of $\mathcal{A}_0$ by

$$Z(T) = \sum_{m=0}^{\infty} \frac{1}{(2\pi\sqrt{-1})^m} \int_{t_{\min}<t_1<\cdots<t_m<t_{\max}} \sum_P (-1)^{\#P\downarrow} \wedge$$

$$\times \frac{\mathrm{d}z_i - \mathrm{d}z_i'}{z_i - z_i'} D_P, \tag{7}$$

where for fixed $t_{\min} < t_1 < t_2 < \cdots < t_m < t_{\max}$ the object P is a choice of unordered pairs of distinct points z_i, z_i' lying in $K \cap \{t = t_i\}$ for $i = 1 \ldots, m$; the summation in (7) is over all such choices; D_P is the chord diagram in $\mathcal{A}_0$ obtained by connecting pairs z_i, z_i' by dashed lines; $\#P\downarrow$ is the number of z_i, z_i' at which the orientation of K is downwards. Here we regard z_i, z_i' as functions of t_i (for more details on the Kontsevich integral see [Bar1]).

The integral $Z(K)$ is not an isotopy invariant. Let $\gamma = p(\phi)$. Kontsevich proved [Kont1] that $\hat{Z}(K) := \gamma^{-s} . Z(K)$, where s is the number of maximum points of the coordinate function t on K, is an isotopy invariant of (unframed) oriented knots. Note that in [Bar1] instead of $\hat{Z}$ another normalization $\tilde{Z} = \gamma . \hat{Z}$ is used. This invariant is as powerful as the set of all invariants of finite type. The relation between $\hat{Z}_f$ and the Kontsevich integral is explained in the following

THEOREM 6. *For a framed oriented knot K*

$$p(\hat{Z}_f(K)) = \hat{Z}(K).$$

This theorem and the trivial generalization for links are proved in [Le-Mu3]. Knowing $\tilde{Z}(K) \in \mathcal{A}_0$ one can also compute $\hat{Z}_f(K)$ (see [Le-Mu3]).

7. Symmetric twisting

An element $D \in \mathcal{P}_2 \otimes \mathbb{C}$ is called *symmetric* if $D^{21} = D$, where D^{21} is obtained from D by permuting the two strings of the support. Suppose that $F \in \mathcal{P}_2 \otimes \mathbb{C}$ satisfies

(T1) $\varepsilon_1(F) = \varepsilon_2(F) = 1$,
(T2) F is symmetric.

Then there exist F^{-1} in $\mathcal{P}_2 \otimes \mathbb{C}$ satisfying (T1, T2). If Φ is an element of $\mathcal{P}_3 \otimes \mathbb{C}$, then

$$\tilde{\Phi} = [1 \otimes F][(\mathrm{id} \otimes \Delta)F]\Phi[\Delta \otimes \mathrm{id})(F^{-1}][F^{-1} \otimes 1] \tag{8}$$

is said to be obtained from Φ by *twisting via F*.

Note that the first two terms in the right hand side of (8) commute with each other, so do the last two terms. If $\Phi \in \mathcal{P}_3 \otimes \mathbb{C}$ is an associator, i.e. a solution of (A1–A4), then it is not difficult to check that $\tilde{\Phi}$ is also an associator.

For a non-associative word w on one symbol $+$ define $\mathcal{F}_w \in \mathcal{P}_{l(w)}$ by induction as follows. Let $\mathcal{F}_\emptyset = 1 \in \mathbb{Q}, \mathcal{F}_+ = 1 \in \mathcal{P}_1, \mathcal{F}_{++} = F \in \mathcal{P}_2 \otimes \mathbb{C}$. For $w = w_1w_2$ let

$$\mathcal{F}_w = [\mathcal{F}_{w_1} \otimes 1^{\otimes l(w_2)}][1^{\otimes l(w_1)} \otimes \mathcal{F}_{w_2}][(\Delta^{l(w_1)-1} \otimes \Delta^{l(w_2)-1})F]. \tag{9}$$

Then (8) implies that $\tilde{\Phi} = \mathcal{F}_{+(++)}\Phi(\mathcal{F}_{(++)+})^{-1}$.

Fix $F \in \mathcal{P}_2 \otimes \mathbb{C}$ satisfying (T1, T2). Consider a new mapping $T \to Z_f^F(T)$ defined for q-tangle diagrams by the same rules as for Z_f, only replacing the values listed in Section 3 for elementary q-tangle diagrams by:

$$Z_f^F(X_{w,v}^+) = Z_f(X_{w,v}^+),$$

$$Z_f^F(X_{w,v}^-) = Z_f(X_{w,v}^-),$$

$$Z_f^F(Y_{w,v}) = [1^{\otimes m} \otimes S_2(F) \otimes 1^{\otimes n}] \times Z_f(Y_{w,v}), \tag{D2a$'$}$$

$$Z_f^F(Y_{w,v}^{-1}) = Z_f(Y_{w,v}^{-1}) \times [1^{\otimes m} \otimes S_1(F^{-1}) \otimes 1^{\otimes n}]. \tag{D2b$'$}$$

Here S_1 and S_2 are respectively the operators which act by reversing the orientation of the first and the second components of the support of chord diagrams in $\mathcal{P}_2$ as described in Section 1. The values of $Z_f^F(T_{w_1w_2w_3,w}^{\pm})$ are defined by the same formulas (D3a,b), replacing Φ_{KZ} by $\tilde{\Phi}$ obtained from Φ_{KZ} by twisting via F.

