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Left cells and domino tableaux in classical Weyl groups

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1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra of classical type, $U(\mathfrak{g})$ its enveloping algebra. The classification of the primitive spectrum Prim $U(\mathfrak{g})$ of $U(\mathfrak{g})$ was first achieved by Joseph in type A_n ([14, 16]; see also [33]) and by Barbasch and Vogan in types B_n, C_n, and D_n [3]. Garfinkle has substantially simplified the Barbasch-Vogan classification [7, 8, 9, 10]. In all of these papers, the starting point is a fundamental result of Duflo which states that the primitive ideals of a fixed (dominant) infinitesimal character λ are all realized as annihilators I_w of simple highest weight modules L_w indexed by an element w of the Weyl group W and λ [6]. Thus classifying primitive ideals of infinitesimal character λ amounts to deciding for each $w, w' \in W$ whether or not $I_w = I_{w'}$. This is done by attaching a complete combinatorial invariant T_w to w depending only on I_w, so that $I_w = I_{w'}$ if and only if $T_w = T_{w'}$. In type A_n, T_w turns out to be a standard Young $(n+1)$-tableau; in the other types T_w is a standard Young $2n$-tableau with special symbol in [3] and a standard domino n-tableau of special shape in [7].

The equivalence classes under the relation defined by $w \sim w'$ if and only if $I_w = I_{w'}$ are called left cells and were originally defined by Kazhdan and Lusztig in a completely different way [20]; the equivalence of their definition and the one above is a consequence of the Kazhdan-Lusztig conjectures. (Earlier Joseph had given a weaker definition of cell [14], which was slightly modified in type D_n by Vogan [34]; the modified definition turns out to coincide with the above one in the classical cases but not in general.) A fundamental property of left cells as defined in [20] is that they span vector spaces which carry the natural structure of left W-modules, or more precisely left modules over the Hecke algebra H of W; prior to [20], Joseph showed (modulo a conjecture later proved by Vogan) that Prim $U(\mathfrak{g})$ also carries a W-module structure. Using some tables of Alvis in the exceptional cases, Lusztig has computed the W-module structure of every left cell [22, 26]. Although the resulting Kazhdan-Lusztig picture of the left cells is quite beautiful (at least in the classical cases), it did not seem to merge well with the Barbasch-Vogan-Garfinkle picture; no one knew how to compute W-module structure from Garfinkle's algorithms.

The purpose of this paper is to remedy this gap by showing that the W-module structure of a left cell can in fact be read off very simply from its standard domino tableau of special shape, using bijections between Weyl group representations and symbols on the one hand and symbols and partitions on the other. We also show that the operators $T_{\alpha\beta}$ and $S_{\alpha\beta}$ used to define Vogan's generalized τ-invariant, which play a fundamental role in Garfinkle's classification of the primitive spectrum, may be lifted to W-module maps between left cells (or actually H-module maps). Furthermore, the operators $T_{\alpha\beta}, S_{\alpha\beta}$ (plus a substitute S_D for $S_{\alpha\beta}$ in type D) generate enough intertwining operators between left cells to enable one in principle to write down Kazhdan-Lusztig bases for every irreducible W- or H-submodule of a left cell (Theorem 4.3). We conclude the paper by showing how to compute explicitly the product of two basis vectors of Lusztig's asymptotic Hecke algebra J ([25, 29]) whenever this product is a third basis vector. As a consequence we get explicit formulas for the socle of the bimodule of Ad $\mathfrak{g}$-finite maps between two simple highest weight modules in many cases and for the behavior of special unipotent representations under the tensor product.

The paper is organized as follows. In Section 2, we recall Lusztig's theory of classical left cells, regarded as modules. Our exposition is a slight variant of that in [23, chs. 4, 5]. We also set up the correspondences between Weyl group representations, symbols, and partitions that we will need in the next section. In Section 3, we show how to read off the W-module structure of a left cell from its tableaux. In the next section, we recall the definitions of the maps $T_{\alpha\beta}, S_{\alpha\beta}$, and S_D on left cells and observe that they induce H-module maps. We then use these intertwining operators to produce basis vectors for irreducible H-representations. Finally, in the last section, we develop the applications promised above to bimodules of maps between simple highest weight modules and tensor products of special unipotent representations.

2. Left cells as modules

Throughout we consider only Weyl groups of types BC and D, as all of our results are trivial in type A. So let W_n be the Weyl group of type BC_n; it acts in the usual way on $\mathbb{C}^n$ by permuting and changing the signs of the coordinates. Let $W'_n \subset W_n$ be the Weyl group of type D_n, consisting of all permutations and even sign changes. We begin by recalling the standard parametrization of irreducible W_n- and W'_n-representations.

PROPOSITION 2.1. *There is a 1-1 correspondence* $(\mathbf{d},\mathbf{f}) \mapsto \pi_{(\mathbf{d},\mathbf{f})}$ *between ordered pairs* $(\mathbf{d},\mathbf{f})$ *of partitions the sums* $|\mathbf{d}|, |\mathbf{f}|$ *of whose parts add to* n, *and irreducible representations of* W_n. *We have* $\pi_{(\mathbf{f}^t,\mathbf{d}^t)} \cong \pi_{(\mathbf{d},\mathbf{f})} \otimes \mathrm{sgn}$, *where* $\mathbf{p}^t$ *denotes the transpose of the partition* $\mathbf{p}$, *and* sgn *denotes the sign representation.*

PROPOSITION 2.2. *There is a correspondence* $(\mathbf{d},\mathbf{f}) \mapsto \pi_{\mathbf{d},\mathbf{f}}$ *between unordered pairs* $(\mathbf{d},\mathbf{f})$ *of partitions with* $|\mathbf{d}|+|\mathbf{f}|=n$ *and irreducible representations of* W_n'. *The correspondence is 1-1 except when* $\mathbf{d}=\mathbf{f}$; *in that case two representations* $\pi^1_{\mathbf{d},\mathbf{f}}, \pi^2_{\mathbf{d},\mathbf{f}}$ *are attached to* $(\mathbf{d},\mathbf{f})$. *As in type BC, we have* $\pi_{\mathbf{f}^t,\mathbf{d}^t} \cong \pi_{\mathbf{d},\mathbf{f}} \otimes \mathrm{sgn}$. *If a representation is twisted by the outer automorphism of* W_n' *induced from the symmetry of its Coxeter graph, the resulting representation is isomorphic to the original one, unless the latter has a numeral, in which case the new representation has the opposite numeral.*

For proofs see, e.g., [30,31]; these papers also give a precise definition of the labels 1 and 2 and show how these labels change when the corresponding representations are tensored with sgn (cf. [5]).

