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Abstract. This note states a structure theorem for compact Kähler manifolds with semipositive Ricci curvature: any such manifold has a finite étale covering possessing a De Rham decomposition as a product of irreducible compact Kähler manifolds, each one being either Ricci flat (torus, symplectic or Calabi-Yau manifold), or Ricci semipositive without non trivial holomorphic forms. Related questions and conjectures concerning the latter case are discussed.

Key words: Compact Kähler manifold, semipositive Ricci curvature, complex torus, symplectic manifold, Calabi-Yau manifold, Albanese map, fundamental group, Bochner formula, De Rham decomposition, Cheeger-Gromoll theorem, nef line bundle, Kodaira-Iitaka dimension, rationally connected manifold

1. Main results

This short note is a continuation of our previous work [DPS93] on compact Kähler manifolds $X$ with semipositive Ricci curvature. Our purpose is to state a splitting theorem describing the structure of such manifolds, and to raise some related questions. The foundational background will be found in papers by Lichnerowicz [Li67], [Li71], and Cheeger-Gromoll [CG71], [CG72]. Recall that a Calabi-Yau manifold $X$ is a compact Kähler manifold with $c_1(X) = 0$ and finite fundamental group $\pi_1(X)$, such that the universal covering $\widetilde{X}$ satisfies $H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}) = 0$ for all $1 \leq p \leq \dim X - 1$. A symplectic manifold $X$ is a compact Kähler manifold admitting a holomorphic symplectic 2-form $\omega$ (of maximal rank everywhere); in particular $K_X = \mathcal{O}_X$. We denote here as usual

$$\Omega_X = \Omega^1_X = T^*_X, \quad \Omega^p_X = \Lambda^p T^*_X, \quad K_X = \det(T^*_X).$$

The following structure theorem generalizes the structure theorem for Ricci-flat manifolds (due to Bogomolov [Bo74a], [Bo74b], Kobayashi [Ko81] and Beauville [Be83]) to the Ricci semipositive case.
STRUCTURE THEOREM. Let $X$ be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then

(i) The universal covering $\tilde{X}$ admits a holomorphic and isometric splitting

$$\tilde{X} \simeq \mathbb{C}^q \times \prod X_i$$

with $X_i$ being either a Calabi-Yau manifold or a symplectic manifold or a manifold with $-K_{X_i}$ semipositive and $H^0(X_i, \Omega_{X_i}^{\otimes m}) = 0$ for all $m > 0$.

(ii) There exists a finite étale Galois covering $\tilde{X} \to X$ such that the Albanese variety $\text{Alb}(\tilde{X})$ is a $q$-dimensional torus and the Albanese map $\alpha : \tilde{X} \to \text{Alb}(X)$ is a locally trivial holomorphic fibre bundle whose fibres are products $\prod X_i$ of the type described in (i), all $X_i$ being simply connected.

(iii) We have $\pi_1(\tilde{X}) \simeq \mathbb{Z}^{2q}$ and $\pi_1(X)$ is an extension of a finite group $\Gamma$ by the normal subgroup $\pi_1(\tilde{X})$. In particular there is an exact sequence

$$0 \to \mathbb{Z}^{2q} \to \pi_1(X) \to \Gamma \to 0,$$

and the fundamental group $\pi_1(X)$ is almost abelian.

Recall that a line bundle $L$ is said to be hermitian semipositive if it can be equipped with a smooth hermitian metric of semipositive curvature form. A sufficient condition for hermitian semipositivity is that some multiple of $L$ is spanned by global sections; on the other hand, the hermitian semipositivity condition implies that $L$ is numerically effective (nef) in the sense of [DPS94], which, for $X$ projective algebraic, is equivalent to saying that $L \cdot C \geq 0$ for every curve $C$ in $X$. Examples contained in [DPS94] show that all three conditions are different (even for $X$ projective algebraic). By Yau’s solution of the Calabi conjecture (see [Au76], [Yau78]), a compact Kähler manifold $X$ has a hermitian semipositive anticanonical bundle $-K_X$ if and only if $X$ admits a Kähler metric $g$ with $\text{Ricci}(g) \geq 0$. The isometric decomposition described in the theorem refers to such Kähler metrics.