THEOREM 7. *The map Z_f^F is well-defined and for every q-tangle diagram T*

$$Z_f^F(T) = \mathcal{F}_{w_t(T)} Z_f(T)[\mathcal{F}_{w_b(T)}]^{-1}. \tag{10}$$

In particular, $Z_f^F(L) = Z_f(L)$ for any tangle diagram L without boundary points.

Proof. We need only to check identity (10). It's sufficient to check for the cases when T are elementary q-tangle diagrams. These cases follows trivially from the definition. □

Note that if $\tilde{\Phi}$ has rational coefficients, i.e. if $\tilde{\Phi} \in \mathcal{P}_3$, then from the definition, the invariant Z_f^F of a framed *link* (not framed q-tangle) has rational coefficients, $Z_f^F(K) \in \mathcal{A}(K)$. Although the coefficients of F may be irrational and in (D2a$'$, D2b$'$) the elements F, F^{-1} are involved, they appear in pairs which annihilate each other in every link diagram.

REMARK. In [Drin1; Drin2] Drinfeld defined twists for quasi-triangular quasi-Hopf algebras. Here we adapt a similar definition for the series of algebras $\mathcal{P}_n$ which play the role of a *single* quasi-triangular quasi-Hopf algebra. If we use the representation of section 10 below then we get a quasi-triangular quasi-Hopf algebra, and the construction of twists here corresponds only to Drinfeld's twist which does not change the co-multiplication. If F is not symmetric, then Δ is replaced by $\tilde{\Delta} = F^{21}\Delta F^{-1} = (F^{21}F^{-1})\Delta$.

8. Uniqueness and rationality of the universal Vassiliev-Kontsevich invariant

THEOREM 8. *If $\Phi, \Phi' \in (\mathcal{P}_3 \otimes \mathbb{C})$ are associators, then Φ is obtained from Φ' by twisting via some $F \in \mathcal{P}_2 \otimes \mathbb{C}$ satisfying (T1, T2).*

As a corollary, from every solution Φ of (A1–A4) we can construct an invariant of framed q-tangles. All such invariants, when restrict to the sets of framed oriented links, are the same and contain all invariants of framed oriented links of finite type. By theorem A'' of [Drin2] there is a solution $\Phi_{\mathbb{Q}}$ with rational coefficients, thus we get

COROLLARY. *The universal Vassiliev-Kontsevich invariant of framed links has rational coefficients in the sense that $\hat{Z}_f(L)$ belongs to $\mathcal{A}(L)$ for every framed link L. The Kontsevich integral of a knot has rational coefficient in the sense that $\tilde{Z}(K) \in \mathcal{A}_0$.*

Proof of Theorem 8. Let

$$\Phi = 1 + \Phi_1 + \cdots + \Phi_n + \cdots$$

$$\Phi' = 1 + \Phi_1' + \cdots + \Phi_n' + \cdots$$

Here Φ_n, Φ_n' are the homogeneous part of grading n. Suppose we already have $\Phi_i = \Phi_i'$ for $0 \leqslant i \leqslant k-1$. Put $\psi = \Phi_k - \Phi_k'$.

Comparing the k-grading parts of (A1–A4) for Φ, Φ' we get:

$$\mathrm{d}\psi = 0, \tag{C1}$$

$$\psi - \psi^{132} - \psi^{213} = 0, \tag{C2}$$

$$\psi^{321} = -\psi, \tag{C3}$$

$$\varepsilon_1(\psi) = \varepsilon_2(\psi) = \varepsilon_3(\psi) = 0, \tag{C4}$$

where $d\colon \mathcal{P}_n \to \mathcal{P}_{n+1}$ is the mapping:

$$d(a) = 1 \otimes a - \Delta_1(a) + \Delta_2(a) - \cdots + (-1)^n \Delta_n(a) + (-1)^{n+1} a \otimes 1. \tag{C5}$$

PROPOSITION 3. *If $\psi \in \mathcal{P}_3 \otimes \mathbb{C}$ of grading k and satisfying (C1–C4) then there is a symmetric element $f \in \mathcal{P}_2 \otimes \mathbb{C}$ of grading k such that $d(f) = \psi$; $\varepsilon_1(f) = \varepsilon_2(f) = 0$.*

Suppose for the moment that this is true. Pick f as in this proposition. Then one can check immediately that the twist by $F = 1 + f$ transfers Φ to $\tilde{\Phi}$ with $\tilde{\Phi}_i = \Phi_i'$ for $0 \leqslant i \leqslant k$.