We now recall Lusztig's well-known method for rewriting the parametrizations of Propositions 2.1 and 2.2. Henceforth it will be convenient to treat the Weyl groups of types B and C separately (for Lie-theoretic reasons), even though these groups are of course isomorphic. Following [21], we define a symbol in type B_n (resp. C_n) to be an arrangement

$$\begin{pmatrix} p_1 & & p_2 & \dots & p_r & & p_{r+1} \\ & q_1 & & & & q_r & \end{pmatrix} \tag{2.3}$$

of non-negative numbers such that $\sum_i(2p_i+1)+\sum_j 2q_j = 2n+1+r(2r+1)$ (resp. $\sum_i 2p_i + \sum_j(2q_j+1) = 2n+r(2r+1)$) and $p_1 < \cdots < p_{r+1}, q_1 < \cdots < q_r$. Define a symbol in type D_n to be an arrangement

$$\begin{pmatrix} p_1 & \dots & p_r \\ q_1 & \dots & q_r \end{pmatrix} \tag{2.4}$$

of non-negative integers such that $\sum_i(2p_i+1)+\sum_j 2q_j = 2n+r(2r-1)$ and $p_1 < \cdots < p_r, q_1 < \cdots < q_r$. We introduce an equivalence relation $\sim$ on symbols as the transitive closure of the 'shift relations'

$$\begin{pmatrix} p_1 & & \dots & p_{r+1} \\ & q_1 & \dots & \end{pmatrix} \sim \begin{pmatrix} 0 & & p_1+1 & & \dots & p_{r+1}+1 \\ & 0 & & q_1+1 & \dots & \end{pmatrix}$$

and

$$\begin{pmatrix} p_1 & \dots & p_r \\ q_1 & \dots & q_r \end{pmatrix} \sim \begin{pmatrix} 0 & p_1+1 & \dots & p_{r+1} \\ 0 & q_1+1 & \dots & q_{r+1} \end{pmatrix}. \tag{2.5(b)}$$

In type D_n, we further extend $\sim$ by decreeing that

$$\begin{pmatrix} p_1 & \dots & p_r \\ q_1 & \dots & q_r \end{pmatrix} \sim \begin{pmatrix} q_1 & \dots & q_r \\ p_1 & \dots & p_r \end{pmatrix}; \tag{2.5(c)}$$

furthermore, if a symbol in type D_n has $p_i = q_i$ for all i, then we attach a numeral 1 or 2 to it.

There are injective maps π'_n from partitions of $2n+1$ (resp. $2n, 2n$) to symbols in type B_n (resp. C_n, D_n), defined as follows: given a partition $\mathbf{p}$, add a zero part to it if necessary to make it have an odd number of parts (resp. an odd number of parts, an even number of parts) and arrange these parts in increasing order. Obtain a new partition $\mathbf{p}'$ (of a number larger than $2n+1$) by adding zero to the first part of $\mathbf{p}$, one to its second part, two to its third, and so on. Enumerate the odd parts of $\mathbf{p}'$ as $2p_1+1 < \cdots < 2p_s+1$ (resp. $2q_1+1 < \cdots < 2q_r+1, 2p_1+1 < \cdots < 2p_s+1$) and its even parts as $2q_1 < \cdots < 2q_r$ (resp. $2p_1 < \cdots < 2p_s, 2q_1 < \cdots < 2q_r$). Assume that $s = r+1$ (resp. $s = r+1, s = r$); that is, restrict the domain of π'_n to partitions for which this condition holds. Then one may form a symbol as in (2.3) (resp. (2.3),(2.4)) out of the p_i and q_i. Take this symbol as the image of $\mathbf{p}$ under π'_n. In case the original partition $\mathbf{p}$ is very even in the usual sense that its terms are all even and occur with even multiplicity, then we attach a numeral 1 or 2 to $\mathbf{p}$ and the same numeral to its image under π'_n. While the domain of π'_n does not contain every partition of $2n+1$ (resp. $2n, 2n$), it does contain every partition corresponding to a nilpotent orbit in the appropriate Lie algebra. In particular, it contains the partitions corresponding to the special orbits; these are just the ones for which the associated symbol is special in the usual sense that $p_1 \leq q_1 \leq p_2 \leq \cdots$. The map π'_n is one-to-one in types B_n, C_n and on very even partitions in type D_n; for other partitions in type D_n, it is two-to-one (thanks to the identification (2.5(c))). Similarly, there are bijections π_n from symbols in type B_n (resp. C_n, D_n) to representations of W_n (resp. W_n, W'_n), obtained as follows. Given a symbol as in (2.3) (resp. (2.3),(2.4)), subtract $i-1$ from p_i and q_i to obtain an ordered (resp. ordered,unordered) pair of partitions $(p'_i), (q'_j)$ the sums of whose parts add to n. Attach a representation of W_n (resp. W_n, W'_n) to this pair as in Proposition 2.1 (resp. 2.1, 2.2). This representation is the image of the symbol under π_n; in type D_n, if the symbol has a numeral attached to it, then the representation has the same numeral. If we set $\pi := \pi_n \pi'_n$ and restrict its domain to partitions corresponding to nilpotent orbits in the appropriate Lie algebra, then it induces a map from nilpotent orbits to Weyl group representations which coincides with (part of) the Springer correspondence [3, 24].

We now recall Lusztig's definition of left cells in [22] (where they are called 'packets'; their coincidence with the left cells of [20] is demonstrated in [26], using the theory of primitive ideals in $U(\mathfrak{g})$).

DEFINITION 2.6. The left cells of W_n or W'_n are the smallest class of representations containing the trivial one and closed under truncated induction from parabolic subgroups and tensoring with sgn.

We do not need to recall the definition of truncated induction here; it suffices to cite the formula from [22] for the representation truncatedly induced from a given irreducible one. Thanks to the transitivity of truncated induction and its well-known behavior in type A, it suffices to show how to induce an irreducible representation π' of W' to $W(= W_n$ or $W_n')$ when W' is a maximal parabolic subgroup whose type A component acts by sgn on π'.

PROPOSITION 2.7. ([22]). *With the above notation, suppose that the type A component of W' has rank $r-1$. Assume that the symbol s' has at least r (not necessarily distinct) terms, using the shift relations as necessary. Then the induced representation π is irreducible if and only if the rth largest term in s' occurs only once; in that case the symbol s of π is obtained from s' by adding one to the r largest terms of the latter. Otherwise π has length two and the two symbols s_1, s_2 of the constituents of π are obtained from s' by adding one to the $r-1$ largest parts and to each of the two parts tied for rth largest in turn. In case W is of type D and W' itself is of type A, then the symbol of π can have either numeral, depending on the choice of W'. Otherwise this symbol has either no numeral or the same numeral as s'.*

We will also need to record the effect on symbols of tensoring with sgn.

PROPOSITION 2.8. ([22]). *With notation as in Proposition 2.7, let m be the largest number occurring in the symbol s of a representation π of W. Then the top row of the symbol s' of $\pi \otimes$ sgn is obtained by listing the integers from 0 to m, omitting $m-a$ whenever a occurs in the bottom row of s. Similarly, the bottom row of s' is obtained by listing the integers from 0 to m, omitting $m-a$ whenever a occurs in the top row of s.*

As mentioned above, there is also a rule for determining the numeral of s' in Proposition 2.8 if s has a numeral, but we will not need it. We now reformulate Propositions 2.7 and 2.8 in terms of partitions.