In view of ‘standard conjectures’ in minimal model theory it is expected that projective manifolds $X$ with no nonzero global sections in $H^0(X, \Omega_{X}^{\otimes m})$, $m > 0$, are rationally connected. We hope that most of the above results will continue to hold under the weaker assumption that $-K_X$ is nef instead of hermitian semipositive. However, the technical tools needed to treat this case are still missing.

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2. Bochner formula and holomorphic differential forms

Our starting point is the following well-known consequence of the Bochner formula.
LEMMA. Let $X$ be a compact $n$-dimensional Kähler manifold with $-K_X$ hermitian semipositive. Then every section of $\Omega_X^{\otimes m}$, $m \geq 1$ is parallel with respect to the given Kähler metric.

Proof. The Lemma is an easy consequence of the Bochner formula

$$\Delta(\|u\|^2) = \|\nabla u\|^2 + Q(u),$$

where $u \in H^0(X, \Omega_X^{\otimes m})$ and $Q(u) \geq m\lambda_0 \|u\|^2$. Here $\lambda_0$ is the smallest eigenvalue of the Ricci curvature tensor. For details see for instance [Ko83].

The following definition of a modified Kodaira dimension $\kappa^+(X)$ is taken from Campana [Ca93]. As the usual Kodaira dimension $\kappa(X)$, this is a birational invariant of $X$. Other similar invariants have also been considered in [BR90] and [Ma93].

DEFINITION. Let $Y$ be a compact complex manifold. We define

(i) $\kappa^+(Y) = \max\{\kappa(\det F) : F$ is a subsheaf of $\Omega_Y^p$ for some $p > 0\}$,

(ii) $\kappa_{++}(Y) = \max\{\kappa(\det F) : F$ is a subsheaf of $\Omega_Y^{\otimes m}$ for some $m > 0\}$.

Here we let as usual $\det F = (\Lambda^r F)^{**}$, where $r = \text{rank } F$ and $\kappa$ is the usual Iitaka dimension of a line bundle.

Clearly, we have $-\infty \leq \kappa(Y) \leq \kappa^+(Y) \leq \kappa_{++}(Y)$ where $\kappa(Y) = \kappa(K_Y)$ is the usual Kodaira dimension. It would be interesting to know whether there are precise relations between $\kappa^+(Y)$ and $\kappa_{++}(Y)$, as well as with the weighted Kodaira dimensions defined by Manivel [Ma93]. The above lemma implies:

PROPOSITION. Let $X$ be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then $\kappa_{++}(X) \leq 0$.

Proof. Assume that $\kappa_{++}(X) > 0$. Then we can find an integer $m > 0$ and a subsheaf $F \subset \Omega_X^{\otimes m}$ with $\kappa(\det F) > 0$. Hence there is some $\mu \in \mathbb{N}$ and $s \in H^0(X, (\det F)^{\mu})$ with $s \neq 0$. Since $\kappa(\det F) > 0$, $s$ must have zeroes. Hence the induced section $\tilde{s} \in H^0(X, \Omega_X^{\otimes m_{\mu r}})$ has zeroes too, $r$ being the rank of $F$. This contradicts the previous Lemma.

COROLLARY. Let $X$ be a compact Kähler manifold with $-K_X$ hermitian semipositive. Let $\phi: X \to Y$ be a surjective holomorphic map to a normal compact Kähler space. Then $\kappa(Y) \leq 0$. (Here $\kappa(Y) = \kappa(\hat{Y})$, where $\hat{Y}$ is an arbitrary desingularization of $Y$.)

Proof. This follows from the inequalities $0 \geq \kappa^+(X) \geq \kappa^+(Y) \geq \kappa(Y)$. For the second inequality, which is easily checked by a pulling-back argument, see [Ca93].
3. Proof of the structure theorem

We suppose here that $X$ is equipped with a Kähler metric $g$ such that $\text{Ricci}(g) \geq 0$, and we set $n = \dim C X$.