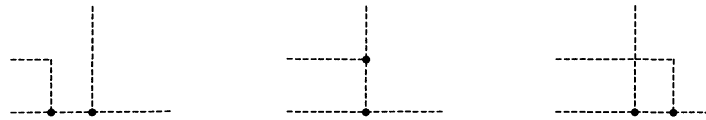


Fig. 5.

Continue the process we can find a element $F \in \mathcal{P}_2 \otimes \mathbb{C}$ satisfying (T1, T2) which transfers Φ into Φ'.

There remains Proposition 3 to prove.

9. Proof of Proposition 3

9.1. OTHER REALIZATIONS OF $\mathcal{P}_n$

A Chinese character ([Bar1]) is a graph whose vertices are either trivalent and oriented or univalent. Here an orientation of a trivalent vertex is just a cyclic order of the three edges incident to this vertex. The trivalent vertices are called *internal*, the univalent vertices are called *external*. The edges of the graph will be represented by dashed lines on the plane. By convention all the orientations in figures are counterclockwise for Chinese characters.

An *n-marked Chinese character* ξ is a Chinese character with at least one external vertex in each connected component, where in addition the external vertices are partitioned into n labeled sets $\Theta_1(\xi), \dots, \Theta_n(\xi)$.

Let $\mathcal{E}_n$ be the vector space over $\mathbb{Q}$ spanned by all n-marked Chinese characters subject to the following identities (see also [Bar1]):

(1) the antisymmetry of internal vertices: $\xi_1 + \xi_2 = 0$, for every two Chinese characters ξ_1 and ξ_2 identical identical everywhere except for the orientation at one internal vertex.
(2) the Jacobi identity: $\xi_1 = \xi_2 + \xi_3$, for every three Chinese characters identical outside a ball in which they differ as in Figure 5.

Let us define linear mappings $\Delta_i: \mathcal{E}_n \to \mathcal{E}_{n+1}$ and $\varepsilon_i: \mathcal{E}_n \to \mathcal{E}_{n-1}$. Suppose the set $\Theta_i(\xi)$ of an n-marked Chinese character $\xi \in \mathcal{E}_n$ contains exactly m vertices. There are 2^m ways of partition $\Theta_i(\xi)$ into an ordered pair of subsets. For each such partition q let ξ_q be the $(n+1)$-marked Chinese character with the same underlying Chinese character as ξ, $\Theta_j(\xi_q) = \Theta_j(\xi)$ if $j < i$, $\Theta_j(\xi_q) = \Theta_{j-1}(\xi)$ if $j \geqslant i+2$, while $\Theta_i(\xi_q), \Theta_{i+1}(\xi_q)$ are two subsets of $\Theta_i(\xi)$ corresponding to the partition q. Define $\Delta_i(\xi)$ as the sum of all 2^m $(n+1)$-marked Chinese characters ξ_q.

If $\Theta_i(\xi) \neq \emptyset$ define $\varepsilon_i(\xi) = 0$. Otherwise define $\varepsilon_i(\xi)$ as the $(n-1)$-marked Chinese character with the same underlying Chinese character and $\Theta_j(\varepsilon_i(\xi)) = \Theta_j(\xi)$ if $j < i$, $\Theta_j(\varepsilon_i(\xi)) = \Theta_{j+1}(\xi)$ if $j \geqslant i$.

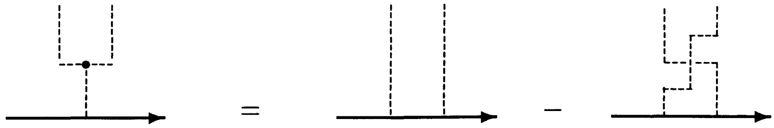


Fig. 6.

The $\mathbb{Z}^n$-grading of an n-marked Chinese character ξ is the tuple $(k_1, \ldots, k_n)$ of integers, where k_i is the number of elements of $\Theta_i(\xi)$. The number $\sum_{i=1}^n k_i$ is called the $\mathbb{Z}$-grading of ξ. Note that all the mappings Δ_i, ε_i respect the $\mathbb{Z}$-grading.

We define the linear mapping $d: \mathcal{E}_n \to \mathcal{E}_{n+1}$ by

$$d(\xi) = 1 \otimes \xi - \Delta_1(\xi) + \Delta_2(\xi) - \cdots + (-1)^n \Delta_n(\xi) + (-1)^{n+1} \xi \otimes 1.$$

Here $1 \otimes \xi$ and $\xi \otimes 1$ are the $(n+1)$-marked Chinese characters gotten from modifying the marking on ξ by setting $\Theta_1(1 \otimes \xi) = \emptyset$, $\Theta_j(1 \otimes \xi) = \Theta_{j-1}(\xi)$ for $2 \leqslant j \leqslant n+1$, $\Theta_{n+1}(1 \otimes \xi) = \emptyset$, $\Theta_j(1 \otimes \xi) = \Theta_j(\xi)$ for $1 \leqslant j \leqslant n$.