LEMMA 2.9. *Under the hypotheses of Proposition 2.7, let $\mathbf{p}$ be the partition corresponding to π' when the latter is restricted to the non-type A component of W'. Write $\mathbf{p} = [p_1, \ldots, p_s]$ with $p_1 \geqslant \cdots \geqslant p_s$ and assume that $s \geqslant r$, by adding zero parts to $\mathbf{p}$ as necessary. Let $p_{r-a+1}, \ldots, p_{r+b}$ enumerate the parts of $\mathbf{p}$ equal to p_r. Set $\mathbf{p}' = [p_1+2, \ldots, p_r+2, p_{r+1}, \ldots, p_s]$, $\mathbf{p}'' = [p_1+2, \ldots, p_{r-1}+2, p_r+1, p_{r+1}+1, p_{r+2}, \ldots, p_s]$. Then either π is irreducible and corresponds to $\mathbf{p}'$, or π has length two and its constituents correspond to $\mathbf{p}'$ and $\mathbf{p}''$. In type B, π is irreducible if and only if either a is even, or $b=0$ and $p_r \neq 0$. In type C, π is irreducible if and only if either a and p_r have opposite parity, or a is even and $b=0$. In type D, π is irreducible if and only if a and p_r have the same parity, or a is even and $b=0$.*

Proof. This is a simple direct calculation from Proposition 2.7 and the correspondence between symbols and partitions. □

LEMMA 2.10. *If a representation π has partition* $\mathbf{p}$, *then the representation* $\pi \otimes \mathrm{sgn}$ *has partition* $\mathbf{p}^t$, *the transpose of* $\mathbf{p}$.

Proof. Let s be the symbol of π and let $s_1 > \cdots > s_j$ enumerate the distinct terms in s. Let s' be the symbol of $\pi \otimes$ sgn. Gather the terms of s into groups, the ith of which consists of the terms equal to s_i or s_{i+1} (note that each term appears at most twice in s). One can easily work out an explicit correspondence between groups of terms in s, in $\mathbf{p}$, in $\mathbf{p}^t$, and in s'. Now the result follows by an easy calculation with the terms in each group. □

We will also need to see how to the inductive constructions of Definition 2.6 behave on the level of subsets of Weyl groups.

PROPOSITION 2.11. *If $w \in W$ represents the left cell $\mathcal{C}$ (regarded as a module), then $w_0 w$ represents the left cell obtained from $\mathcal{C}$ by tensoring with sgn, where w_0 is the longest element of W. If W' is a parabolic subgroup of W and w' represents a left cell $\mathcal{C}'$ of W', then $w_0 w_0' w'$ represents the left cell obtained from $\mathcal{C}'$ by truncated induction, where w_0' is the longest element of W'.*

Proof. Both assertions follow from [23, ch. 5]; cf. also [13, 14.17]. □

Now we are ready to head towards Lusztig's characterization of left cells in the classical case. This appeared first in [22] and was reformulated in [23] and [28]. Here we modify the treatment in [23] slightly. We have mentioned above that a symbol is said to be special if it is equivalent to one of the form (2.3) or (2.4) with $p_1 \leq q_1 \leq p_2 \leq \cdots$. We will attach a family of left cells to each special symbol; then the totality of left cells will simply be the union of the families.

Given a special symbol s, let $s_1 < \cdots < s_m$ enumerate the terms appearing only once in s. Let $T = \{t_1, \ldots, t_p\}$ (resp. $B = \{b_1, \ldots, b_q\}$ consist of the s_i appearing in the top (resp. bottom) row of s, with the t_i and b_j labelled in increasing order. Then one easily checks that $p = q + 1$ in type B or C, while $p = q$ in type D. The Cartesian product $\mathcal{P}(T) \times \mathcal{P}(B)$ of the power sets of T and B becomes a vector space over the field $\mathbb{F}_2$ of two elements if addition is defined via the symmetric difference and scalar multiplication in the obvious way. We now define two subspaces L and R of $\mathcal{P}(T) \times \mathcal{P}(B)$ and set up a perfect pairing $\langle \cdot, \cdot \rangle$ between them. Take L' (resp. R) to be the span of all $\ell_i := (t_i, b_i)$ with $1 \leq i \leq q$ (resp. all $r_i := (t_{i+1}, b_i)$ with $1 \leq i \leq \min(q, p-1)$); here we are identifying singleton subsets $\{x\}$ with their unique elements x. If s is of type D_n, so that $p = q$, let L be the quotient of L' by the span of $\sum \ell_i = (T, B)$; otherwise, let $L = L'$. We define the pairing $\langle \cdot, \cdot \rangle$ by decreeing that two basis vectors ℓ_i, r_j are orthogonal if and only if the corresponding singletons are all disjoint. Thus $\langle \ell_i, r_j \rangle = 1$ if $j = i$ or $j = i - 1$ and $\langle \ell_i, r_j \rangle = 0$ otherwise. It is easy to see that $\langle \cdot, \cdot \rangle$ is indeed a perfect pairing.

Define a subspace S of L or R to be *smooth* if it is spanned by sums of consecutive basis vectors ℓ_i or r_j. For example, if $p = 4$, then R has exactly one nonsmooth subspace, spanned by $r_1 + r_3$. We say that S is *supersmooth* if both S and its $\langle \cdot, \cdot \rangle$-orthogonal $S^\perp$ are smooth. If $p = 4$, then R has exactly one smooth but not supersmooth subspace, spanned by $r_1 + r_2$ and $r_2 + r_3$ (its orthogonal is spanned by $\ell_1 + \ell_3$).

At last we are ready to characterize the left cells.