(i) Let $(\tilde{X}, g) \simeq \prod (X_i, g_i)$ be the De Rham decomposition of $(\tilde{X}, g)$, induced by a decomposition of the holonomy representation in irreducible representations. Since the holonomy is contained in $U(n)$, all factors $(X_i, g_i)$ are Kähler manifolds with irreducible holonomy and holonomy group $H_i \subset U(n_i)$, $n_i = \dim X_i$. By Cheeger-Gromoll \cite{CG71}, there is possibly a flat factor $X_0 = \mathbb{C}$ and the other factors $X_i$, $i \geq 1$, are compact. Also, the product structure shows that $-K_{\tilde{X}}$ is hermitian semipositive. It suffices to prove that $\kappa_+(X_i) = 0$ implies that $X_i$ is a Calabi-Yau manifold or a symplectic manifold. In view of Section 2, the condition $\kappa_+(X_i) = 0$ means that there is a nonzero section $u \in H^0(X_i, \Omega^m_{X_i})$ for some $m > 0$. Since $u$ is parallel by the lemma, it is invariant under the holonomy action, and therefore the holonomy group $H_i$ is not the full unitary group $U(n_i)$ (indeed, the trivial representation does not occur in the decomposition of $(\mathbb{C}^{n_i})^\otimes m$ in irreducible $U(n_i)$-representations, all weights being of length $m$). By Berger’s classification of holonomy groups \cite{Bg55} there are only two remaining possibilities, namely $H_i = SU(n_i)$ or $H_i = Sp(n_i/2)$. The case $H_i = SU(n_i)$ leads to $X_i$ being a Calabi-Yau manifold. The remaining case $H_i = Sp(n_i/2)$ implies that $X_i$ is symplectic (see e.g. \cite{Be83}).

(ii) Set $X' = \prod_{i \geq 1} X_i$. The group of covering transformations acts on the product $\tilde{X} = \mathbb{C}' \times \tilde{X}'$ by holomorphic isometries of the form $x = (z, x') \mapsto (u(z), v(x'))$. At this point, the argument is slightly more involved than in Beauville’s paper \cite{Be83}, because the group $G'$ of holomorphic isometries of $X'$ need not be finite ($X'$ may be for instance a projective space); instead, we imitate the proof of ([CG72], Theorem 9.2) and use the fact that $X'$ and $G' = \text{Isom}(X')$ are compact. Let $E_q = \mathbb{C}' \rtimes U(q)$ be the group of unitary motions of $\mathbb{C}'$. Then $\pi_1(X)$ can be seen as a discrete subgroup of $E_q \times G'$. As $G'$ is compact, the kernel of the projection map $\pi_1(X) \to E_q$ is finite and the image of $\pi_1(X)$ in $E_q$ is still discrete with compact quotient. This shows that there is a subgroup $\Gamma$ of finite index in $\pi_1(X)$ which is isomorphic to a crystallographic subgroup of $\mathbb{C}'$. By Bieberbach’s theorem, the subgroup $\Gamma_0 \subset \Gamma$ of elements which are translations is a subgroup of finite index. Taking the intersection of all conjugates of $\Gamma_0$ in $\pi_1(X)$, we find a normal subgroup $\Gamma_1 \subset \pi_1(X)$ of finite index, acting by translations on $\mathbb{C}'$. Then $\tilde{X} = \tilde{X}/\Gamma_1$ is a fibre bundle over the torus $\mathbb{C}'/\Gamma_1$ with $X'$ as fibre and $\pi_1(X') = 1$. Therefore $\tilde{X}$ is the desired finite étale covering of $X$.

(iii) is an immediate consequence of (ii), using the homotopy exact sequence of a fibration.

\begin{corollary}
Let $X$ be a compact Kähler manifold with $-K_X$ hermitian semipositive. If $\tilde{X}$ is indecomposable and $\kappa_+(X) = 0$, then $X$ is Ricci-flat.
\end{corollary}
COROLLARY 2. Let $X$ be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then, if $\hat{X} \to X$ is an arbitrary finite étale covering

$$\kappa_+(X) = -\infty \iff \kappa_+(\hat{X}) = -\infty \iff \forall \hat{X} \to X, \forall p \geq 1, \quad H^0(\hat{X}, \Omega^p_{\hat{X}}) = 0.$$ 

If $\kappa_+(X) = -\infty$, then $\chi(X, \mathcal{O}_X) = 1$ and $X$ is simply connected.