Now we define a linear mapping $\chi: \mathcal{E}_n \to \mathcal{P}_n$ as follows. First we define $\chi'(\xi)$ for an n-marked Chinese character ξ of $\mathbb{Z}^n$-grading $(k_1, \ldots, k_n)$. There are $k_i!$ ways to put vertices from $\Theta_i(\xi)$ on the ith string and each of the $k_1! \ldots k_n!$ possibilities gives us an element of $\mathcal{P}_n$. Summing up all such elements, we get $\chi'(\xi)$.

Now we use the STU relation indicated in Figure 6 to convert every diagram appearing in $\chi'(\xi)$ into a chord diagram, obtaining in total $\chi(\xi)$ from $\chi'(\xi)$.

THEOREM 9. *The linear mapping χ is well-defined and is an isomorphism between vector spaces $\mathcal{E}_n$ and $\mathcal{P}_n$ commuting with all the operators Δ_i, ε_i.*

REMARK. χ, however, does not preserve gradings.

The proof for the case $n = 1$ is presented in [Bar1, Theorems 6 & 8]. This proof does not concern the support of chord diagrams except for the first step of the induction which is trivial in our case (see also [Bar2]).

Consider the following subspaces $\mathcal{G}_n$ of $\mathcal{E}_n \otimes \mathbb{C}$, $\mathcal{G}_n = \cap_{i=1}^n \ker(\varepsilon_i)$. It can be checked that $d(\mathcal{G}_n) \subset \mathcal{G}_{n+1}$. We will now study the homology of the following chain complex:

$$0 \xrightarrow{d} \mathcal{G}_1 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{G}_n \xrightarrow{d} \mathcal{G}_{n+1} \xrightarrow{d} \cdots. \qquad (*)$$

Note that d preserves the $\mathbb{Z}$-grading, hence it suffices to consider the part of $\mathbb{Z}$-grading m of the complex.

$$0 \xrightarrow{d} \mathcal{G}_1^m \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{G}_n^m \xrightarrow{d} \mathcal{G}_{n+1}^m \xrightarrow{d} \cdots, \qquad (*_m)$$

where $\mathcal{G}_n^m$ is the homogeneous part of $\mathbb{Z}$-grading m of $\mathcal{G}_n$. We will find a geometric interpretation of this complex.

9.2. A SIMPLICIAL COMPLEX OF THE CUBE

Let I^m be the m-dimensional cube,

$$I^m = \left\{ \sum_{i=1}^{m} \lambda_i v_i \quad | \quad \lambda_i \in [0,1] \right\},$$

where $v_1, \dots, v_m$ form a base of $\mathbb{R}^m$. We partition I_m into $m!$ m-simplexes: a permutation $(i_1, \dots, i_m)$ of $(1, \dots, m)$ gives rise to the m-simplex which is the convex hull of $m+1$ points $0, v_{i_1}, v_{i_1} + v_{i_2}, \dots, v_{i_1} + \cdots + v_{i_m}$. This turns I^m into a simplicial complex, denoted by $C(I^m)$. The space $C_k(I^m)$ is the vector space over $\mathbb{C}$ spanned by all the k-facets of all $m!$ above m-simplexes. The boundary $\partial(I^m)$ is a simplicial subcomplex. The space $C_k(\partial(I^m))$ is spanned by all k-facets which lie entirely in $\partial(I^m)$.

Let C_k be the vector space over $\mathbb{C}$ spanned by all tuples $(\theta_1, \dots, \theta_k)$ which are partitions of the set $\{1, 2, \dots, m\}$, each θ_i non-empty. Define $\partial\colon C_k \to C_{k-1}$ by

$$\partial(\theta_1, \theta_2, \dots, \theta_k) = (\theta_1 \cup \theta_2, \theta_3, \dots, \theta_k) - (\theta_1, \theta_2 \cup \theta_3, \dots, \theta_k)$$
$$+ \cdots + (-1)^{k-1}(\theta_1, \dots, \theta_{k-1} \cup \theta_k).$$

Then the chain complex (C_*, ∂) is isomorphic to the quotient complex $C(I^m)/C(\partial(I^m))$. In fact, the mapping which sends $(\theta_1, \dots, \theta_k)$ to the k-simplex with vertices $0, v_{\theta_1}, v_{\theta_1} + v_{\theta_2}, \dots, v_{\theta_1} + \cdots + v_{\theta_k}$ is an isomorphism between these two complexes, where $v_\theta = \sum_{j \in \theta} v_j$.

Let E_m be the dual chain complex of (C_*, ∂), $E_m = (C^*, d)$. Using the above base of C_k, we can identify C_k^* with C_k with the same base. Then the co-boundary d can be written explicitly as

$$d(\theta_1, \theta_2, \dots, \theta_k) = (d(\theta_1), \theta_2, \dots, \theta_k) - (\theta_1, d(\theta_2), \dots, \theta_k)$$
$$+ \cdots + (-1)^{k-1}(\theta_1, \theta_2, \dots, d(\theta_k)),$$

where for a non-empty subset θ of $\{1, 2, \dots, m\}$ we set $d(\theta) = \sum(\theta', \theta'')$, the summation is over all possible partitions of θ into an ordered pair of non-empty subsets.