THEOREM 2.12. ([23]). *Given a special symbol s, the left cells in the family (or double cell) of s are parametrized by supersmooth subspaces of the space L (or equivalently the space R) defined above. Given such a subspace S, the left cell $\mathcal{C}$ corresponding to S consists of the representations with the following symbols: for each $(X, Y) \in S + S^\perp$, transfer the elements of X from the top to the bottom row of s, and similarly transfer the elements of Y from the bottom to the top row of s. In type D_n, if s has equal rows and a numeral, then each of the two representations with the symbol s lies in a left cell by itself.*

For example, if

$$s = \begin{pmatrix} 0 & & 2 & & 4 \\ & 1 & & 3 & \end{pmatrix}$$

then the double cell corresponding to s has exactly five (isomorphism types of) left cells, corresponding to the five subspaces of L (all of which turn out to be supersmooth). The symbols of the representations in the left cell attached to L itself are

$$\begin{pmatrix} 0 & & 2 & & 4 \\ & 1 & & 3 & \end{pmatrix}\begin{pmatrix} 1 & & 2 & & 4 \\ & 0 & & 3 & \end{pmatrix}\begin{pmatrix} 0 & & 3 & & 4 \\ & 1 & & 2 & \end{pmatrix}\begin{pmatrix} 1 & & 3 & & 4 \\ & 0 & & 2 & \end{pmatrix}$$

while those in the left cell attached to the subspace $\mathbb{F}_2\ell_2$ (whose orthogonal is spanned by $r_1 + r_2$) are

$$\begin{pmatrix} 0 & & 2 & & 4 \\ & 1 & & 3 & \end{pmatrix}\begin{pmatrix} 0 & & 3 & & 4 \\ & 1 & & 2 & \end{pmatrix}\begin{pmatrix} 0 & & 1 & & 3 \\ & 2 & & 4 & \end{pmatrix}\begin{pmatrix} 0 & & 1 & & 2 \\ & 3 & & 4 & \end{pmatrix}.$$

The last of these symbols was computed by observing that $r_1 + r_2 + \ell_2 = (4, 1)$.

Of course the analogous result to Theorem 2.12 holds for right cells. Theorem 2.12 enables us to put an obvious structure of elementary abelian 2-group on the representations in a left or right cell, or on the representations common to a left and a right cell. We will use this group structure in Section 5. For now, we note that any family (or double cell) contains two distinguished left cells $\mathcal{C}_0, \mathcal{C}_L$, corresponding respectively to the supersmooth subspaces $0, L$ of L. From the formulas for the Springer correspondence in [24], one can check that in types B and D (resp. type C), the cell $\mathcal{C}_L$ (resp. $\mathcal{C}_0$) consists exactly of the representations attached by Springer to the nilpotent orbit in the Lie algebra $\mathfrak{g}$ whose (special) symbol coincides with that of the family. We therefore call these cells $\mathcal{C}_L$ or $\mathcal{C}_0$ *Springer cells*. Following

Joseph, whenever $\mathcal{C}_L$ (resp. $\mathcal{C}_0$) is a Springer cell, we call the 'opposite' cell $\mathcal{C}_0$ (resp. $\mathcal{C}_L$) a *Lusztig cell.* In general, Lusztig attaches an elementary abelian 2-group to every family of left cells and a subgroup of this group to every left cell in the family ([23, 28]); this subgroup is the whole group exactly when the cell is Lusztig in the above sense. Note that any Lusztig cell has only the special representation in common with the corresponding Springer cell. It turns out that an analogue of the Springer cell can be attached to non-Lusztig cells as well.

PROPOSITION 2.13. *Given any left cell $\mathcal{C}$, there is another left cell $\mathcal{C}'$ in the family $\mathcal{D}$ of $\mathcal{C}$ that has only the special representation in common with $\mathcal{C}$.*

Proof. In general, if $\mathcal{C}_1, \mathcal{C}_2$ are any two left cells in $\mathcal{D}$, corresponding to the supersmooth subspaces S_1, S_2 of L via Theorem 2.12, then one easily checks that representations common to $\mathcal{C}_1$ and $\mathcal{C}_2$ are parametrized by elements of $(S_1 \cap S_2) + (S_1^{\perp} \cap S_2^{\perp})$. Hence it suffices to locate a supersmooth subspace of L complementary to the one (call it S) corresponding to $\mathcal{C}$. Extend a basis of S to a basis of L by adding vectors of the form $\ell_1 + \cdots + \ell_i$. Let S' be the span of the added vectors. Since $\mathbb{F}_2(\ell_1 + \cdots + \ell_i)^{\perp}$ is spanned by the r_j with $j \neq i$, it follows that S' is supersmooth, as desired. □

In [32, 6.9] it was claimed that the $\mathcal{C}'$ of Proposition 2.13 is unique (when regarded as a W-module); this actually holds only for Lusztig and Springer cells. However, these cases suffice for the applications in that paper (cf. [1, Sect. 5]). We conclude this section with a characterization of supersmooth subspaces that we will need in Section 4.

LEMMA 2.14. *Retain the above notation. A subspace S of L is supersmooth if and only if it is spanned by a set of sums $\ell_i + \cdots + \ell_j$ of consecutive ℓ_k such that, if $\ell_i + \cdots + \ell_j, \ell_{i'} + \cdots + \ell_{j'}$ are two sums in the set, then the intervals $[i, j], [i', j']$ are either one contained in the other or disjoint.*

Proof. One computes that $\mathbb{F}_2(\ell_i + \cdots + \ell_j)^{\perp}$ is spanned by all r_k with $k \neq i-1, j$, together with $r_{i-1} + r_j$ if $i > 1$. Thus if the intervals $[i, j], [i', j']$ overlap but neither is contained in the other, then a sum of consecutive r_k is orthogonal to both $\ell_i + \cdots + \ell_j$ and $\ell_{i'} + \cdots + \ell_{j'}$ if and only if it involves all or none of the indices $i-1, i'-1, j, j'$. By contrast, an arbitrary sum of r_k's is orthogonal to both of these sums if and only if it involves both or none of the indices $i-1, j$, and both or none of the indices $i'-1, j'$. Now the necessity of the stated condition is clear, and its sufficiency is easy to check as well. □

3. Standard domino tableaux and W-module structure

We turn now to our recipe for computing the W-module structure of a left cell from Garfinkle's standard tableaux. We begin by summarizing the basic properties of these tableaux [7, 8, 9, 10]. Given an element w of W_n or W'_n, Garfinkle constructs an ordered pair of standard domino tableaux $(T_L(w), T_R(w))$ of the same shape in such a way that w can be recovered from the pair $(T_L(w), T_R(w))$. A domino tableau in type C or D is simply an arrangement of numbered horizontal and vertical dominos having the same shape as a Young tableau such that domino labels increase as one moves downward or to the right. (The definition in type B is slightly different: there domino tableaux consist of dominos as above plus a single square in the upper left corner, always numbered 0.) A domino tableau is called standard if the domino labels are precisely the integers from 1 to n for some n, each occurring once. The procedure for constructing $(T_L(w), T_R(w))$ from w is similar to the Robinson-Schensted insertion algorithm (where w is first replaced by the sequence $w(1, \ldots, n)$ of signed integers), but involves a more complicated kind of 'bumping', as dominos may be horizontal or vertical and may change their orientations when subsequent dominos are added. As in type A, we have $T_R(w) = T_L(w^{-1})$, but this time we do *not* have $I_w = I_{w'}$ if and only if $T_L(w) = T_L(w')$ (in the notation of Section 1). The decomposition of W_n and W'_n by left domino tableaux is strictly finer than the left cell decomposition (we hope to study it in a future paper). Thus one must introduce an equivalence relation $\approx$ on tableaux, as follows. The dominos in a tableau can be grouped into 'cycles', some of which are called 'open' and the others 'closed'. For each open cycle, there is a procedure called 'moving the tableau through the cycle', which involves changing the positions of the dominos in the cycle (but no others). Moreover, the set of squares involved in moving through one open cycle is disjoint from the corresponding set for any other, so that it is possible to move through any set of open cycles simultaneously. For any two tableaux T_1, T_2, we say that $T_1 \approx T_2$ if it is possible to get from T_1 to T_2 by moving through open cycles. The symmetry of this relation is guaranteed because moving through the same open cycle twice always leads back to the original tableau. Now the classification theorem states that $I_w = I_{w'}$ if and only if $T_L(w) \approx T_L(w')$. Garfinkle actually expresses this result slightly differently: she picks a distinguished representative in every $\approx$-equivalence class, namely the one with 'special shape' in her terminology, and then classifies primitive ideals by domino tableaux of special shape. The definitions of special shape and open cycle depend on the type B, C, or D of the tableau. We will use a different representative in each equivalence class in §5. For now, we need a preliminary result. Recall the notion of extended open cycles of one tableau relative to another (of the same shape) [8, 2.3.1].