Proof. The equivalence of all three properties is a direct consequence of the structure theorem. Now, any étale covering $\hat{X} \to X$ satisfies $\kappa_+(\hat{X}) = \kappa_+(X) = -\infty$, hence $\chi(\hat{X}, \mathcal{O}_{\hat{X}}) = \chi(X, \mathcal{O}_X) = 1$ (by Hodge symmetry we have $h^p(X, \mathcal{O}_X) = 0$ for $p \geq 1$, whilst $h^0(X, \mathcal{O}_X) = 1$). However, if $d$ is the covering degree, the Riemann-Roch formula implies $\chi(\hat{X}, \mathcal{O}_{\hat{X}}) = d \chi(X, \mathcal{O}_X)$, hence $d = 1$ and $X$ must be simply connected.

4. Related questions for the case $-K_X$ nef

In order to make the structure theorem more explicit, it would be necessary to characterize more precisely the manifolds for which $\kappa_+(X) = -\infty$. We expect these manifolds to be rationally connected, even when $-K_X$ is just supposed to be nef.

CONJECTURE. Let $X$ be a compact Kähler manifold such that $-K_X$ is nef and $\kappa_+(X) = -\infty$. Then $X$ is rationally connected, i.e. any two points of $X$ can be joined by a chain of rational curves.

Campana even conjectures this to be true without assuming $-K_X$ to be nef.

Another hope we have is that a similar structure theorem might also hold in the case $-K_X$ nef. A small part of it would be to understand better the structure of the Albanese map. We proved in [DPS93] that the Albanese map is surjective when $\dim X \leq 3$, and if $\dim X \leq 2$ it is well-known that the Albanese map is a locally trivial fibration. It is thus natural to state the following

PROBLEM. Let $X$ be a compact Kähler manifold with $-K_X$ nef. Is the Albanese map $\alpha: X \to \text{Alb}(X)$ a smooth locally trivial fibration?

The following simple example shows, even in the case of a locally trivial fibration, that the structure group of transition automorphisms need not be a group of isometries, in contrast with the case $-K_X$ hermitian semipositive.

EXAMPLE 1 (see [DPS94], Example 1.7). Let $\mathbb{C} = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic
curve, and let $E \rightarrow C$ be the flat rank 2 bundle associated to the representation $\pi_1(C) \rightarrow \text{GL}_2(\mathbb{C})$ defined by the monodromy matrices

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}.
$$

Then the projectivized bundle $X = \mathbb{P}(E)$ is a ruled surface over $C$ with $-K_X$ nef and not hermitian semipositive (cf. [DPS94]). In this case, the Albanese map $X \rightarrow C$ is a locally trivial $\mathbb{P}^1$-bundle, but the monodromy group is not relatively compact in $\text{GL}_2(\mathbb{C})$, hence there is no invariant Kähler metric on the fibre.

**EXAMPLE 2.** The following example shows that the picture is unclear even in the case of surfaces with $\kappa_+(X) = -\infty$. Let $\mathbf{p} = (p_1, \ldots, p_9)$ be a configuration of 9 points in $\mathbb{P}^2$ and let $\pi: X_{\mathbf{p}} \rightarrow \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ with center $\mathbf{p}$. Here some of the points $p_i$ may be infinitely near: as usual, this means that the blowing-up process is made inductively, each $p_i$ being an arbitrary point in the blow-up of $\mathbb{P}^2$ at $(p_1, \ldots, p_{i-1})$. There is always a cubic curve $C$ containing all 9 points ($C$ is even unique if $\mathbf{p}$ is general enough). The only assumption we make is that $C$ is nonsingular, and we let $C = \{Q(x_0, z_1, z_2) = 0\} \subset \mathbb{P}^2$, deg $Q = 3$. Then $C$ is an elliptic curve and $-K_{X_{\mathbf{p}}} = \pi^*\mathcal{O}(3) - \sum E_i$ where $E_i = \pi^{-1}(p_i)$ are the exceptional divisors. Clearly $Q$ defines a section of $-K_{X_{\mathbf{p}}}$, of divisor equal to the strict transform $C'$ of $C$, hence $-K_{X_{\mathbf{p}}} \simeq \mathcal{O}(C')$, and $(-K_{X_{\mathbf{p}}})^2 = (C')^2 = C^2 - 9 = 0$. Therefore $-K_{X_{\mathbf{p}}}$ is always nef.