PROPOSITION 4. *The homology of the chain complex E_m is given by $H_m(E_m) = \mathbb{C}$, $H_i(E_m) = 0$ for $0 \leqslant i \leqslant m-1$.*

This follows from the fact that the homology of E_m is the reduced cohomology of $I^m/\partial(I^m)$.

Since every tuple $(\theta_1, \dots, \theta_k) \in C_k$ is a partition of $\{1, 2, \dots, m\}$, the symmetric group $\mathfrak{S}_m$ acts naturally on C_k. In the simplicial complex $C(I^m)$ this corresponds to the action: $(v_1, \ldots, v_m) \to (v_{\sigma(1)}, \ldots, v_{\sigma(m)})$ for $\sigma \in \mathfrak{S}_m$. On (co)homology the action is trivial.

PROPOSITION 5. *For every right $\mathfrak{S}_m$-module N*

$$H(N \otimes_{\mathfrak{S}_m} E_m) = N \otimes_{\mathfrak{S}_m} H(E_m).$$

Proof. This result is well-known (it was used implicitly in [Drin1]). The proof reduces to the cases of irreducible representations of $\mathfrak{S}_m$.

Consider an irreducible representation N_λ of $\mathfrak{S}_m$ parametrized by a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0$, $\sum_{i=1}^k \lambda_i = m$. The symmetric group $\mathfrak{S}_m$ acts on the complex E_m and this action is compatible with the chain map. So we can split $E_m = \oplus_\lambda E_{m,\lambda}$, where $E_{m,\lambda}$ is isomorphic to a direct sum of several (say m_λ) copies of $N_\lambda^\star$ as a left $\mathfrak{S}_m$-module, where $N_\lambda^\star$ is the contragradient left $\mathfrak{S}_m$-module of N_λ, given by the transpose matrices. Then, $N_\lambda \otimes_{\mathfrak{S}_m} E_m \cong \operatorname{Hom}_{\mathfrak{S}_m}(N_\lambda^\star, E_m)$ and so, by Schur's lemma, $N_\lambda \otimes_{\mathfrak{S}_m} E_m \cong N_\lambda \otimes_{\mathfrak{S}_m} E_{m,\lambda} \cong E_{m,\lambda}/\mathfrak{S}_m$. Since $\mathfrak{S}_m$ acts on $H(E_m)$ trivially, $H(E_{m,\lambda}) = 0$ if N_λ is not the trivial module. Hence $H(N_\lambda \otimes_{\mathfrak{S}_m} E_m) = 0$ if N_λ is not the trivial module. If N_λ is the trivial module, we have $H(N_\lambda \otimes_{\mathfrak{S}_m} E_m) = H(E_m)$. □

9.3. PROOF OF PROPOSITION 3

Denote the homogeneous part of $\mathbb{Z}^m$-grading $(1, 1, \ldots, 1)$ of $\mathcal{E}_m$ by Γ_m. The symmetric group $\mathfrak{S}_m$ acts on the right on Γ_m by permuting the m strings.

PROPOSITION 6. *The chain complex $(*_m)$ is isomorphic to the chain complex $\Gamma_m \otimes_{\mathfrak{S}_m} E_m$.*

Proof. An element ξ of $\mathcal{E}_m$ of $\mathbb{Z}^m$-grading $(1, \ldots, 1)$ is just a Chinese character with m external vertices which are numbered from 1 to m. We map an element $\xi \otimes (\theta_1, \ldots, \theta_k)$ to the element η of $\mathcal{G}_k^m$ with the same underlying Chinese character as that of ξ, only $\Theta_i(\eta)$ is the set of external vertices whose numbers are in θ_i. It can be verified at once that this is an isomorphism between the two complexes. □

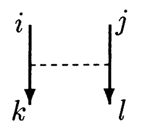
PROPOSITION 7. *Suppose $\psi \in \mathcal{G}_3$ satisfying:*

$$d\psi = 0, \tag{C1$'$}$$

$$\psi - \psi^{213} - \psi^{132} = 0, \tag{C2$'$}$$

$$\psi = -\psi^{321}. \tag{C3$'$}$$

Then there is a symmetric element $f \in \mathcal{G}_2$ such that $df = \psi$.



$\Longrightarrow r^{ij}_{kl}$

Fig. 7.

($f \in \mathcal{G}_2$ is symmetric if $f^{21} = f$, by definition.)

Proof. It suffices to consider the case when ψ is homogeneous. Since $d\psi = 0$, if the $\mathbb{Z}$-grading k of ψ is greater than 3 then by the previous proposition there is $f' \in \mathcal{G}_2^k$ such that $df' = \psi$.