LEMMA 3.1. *Let $\mathcal{D}$ be a double cell containing a left cell $\mathcal{C}$ and a right cell $\mathcal{R}$. Let $T_{\mathcal{C}}, T_{\mathcal{R}}$ be the standard domino tableaux of special shape corresponding to $\mathcal{C}, \mathcal{R}$. Then the number of elements in the intersection $\mathcal{C} \cap \mathcal{R}$ equals $2^{\max(0,m-1)}$, where m is the number of extended open cycles of T_L relative to T_R.*

Proof. Assume first that we are in type B or C. The possible left tableaux $T_L(w)$ (resp. right tableaux $T_R(w)$) of elements w of the given intersection are obtained from $T_{\mathcal{C}}$ (resp. $T_{\mathcal{R}}$) by moving through open cycles, none of which can involve the upper left corner of the tableau (since it must not be vacated in type C or occupied by a domino in type B). Conversely, a pair of tableaux (T, T') obtained as in the last sentence arise from a Weyl group element w, necessarily lying in the relevant intersection, if and only if they have the same shape. Since the upper left corners of $T_{\mathcal{C}}, T_{\mathcal{R}}$ always belong to open cycles [7, Sect. 5], the result follows at once from the definition of extended open cycles. In type D, one must work a little harder. All of the above reasoning goes through, except that (1) the element w of W_n corresponding to a given pair (T_1, T_2) of tableaux of the same shape need not lie in W'_n, and (2) the upper left corner of a tableau is always occupied by a domino and never vacated in the course of moving the tableau through open cycles. Thus one gets exactly 2^m elements v of W_n with left and right tableaux equivalent to $T_{\mathcal{C}}, T_{\mathcal{R}}$, respectively, and it suffices to show that exactly half of these lie in W'_n. To this end, let $c \in W_n \backslash W'_n$ act on $\mathbb{C}^n$ by changing the sign of the first coordinate. Then [10] shows that the left tableaux of w and cw are $\approx$-equivalent whenever $w \in W'_n$, where of course the open cycles of cw are defined relative to type D even though $cw \notin W'_n$. As for the right tableaux of w and cw, they either coincide or differ only by moving the domino labelled 1 through its *closed* cycle, up to $\approx$-equivalence. So if the domino labelled 1 in $T_{\mathcal{R}}$ belongs to an open cycle, then elements v of W_n as above come in pairs $\{w, cw\}$ and the desired result follows. So assume that this domino belongs to a closed cycle c_1 instead and let $T'_{\mathcal{R}}$ be obtained from $T_{\mathcal{R}}$ by moving through c_1. Then there are clearly just 2^m elements v' of W_n with left and right tableaux equivalent to $T_{\mathcal{C}}, T'_{\mathcal{R}}$. Elements v as above not lying in W'_n correspond bijectively to elements v' as above lying in W'_n under the map $w \mapsto cw$, and vice versa. Thus, of the 2^{m+1} elements v or v' as above, exactly 2^m of them lie in W'_n. The tableau $T'_{\mathcal{R}}$ also has special shape, and the right cell $\mathcal{R}'$ corresponding to it is obtained from $\mathcal{R}$ as a module by twisting every representation by conjugation by c. It follows from Proposition 2.2 and Theorem 2.12 that $\mathcal{R}'$ is isomorphic to $\mathcal{R}$ as a W'_n-module, *unless* $\mathcal{R}$ consists of a single representation with a numeral. If $\mathcal{R}' \cong \mathcal{R}$, then $\mathcal{C} \cap \mathcal{R}, \mathcal{C} \cap \mathcal{R}'$ have the same cardinality [23, 12.15], and the desired result follows. If $\mathcal{R}$ consists of a single representation with a numeral, then [23, 12.15] applies again and shows that $\mathcal{C} \cap \mathcal{R}$ is a singleton while $\mathcal{C} \cap \mathcal{R}'$ is empty. It follows that $m = 0$ in this case. Hence Lemma 3.1 holds in all cases. □

Now we are ready to compute W-module structure from domino tableaux. Given a tableau T, let $T_1, \ldots, T_k$ enumerate the tableaux obtained from T by moving through open cycles and of the same type as T (so not involving the upper left corner of T, if it lives in type B or C). For $1 \leq i \leq k$, let $\mathbf{p}_i$ be the partition corresponding to the shape of T_i. In case $\mathbf{p}_i$ is very even and T_i lives in type D, then we also attach a numeral 1 or 2 to $\mathbf{p}_i$, according as the number of vertical dominos in T_i is congruent to 0 or 2 modulo 4. Whenever two tableaux have exactly the same set of partitions $\mathbf{p}_i$ and numerals attached to them by the above recipe, then we say they are *module equivalent*. The terminology is justified by

THEOREM 3.2. *Retain the above notation and let $\mathcal{C}$ be the left cell corresponding to T. Then the constituents of $\mathcal{C}$, when regarded as a W-module, are precisely those corresponding to $\mathbf{p}_1, \ldots, \mathbf{p}_k$ via the map π of Section 2. In types B and C, each constituent appears exactly once in the list $\pi(\mathbf{p}_1), \ldots, \pi(\mathbf{p}_k)$ and the number of w in $\mathcal{C}$ (regarding the latter now as a set) with left tableau T_i equals the dimension of $\pi(\mathbf{p}_i)$. In type D, the list $\pi(\mathbf{p}_1), \ldots, \pi(\mathbf{p}_k)$ contains each constituent of $\mathcal{C}$ exactly twice, unless $k = 1$. The number of $w \in \mathcal{C}$ such that the left tableau of w or wc is T_i again equals the dimension of $\pi(\mathbf{p}_i)$.*

Proof. By Lemma 3.1, the definition of open cycle, and [23, 12.15], we see that two left cells $\mathcal{C}, \mathcal{C}'$ are isomorphic as W-modules if and only if their standard tableaux T, T' of special shape are module equivalent. The first assertion thus follows in general if it can be checked for one left cell in each W-module equivalence class. Thanks to Definition 2.6 and Proposition 2.11, we have an inductive recipe for producing one left cell in each such equivalence class, together with a representative of each cell. Applying the definition of open cycle to each of these representatives and the formulas for truncated induction and tensoring with sgn on the level of partitions (Lemmas 2.9 and 2.10), we see that the first assertion holds in all cases. We remark that we took $c \in W_n \backslash W_n'$ to change the sign of the first rather than the last coordinate because Garfinkle makes a nonstandard choice of positive roots in type D_n in [10].