It is easy to see that $-mK_{X_{\mathbf{p}}}$ may be generated or not by sections according to the choice of the 9 points $p_i$. In fact, if $p_i'$ is the point of $C'$ lying over $p_i$, we have

$$-K_{X_{\mathbf{p}}}|_{C'} = \pi^*(\mathcal{O}(3))|_{C'} \otimes \mathcal{O}\left(-\sum p_j\right) = \pi^*(\mathcal{O}(3)|_C \otimes \mathcal{O}\left(-\sum p_j\right)).$$

Since $C' \simeq C$ is an elliptic curve and $-K_{X_{\mathbf{p}}}|_{C'}$ has degree 0, there are nonzero sections in $H^0(C', -mK_{X_{\mathbf{p}}}|_{C'})$ if and only if $L_{\mathbf{p}} = \mathcal{O}(3)|_C \otimes \mathcal{O}(\sum p_j)$ is a torsion point in $\text{Pic}^0(C)$ of order dividing $m$. Such sections always extend to $X_{\mathbf{p}}$. Indeed, we may assume that $m$ is exactly the order. Then $\mathcal{O}(C') \otimes \mathcal{O}(\sum p_j) = \mathcal{O}(C')$ admits a filtration by its subsheaves $\mathcal{O}(kC')$, $0 \leq k \leq m - 1$, and the $H^1$ groups of the graded pieces are $H^1(X_{\mathbf{p}}, \mathcal{O}_{X_{\mathbf{p}}}) = 0$ for $k = 0$ and

$$H^1(C', \mathcal{O}(kC')|_{C'}) = H^0(C', \mathcal{O}(-kC')) = 0 \quad \text{for} \quad 0 < k < m.$$

Therefore $H^1(X_{\mathbf{p}}, \mathcal{O}(\sum p_j) \otimes \mathcal{O}(\sum p_j)) = 0$, as desired. In particular, $-K_{X_{\mathbf{p}}}$ is hermitian semipositive as soon as $L_{\mathbf{p}}$ is a torsion point in $\text{Pic}^0(C)$. In this case, there is a polynomial $R_m$ of degree $3m$ vanishing of order $m$ at all points $p_i$, such that the rational function $R_m/Q^m$ defines an elliptic fibration $\varphi: X_{\mathbf{p}} \rightarrow \mathbb{P}^1$; in this fibration $C$ is a multiple fibre of multiplicity $m$ and we have $-mK_{X_{\mathbf{p}}}$ =
An interesting question is to understand what happens when \( L_p \) is no longer a torsion point in \( \text{Pic}^0(C) \) (this is precisely the situation considered by Ogus [Og76] in order to produce a counterexample to the formal principle for infinitesimal neighborhoods). In this situation, we may approximate \( p \) by a sequence of configurations \( p_m \subset C \) such that the corresponding line bundle \( L_{p_m} \) is a torsion point of order \( m \) (just move a little bit \( p \) and take a suitable \( p_m \in C \) close to \( p \)). The sequence of fibrations \( X_{p_m} \rightarrow \mathbb{P}_1 \) does not yield a fibration \( X_p \rightarrow \mathbb{P}_1 \) in the limit, but we believe that there might exist instead a holomorphic foliation on \( X_p \). In this foliation, \( C \) would be a closed leaf, and the generic leaf would be nonclosed and of conformal type \( C \) (or possibly \( C^* \)). If indeed the foliation exists and admits a smooth invariant transversal volume form, then \( -K_{X_p} \) would still be hermitian semipositive. We are thus led to the following question.

**QUESTION.** Let \( X \) be compact Kähler manifold with \( -K_X \) nef and \( X \) rationally connected. Is then \( -K_X \) automatically hermitian semipositive? In particular, is it always the case that \( \mathbb{P}_2 \) blown-up in 9 points of a nonsingular cubic curve has a semipositive anticanonical bundle?

**References**