If $k = 3$, then the $\mathbb{Z}^3$-grading of ψ must be (1, 1, 1), i.e $\psi \in \Gamma_3$. Consider $f_1, f_2 \in \mathcal{G}_2^3$ with the same underlying Chinese character as ψ, only $\Theta_1(f_1) = \Theta_1(\psi) \cup \Theta_2(\psi)$, $\Theta_2(f_1) = \Theta_3(\psi)$, $\Theta_1(f_2) = \Theta_1(\psi)$, $\Theta_2(f_1) = \Theta_2(\psi) \cup \Theta_3(\psi)$. Put $f' = (f_1 - f_2)/3$. Then using $(C2')$ one checks easily that $df' = \psi$.

In both cases we have $df' = \psi$ for some element $f' \in \mathcal{G}_2$. Note that $d(g^{21}) = -(dg)^{321}$ for every $g \in \mathcal{G}_2$. The sum $f = (f' + (f')^{21})/2$ is a symmetric element. Using $(C3')$ we see that $df = \psi$. □

Now Proposition 3 follows from this proposition and Theorem 9.

10. Invariants of framed oriented links coming from quantum groups

Suppose for $1 \leqslant i, j, k, l \leqslant N$ there are given complex numbers r_{ij}^{kl}. By a state of a chord diagram D in $\mathcal{A}$ we mean a map from the set of all arcs of the loop divided by vertices of chords to the set $\{1, 2, \ldots, N\}$. For a fixed state we associate to every chord of D a number as indicated in Figure 7. Take the product of all the numbers associated to all the chords, and then sum up over all the possible states to get a number. This number is well-defined (because of 4-term relation) iff (cf.[Lin; Bar1]):

(a) $r_{ij}^{kl} = r_{ji}^{lk}$,
(b) $[r^{(12)}, r^{(13)}] + [r^{(12)}, r^{(23)}] = 0$.

Where in (b) we view r as a linear mapping from $\mathbb{C}^N \otimes \mathbb{C}^N$ to $\mathbb{C}^N \otimes \mathbb{C}^N$, and $r^{(ij)}$ is the linear mapping from $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ to $\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N$ which is as r on the ith and jth components while leaves the rest unchanged. The equation (b) is the linearized classical Yang-Baxter equation ([Drin3]).

Suppose r satisfies (a), (b). Multiplying r by a formal parameter h and applying the above procedure we get for every diagram $D \in \mathcal{A}$ an element $W_r(D)$ in $\mathbb{C}[h]$. If K is a framed link then $W_r(\hat{Z}_f(K)) \in \mathbb{C}[[h]]$ is an isotopy invariant.

Now suppose $\mathfrak{g}$ is a classical simple Lie algebra, $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$ is a representation. Let t be the symmetric invariant element in $\mathfrak{g} \otimes \mathfrak{g}$ corresponding to the Killing form. Then it can be checked easily that $\rho(t) \in \mathrm{End}(V) \otimes \mathrm{End}(V)$ satisfies both equations (a), (b). Hence we can get an invariant of framed links $\kappa_{\mathfrak{g},\rho} = W_{\rho(t)}(\hat{Z}_f)$ by the above procedure.

On the other hand, for every representation $\rho: \mathfrak{g} \to \operatorname{End}(V)$, using the universal R-matrix, one can construct another invariant $\tau_{g,\rho}$ of framed links by the Reshetikhin-Turaev method (cf. [Re-Tu; Tu1]). Actually this method gives a representation of *tangles* rather than q-tangles and can be summarized as follows. There is a structure of *ribbon Hopf algebra* ([Re-Tu]) on the h-adic completion $\hat{U}\mathfrak{g}$ of $U\mathfrak{g} \otimes \mathbb{C}[[h]]$, where $U\mathfrak{g}$ is the universal enveloping algebra of $\mathfrak{g}$. The R-matrix of this ribbon Hopf algebra was constructed by Drinfeld and Jimbo [Drin3; Jim] and has expansion $R^{21}R = 1 + ht + O(h^2)$. There is a standard procedure (see [Re-Tu]) which associates to every representation $\rho: \mathfrak{g} \to \operatorname{End}(V)$ an invariant $\tau_{\mathfrak{g},\rho}$ of framed oriented links.

THEOREM 10. *The two invariants $\kappa_{\mathfrak{g},\rho}$ and $\tau_{\mathfrak{g},\rho}$ of framed oriented links are the same, assuming both are normalized in such a way that both take the value 1 on the empty knot.*

Proof. To see that the two invariants κ, τ are the same (problem 4.9 in [Bar1]) we can proceed as follows. Let $g_1, \dots, g_n$ be an orthonormal base with respect to the Killing form. We will first define a linear mapping $\mu: \mathcal{P}_m \to \hat{U}\mathfrak{g}^{\otimes m}[[h]]$. Suppose that the vertices of a chord diagram $D \in \mathcal{P}_m$ are $a^i_1, \dots, a^i_{k_i}$ on the ith string (the order follows the orientation of the string). *A state* is a mapping σ from the set of all vertices $\{a^j_i\}$ to $\{1, 2, \dots, n\}$ which takes the same value on the two vertices of every chord (n is the dimension of $\mathfrak{g}$). Let

$$\mu(D) = h^{(\#\text{ of vertices})/2} \sum_{\text{states } \sigma} g_{\sigma(a^1_1)} \cdots g_{\sigma(a^1_{k_1})} \otimes \cdots \otimes g_{\sigma(a^m_1)} \cdots g_{\sigma(a^m_{km})}.$$

This is a well-defined linear mapping (see also [Bar1]).