Turning now to the proof of the second assertion, let $\mathcal{C}, \mathcal{R}$ be arbitrary left and right cells lying in the same double cell $\mathcal{D}$. As w runs over the intersection $\mathcal{C} \cap \mathcal{R}$, its left tableau $T_L(w)$ must always have a shape corresponding via π to a representation in $\mathcal{C}$, and the right tableau $T_R(w)$ must behave similarly with respect to $\mathcal{R}$. In type D_n, similar results hold for wc, by the facts mentioned in the proof of Lemma 3.1 about its tableaux in terms of those of w. But the left and right tableaux of any element have the same shape. Furthermore, there cannot be distinct tableaux $T_L(w)$ for $w \in \mathcal{C} \cap \mathcal{R}$ of the same shape, since each $T_L(w)$ is $\approx$-equivalent to a fixed tableau of special shape. It follows that, in types B and C, the common shapes of $T_L(w), T_R(w)$ as w runs over the relevant intersection parametrize the representations common to $\mathcal{C}$ and $\mathcal{R}$ bijectively. In type D, the common shapes of $T_L(w), T_R(w)$ and $T_L(wc), T_R(wc)$ parametrize the representations common to $\mathcal{C}$ and $\mathcal{R}$ in a two-to-one fashion. In all cases, holding $\mathcal{C}$ fixed and letting $\mathcal{R}$ run through all the right cells in $\mathcal{D}$, we get the desired result by [23, 12.15]. □

Of course the analogous result holds for right and double cells. Theorem 3.2 allows one to attach representations of W to elements of one-sided cells $\mathcal{C}$ in a manner consistent with the module structure of the cell. In particular, in types B and C, we get an injective map from $\mathcal{C} \cap \mathcal{C}^{-1}$ to a subset of $\hat{W}$ that carries a natural structure of elementary abelian 2-group, by the remarks after Theorem 2.12. We could use this map to transfer the group structure to $\mathcal{C} \cap \mathcal{C}^{-1}$. Now we will see in Section 5 that Lusztig has also defined a natural elementary abelian 2-group structure on $\mathcal{C} \cap \mathcal{C}^{-1}$, which is unfortunately *not* the same (in general) as the one just described. We will describe the difference between these two structures precisely in Section 5. In type D, matters are more complicated, for the map π from partitions to representations is (generically) two-to-one. Let T be a tableau and T' the tableau obtained from T by moving through all of its open cycles. Then one easily checks that the shapes of T and T' parametrize the same representation. If $w \in W_n'$ belongs to the intersection $\mathcal{C} \cap \mathcal{R}$ of the left cell $\mathcal{C}$ and right cell $\mathcal{R}$, then any other $v \in \mathcal{C} \cap \mathcal{R}$ has left tableau $T_L(v)$ obtainable from $T_L(w)$ by moving through an *even* number of cycles. Thus the map from a typical intersection $\mathcal{C} \cap \mathcal{C}^{-1}$ to $\hat{W}$ coming from Theorem 3.2 is injective if and only if the tableau corresponding to $\mathcal{C}$ has an odd number of open cycles (or no open cycles at all).

We also remark that Theorem 3.2 shows that the open orbit in the associated variety of a typical primitive ideal I_w in the classical case may be read off from the shape of its corresponding (left) tableau T of special shape (unless this orbit lives in type D and is very even, in which case one must also look at the number of vertical dominos in T, as mentioned above).

4. Wall-crossing functors and Hecke module equivalences

In Joseph's classification of primitive ideals in type A_n a crucial (and often overlooked) role is played by a simple set of generators discovered by Knuth for the equivalence relation of having the same Robinson-Schensted left tableau. Analogues of the Knuth generators in types B, C, D were discovered by Joseph [14] and Vogan [34]. Joseph showed that they furnish simple sufficient conditions for two elements to lie in the same left cell; Vogan then observed that they can be turned around to furnish necessary conditions as well, using τ-invariants. The key to Garfinkle's classification of primitive ideals in types B, C, D lies in her discovery that these necessary and sufficient conditions coincide in these types. Although the statement of this coincidence does not involve domino tableaux, its proof relies on them in a crucial way [8, 9, 10]. We now define the (dual) Knuth map $T_{\alpha\beta}$ (which makes sense in any classical type) and two analogues $S_{\alpha\beta}, S_D$ (which make sense in types BC and D, respectively) and show that they have the properties asserted of them in the introduction.

Let α, β be simple roots spanning a subsystem of type A_2. The *wall-crossing operator* $T_{\alpha\beta}$ is defined on Weyl group elements w whose τ-invariant contains exactly one of α and β where u is uniquely defined by the following properties: first, $u \in wW'$, where W' is the parabolic subgroup of W generated by the reflections s_α, s_β through α, β; second, u and w have different lengths; and third, the τ-invariants of u and w meet $\{\alpha, \beta\}$ in disjoint singletons. Then $T_{\alpha\beta}$ may also be defined on simple highest weight modules (or simple Harish-Chandra modules over some real group) via a composite of translation functors, whence it also induces a well-defined order-preserving map on primitive ideals [34].

In type A_n the various maps $T_{\alpha\beta}$ suffice to classify the primitive spectrum as a set, and even (conjecturally) as an ordered set as well. In types B_n and C_n, however, these maps fail to take account of the short or long simple root at the extreme right end of the Dynkin diagram. One therefore needs to define a substitute for $T_{\alpha\beta}$ if α, β are simple roots spanning a subsystem of type B_2. Although the paper [34] does actually define a map that it calls $T_{\alpha\beta}$ in this case as well, it turns out that the correct analogue of the map $T_{\alpha\beta}$ of the last paragraph is a map defined in a later paper [35] and called $S_{\alpha\beta}$ there. Like the map $T_{\alpha\beta}$, its domain consists of all $w \in W$ whose τ-invariant meets $\{\alpha, \beta\}$ in a singleton, but now the second and third requirements to specify the image u of w under $S_{\alpha\beta}$ are different. The second one now states that the length difference between u and w should be even in any event and nonzero if possible. The third one states that the τ-invariants of u and w should meet $\{\alpha, \beta\}$ in the *same* singleton. Then $S_{\alpha\beta}$ (unlike the $T_{\alpha\beta}$ of [34]) is a well-defined single-valued map that can also be defined on simple highest weight or Harish-Chandra modules by translation functors. Like the $T_{\alpha\beta}$ of the last paragraph, it induces an order-preserving map on primitive ideals [35].