Drinfeld proved that ([Drin1; Drin2]) there is another structure on $\hat{U}\mathfrak{g}$ which makes $\hat{U}\mathfrak{g}$ a *quasi-triangular quasi-Hopf algebra* (not Hopf algebra), with the usual co-multiplication of the universal enveloping algebra, $R = \exp(ht/2)$, $\Phi = \Phi_{KZ}(t^{12}, t^{23})$. Moreover this quasi-triangular quasi-Hopf algebra is a *ribbon quasi-Hopf algebra* (see the definition of ribbon quasi-Hopf algebra in [Al-Co]), the ribbon element is $v = \exp(-\sum_{i=1}^n g_i g_i/2)$.

The series of algebras $\mathcal{P}_n$ is not a ribbon quasi-Hopf algebra, but we have defined operators Δ, ε, elements Φ, R for them. It is easy to see that the mapping μ commutes with $\Delta, \varepsilon, \Phi, R$, and the invariant $\kappa_{\mathfrak{g},\rho}$ is exactly the invariant of oriented framed links obtained by the standard procedure (see [Al-Co]) using the ribbon quasi-Hopf algebra and the representation $\rho: \mathfrak{g} \to \operatorname{End}(V)$.

Drinfeld [Drin1] proved that the above two structures on $\hat{U}\mathfrak{g}$: (1) ribbon Hopf algebra structure and (2) ribbon quasi-Hopf algebra structure are gauge equivalent, i.e. one can be obtained from the other by a (non-symmetric) twist (see also [Koh; Kas]). Their categories of representations are equivalent, hence the two invariants

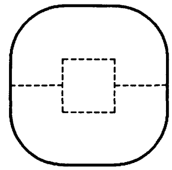


Fig. 8.

κ and τ are the same (up to constant). □

Suppose that $\{\alpha_1, \dots, \alpha_r\}$ is a system of roots of $\mathfrak{g}$ and that $H_1, \dots, H_r$ are the corresponding coroots. Let $\delta = \sum_{i=1}^{r} c_i\alpha_i$, $c_i \in \mathbb{Q}$ be the half-sum of positive roots. Put $H = \sum_{i=1}^{r} c_i H_i$. The value of $\tau_{\mathfrak{g},\rho}$ on the trivial knot is trace$(\exp(h\rho(H))$ (see [Re-Tu]). Applying Theorem 10 we get:

$$\mathrm{trace}(\exp(h\rho(H))) = W_{\rho(t)}(\phi^{-1}). \tag{12}$$

Using formula (1) we can express ϕ^{-1}, and hence the RHS of (12), in terms of the multiple zeta values $\zeta(i_1, \dots, i_k)$ (see also [Le-Mu2]). The coefficient of $h^n, n \in \mathbb{Z}$ of the LHS of (12) are rational, hence from (12) we can get many relations between multiple zeta values.

The cases $\mathfrak{g} = \mathrm{sl}_N$ or so_N and ρ is the fundamental representation were treated in detail in [Le-Mu1; Le-Mu2]. In these papers we need not use Drinfeld's results to prove Theorem 10, instead we used the explicit formula of the Kontsevich integral. Even in these cases the identities between multiple zeta values we obtain seem far from trivial. The famous Euler formula expressing $\zeta(2n)$ in terms of the Bernoulli numbers is among them.

The first two terms of ϕ^{-1} are (see, for example, [Le-Mu2]):

$$\phi^{-1} = 1 + D/48 + (\text{terms of higher grading}), \tag{13}$$

where D is depicted in Figure 8. Hence, comparing the coefficient of h^2 in both sides of (12), we get:

$$\mathrm{trace}(\rho(H)^2/2) = \dim(\rho)C_\rho/48,$$

where $\dim(\rho)$ is the dimension of the representation and C_ρ is its Casimir number. Putting $\rho = Ad$, the adjoint representation, we get

$$(\delta, \delta) = \dim(\mathfrak{g})/24. \tag{14}$$

This is the strange formula of Freudental-de Vries (see [F-dV], Section 47.11). It is interesting to notice that we have deduced this formula by using knot theory.