In type D_n things are more complicated. Although there is only one root length, the maps $T_{\alpha\beta}$ fail to generate the right cells, even if $n = 4$ [34]. So let the simple roots $\alpha, \beta, \gamma, \delta$ span a subsystem of type D_4 with α the inner root. (It does not matter how we label the outer roots β, γ, δ; moreover, the choice of $\{\alpha, \beta, \gamma, \delta\}$ is unique if $\mathfrak{g}$ is simple. This is why we will suppress it from the notation.) Assume that $w \in W$ belongs to the set $\mathcal{S}$ of elements satisfying hypothesis $\mathcal{D}$ of [11],so that in particular the τ-invariant of w meets $\{\alpha, \beta, \gamma, \delta\}$ precisely in $\{\alpha\}$; note that the τ-invariant of [11] coincides with the *left* τ-invariant of [8] in this situation. We now define a map S_D on elements w as above via $S_D(w) := u$, where $u \in \mathcal{S}$ is uniquely specified by the requirement that it also satisfy hypothesis $\mathcal{D}$, differ from w if possible, and lie in a common diagram with w of type 8-2 or 8-6, in the sense of [11]. Using the main theorem of [11], one checks that S_D, like $T_{\alpha\beta}$ and $S_{\alpha\beta}$, may be defined on simple highest weight or Harish-Chandra modules by a composite of translation functors. Hence S_D, like $T_{\alpha\beta}$ and $S_{\alpha\beta}$, induces an order-preserving map on primitive ideals.

Recall now the definition, canonical basis $\{T_w : w \in W\}$ and Kazhdan-Lusztig basis $\{C_w : w \in W\}$ of the Hecke algebra H corresponding to W [20]. Following Kazhdan and Lusztig, we take the ring $A := \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of Laurent polynomials in an indeterminate $q^{1/2}$ as the base ring of H (originally H was defined to have base ring $\mathbb{Z}[q]$). We let F denote the fraction field of A and H_F the algebra obtained from H by extending the scalars to F. Given a left cell $\mathcal{C}$, recall that the F-span $[\mathcal{C}]$ (resp. the $\mathbb{Q}$-span $\langle\mathcal{C}\rangle$) of the C_w for $w \in \mathcal{C}$ carries the natural structure of a left H_F-module (resp. left W-module); more precisely, there is an explicit formula for the left action of T_s on C_w whenever $s \in W$ is a simple reflection and $w \in C$ which involves only structure constants in $\mathbb{Z}[q^{1/2}]$ and depends only on the W-graph of $\mathcal{C}$ [20, 1.3]. Finally, given a left cell $\mathcal{C}$ and one of the maps $T := T_{\alpha\beta}, S_{\alpha\beta}$, or S_D, recall (as noted above) that T is defined at one element of $\mathcal{C}$ if and only if it is defined at every element of $\mathcal{C}$ and in that case it sends $\mathcal{C}$ to a single left cell $\mathcal{C}'$. We extend T to an F-module map defined on $[\mathcal{C}]$ in the obvious way.

THEOREM 4.1. *If $\mathcal{C}$ is a left cell and a map $T := T_{\alpha\beta}, S_{\alpha\beta}$, or S_D, is defined on $\mathcal{C}$, then the induced map on $[\mathcal{C}]$ is left H_F-equivariant.*

Proof. From the discussion of $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ above we see that T is given by a composition of right multiplication by various elements T_s with s a simple reflection, subtraction of a multiple of the identity map, projection to certain left cells, and scalar multiplication (one needs to use the results in [11] to verify this in the case of S_D). All of these maps respect the left H_F-action. □

This fact was already observed in [20] for the maps $T_{\alpha\beta}$, where it was used to show that left cells in type A_n are irreducible as W-modules. For the map $S_{\alpha\beta}$ it is implicit in [35]; for S_D it is new. If the simple roots α and β span a subsystem of type B_2, then we have mentioned above that Vogan has defined a map which he denotes by $T_{\alpha\beta}$ in [34]; we will however denote it by $T'_{\alpha\beta}$ to avoid ambiguity. It is neither injective nor single-valued, but it induces a single-valued left H_F-equivariant map sending a typical C_w on which it is defined either to another C_u or to a sum $C_v + C_{v'}$. Similarly, the map S_D may be modified to a new map T_D with the same property as $T'_{\alpha\beta}$. The maps $T'_{\alpha\beta}$ and T_D can also be defined on the level of left cells, but even on this level they are not single-valued. A crucial result in the program of [9, 10], appearing in [9] as Theorem 3.2.2, asserts that one can get from any left cell to any other in the same double cell by a sequence of the maps $T_{\alpha\beta}, T'_{\alpha\beta}$, and T_D. We will need the analogue of this result for $T_{\alpha\beta}, S_{\alpha\beta}$, and S_D.

THEOREM 4.2. *Let $w_1, w_2 \in W$ belong to the same right cell $\mathcal{R}$ and left cells $\mathcal{C}_1, \mathcal{C}_2$ that are isomorphic as W-modules. Then there is a sequence of maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ sending w_1 to w_2.*

Proof. Assume first that W is of type B or C. We imitate the proof of Theorem 3.2.2 in [9], proceeding by induction on the rank of W. That proof is broken down into a proposition (3.2.4) and a sequence of lemmas (3.2.6–3.2.9). In our situation we must strengthen both the hypothesis and the conclusion of Lemma 3.2.9. The new hypothesis states that we are given a tableau T_1 and an extremal position P' in it such that there is another tableau $\widetilde{T_1}$ module equivalent to T_1 having its domino with largest label in position P'. The new conclusion replaces the sequence of maps in the old conclusion with a sequence of maps $T_{\alpha\beta}$ and $S_{\alpha\beta}$; indeed, of course, Garfinkle's map $T_{\alpha\beta}$ (which coincides with Vogan's in [34]) must be replaced by our map $S_{\alpha\beta}$ throughout whenever α and β have different lengths. Lemma 3.2.8 must also be strengthened. Given a tableau shape S and an extremal position P in it, there is a standard domino tableau T of shape S whose domino D with largest label is in position P and whose cycle structure in the sense of [8] may be any of the possible ones for a tableau of this shape, subject only to the constraint that D may be forced to lie in an open cycle by itself. This is easily proved by induction on the size of S. Now the new versions of Theorem 3.2.2 and Proposition 3.2.4 are easily verified if $r = 2$ (in the notation of [9]). In general, the arguments of [9] can now be carried over to our situation. A similar strategy, using [10], takes care of the case when W is of type D; there the base case is $r = 4$ and we replace the map $S_{\alpha\beta}$ by S_D. □

Unfortunately Theorem 4.2 fails for the exceptional Weyl groups; there are left cells $\mathcal{C}$ in every such group such that the self-intertwining operators on $\mathcal{C}$ sending basis vectors to basis vectors cannot act transitively on $\mathcal{C} \cap \mathcal{C}^{-1}$. The reason is that the finite group attached by [23] to the double cell $\mathcal{D}$ containing $\mathcal{C}$ is not an $\mathbb{F}_2$-vector space in these cases (as mentioned in Section 5, it is a symmetric group instead). We are now ready for the main result of this paper.