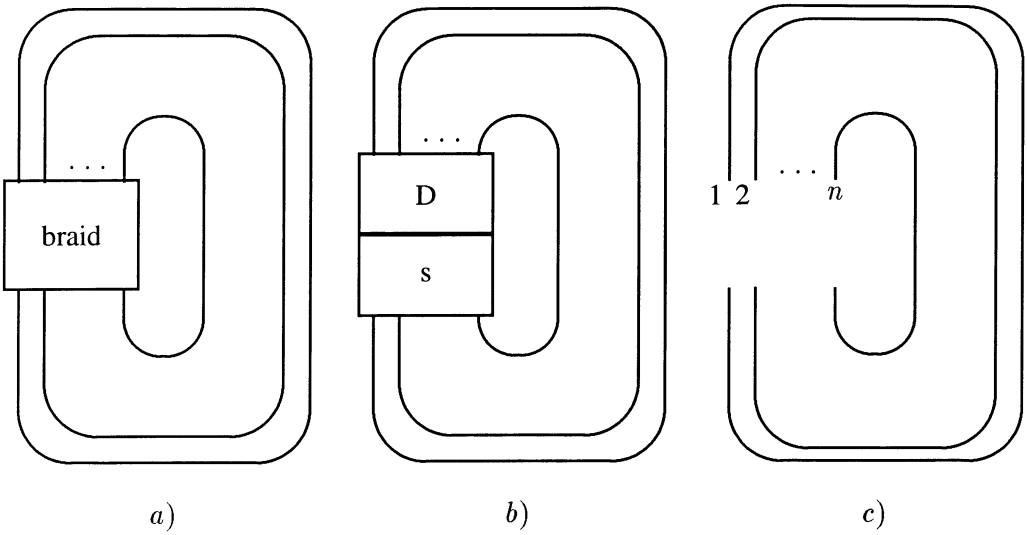


Fig. 9.

11. Computing the universal Vassiliev-Kontsevich invariant using braids

It is well known that every framed oriented link is the closure of a braid (Figure 9a). For a comprehensive treatment of braids, see [Bir2]. Let $\langle\beta\rangle$ be the framed oriented link obtained from a braid β by closing. The braid group B_n on n strands, with the standard generators $\sigma_1, \ldots, \sigma_{n-1}$, is the semi-direct product of the pure braid group and the symmetric group $\mathfrak{S}_n$. Denote by sym the projection sym: $B_n \to \mathfrak{S}_n$.

Regarding every pure braid as a q-tangle, where the top and bottom words are defined by the standard order, we get a representation of the pure braid group into $\mathcal{P}_n$ which can be extended to a representation of B_n into $\mathcal{P}_n \rtimes \mathbb{C}[\mathfrak{S}_n]$. Here $\mathcal{P}_n \rtimes \mathbb{C}[\mathfrak{S}_n]$ is the algebra generated by pairs $(D, s), D \in B_n, s \in \mathbb{C}[\mathfrak{S}_n]$, with the multiplication $(D_1, s_1)(D_2, s_2) = (D_1 \times s_1(D_2), s_1 s_2)$ and bi-linear relation (group $\mathfrak{S}_n$ acts on $\mathcal{P}_n$ by permuting the strings). The representation is given by

$$\rho(\sigma_1) = (\mathrm{e}^{\Omega_{12}/2}, \mathrm{sym}(\sigma_1)),$$

$$\rho(\sigma_i) = ((\Delta^{i-2} \otimes \mathrm{id} \otimes \mathrm{id})(\Phi^{-1} \times \mathrm{e}^{\Omega_{23}/2} \times \Phi^{132}) \otimes 1^{\otimes(n-i-1)}, \mathrm{sym}(\sigma_i)),$$

for $2 \leqslant i \leqslant n-1$ (see also [Al-co, Drin2, Koh, Piu1]). We will compute $\hat{Z}_f(\langle\beta\rangle)$ via $\rho(\beta)$.

Denote by $\mathcal{O}_n$ the space of chord diagrams on n numbered loops. Let $\mathcal{O} = \oplus_{n=0}^{\infty} \mathcal{O}_n \otimes \mathbb{C}$. There is a natural linear mapping from $\mathcal{P}_n \rtimes \mathbb{C}[\mathfrak{S}_n] \to \mathcal{O}$, $(D, s) \to \langle(D, s)\rangle$ by closing as in Figure 9b. For $n = 1$ this is the isomorphism between $\mathcal{A}$

and $\mathcal{P}_1$ described in Section 1. Let $\nu \in \mathcal{P}_1$ be the element such that $\langle \nu \rangle = \phi^{-1}$; it does not depends on the associator Φ and has rational coefficients.

THEOREM 11. *$\hat{Z}_f(\langle \beta \rangle) = \langle \rho(\beta)(c_n, 1) \rangle = \langle (c_n, 1)\rho(\beta) \rangle$, where $c_n = \Delta^{n-1}(\nu) \in \mathcal{P}_n$.*

Note that c_n lies in the center of $\mathcal{P}_n$ (Theorem 1) and $\langle (c_n, 1) \rangle$ is the chord diagram without any chord.

Theorem 11 can be proved by evaluating $\hat{Z}_f$ of the object T_n in Figure 9c. It is not a tangle, but one can define $Z_f(T_n) \in \mathcal{P}_n$ in the obvious way.

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