THEOREM 4.3. *The algebra H_F is semisimple Artinian. Its simple (left) modules are all defined over F and correspond bijectively to simple W-modules over $\mathbb{Q}$. Given a simple $\mathbb{Q}W$-module I, realized as a constituent of some left cell representation $\langle\mathcal{C}\rangle$, one can construct an explicit basis of the corresponding H_F-module whose elements are linear combinations of basis vectors C_w with coefficients ± 1. The structure constants with respect to this basis lie in A. In particular, specializing at $q = 1$, one obtains a canonical basis for every simple $\mathbb{Q}W$-module such that W acts on the basis by integral matrices.*

Proof. The first two assertions follow at once from the Benson-Curtis-Lusztig theorem: the algebra H_F is in fact isomorphic to the group algebra FW. We will also see below that we can recover at least these two assertions without invoking this theorem. Given a left cell $\mathcal{C}$, let $\mathcal{R}$ be a right cell meeting $\mathcal{C}$ nontrivially. By [23, 12.15], one knows that the elements of $\mathcal{C} \cap \mathcal{R}$ are parametrized by the representations common to $\langle\mathcal{C}\rangle$ and $\langle\mathcal{R}\rangle$. More precisely, the arguments of [19, 2.8] show that the F-span $[\mathcal{C} \cap \mathcal{R}]$ of the C_w with $w \in \mathcal{C} \cap \mathcal{R}$ generates the H_F-submodule of $\mathcal{C}$ corresponding to the sum of these common representations. For each constituent J of $[\mathcal{C}]$, we will construct a weighted sum of C_w lying in J. Repeating the construction for every right cell $\mathcal{R}$ with J a submodule of $[\mathcal{R}]$, we get a basis of J of the desired type.

We begin by considering all compositions of maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ defined on $\mathcal{C}$ and mapping it into itself. Each such composition induces a permutation σ of $\mathcal{C} \cap \mathcal{R}$; the set Σ of permutations obtained in this way is obviously a subgroup of the symmetric group S_k on $k := \#(\mathcal{C} \cap \mathcal{R})$ letters. Note that k is a power of 2, by Theorem 2.12. Every $\sigma \in \Sigma$ induces a linear map on $[\mathcal{C}]$ that multiplies every constituent of the latter by a scalar, which must be a root of unity in F. As the only such roots of unity are ± 1, we see that σ must be an involution (or the identity). Thus Σ must be an elementary abelian 2-subgroup of S_k acting transitively on the k letters, by Theorem 4.2. There is only one such subgroup, up to conjugacy; it may be described geometrically as the symmetry group of a $\log_2 k$-dimensional parallelepiped whose edges have distinct lengths, identifying the k letters with the k vertices of the parallepiped. It follows that Σ acts on $[\mathcal{C} \cap \mathcal{R}]$ (or $\langle\mathcal{C} \cap \mathcal{R}\rangle$) by the left regular representation, so that this space decomposes uniquely as the sum of one-dimensional subspaces $\mathcal{S}$, each preserved by Σ and lying in some constituent J as above. It only remains to decide which subspace $\mathcal{S}$ lies in which constituent J. This is done by induction on the 'complexity' of J, which is defined as follows. We know from Theorem 2.12 that there are finite-dimensional $\mathbb{F}_2$-vector spaces L, R attached to $\mathcal{C}$ endowed with a perfect pairing $\langle\cdot,\cdot\rangle$ and canonical respective bases $\{\ell_i\}, \{r_j\}$ such that J identifies with a sum $\ell + r$ with $\ell \in S$, a supersmooth subspace of L, and $r \in S^\perp$. Let t be the least number of sums of consecutive ℓ_i in $\mathcal{S}$ adding up to ℓ and let u be the corresponding number for r and $\mathcal{S}^\perp$. Then the complexity of J is defined to be $t+u$. Assume now that we can compute exactly which subspaces $\mathcal{S}$ lie in which submodules J' whenever J' has complexity less than m and suppose that J has complexity exactly m. Define the sum $\ell + r$ and integers t, u as above. Thanks to Lemma 2.14 we can find t sums of consecutive ℓ_i spanning a supersmooth subspace S_1 of L and adding to ℓ. Similarly there are u sums of consecutive r_j spanning a supersmooth subspace S_2 of R and adding to r. Let $\mathcal{R}_1, \mathcal{R}_2$ be the right cells corresponding to S_1, S_2 and let $\{\mathcal{S}_i\}, \{\mathcal{S}_i'\}$ be the sets of subspaces produced as above from the intersections $\mathcal{C} \cap \mathcal{R}_1, \mathcal{C} \cap \mathcal{R}_2$. Enumerate the subspaces $\mathcal{S}_i$ that are conjugate under H_F to subspaces $\mathcal{S}_j'$ as $\mathcal{S}_1'', \mathcal{S}_2'', \ldots$. Using Theorem 2.12, we see that all but one of the subspaces $\mathcal{S}_i''$ lies inside submodules of complexity less than m, whence we can inductively identify these submodules. The unique exceptional $\mathcal{S}_i''$ lies in J, and now we can say that an arbitrary subspace $\mathcal{S}$ lies in J if and only if it is conjugate under H_F to this subspace $\mathcal{S}_i''$. Thus we can 'place' all the subspaces $\mathcal{S}$ arising in the first part of the argument, and this suffices to complete the proof. □

Actual computations of course become quite tedious as soon as k is large, but one should note that the recipes in [9] and [10] enable one to evaluate the operators $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ directly on ordered pairs of standard domino tableaux, without passing to Weyl group elements. We hope to pursue the applications of Theorem 4.3 in a future paper; for now we mention just two of them. First, it is clear that this theorem puts severe and explicit constraints on the behavior of the Jacquet functor from a double cell of simple Harish-Chandra modules over a classical real reductive group to a right cell in W, since Casian and Collingwood have shown that this functor may be viewed as a Hecke module map. For example, Collingwood has shown that this functor takes a certain 14-dimensional double cell of SO(6,2) to a certain 10-dimensional right cell of W_4'. Using Theorem 4.3, one can show that the range of this map is the unique 8-dimensional submodule of this right cell, corresponding to the 8-dimensional special representation of W_4'; the theorem also provides a basis of this submodule. (One can identify the functor in this case with a map T_D mentioned above.) Second, one can now attempt to relate the Kazhdan-Lusztig bases of irreducible $\mathbb{Q}$-representations of W_n and W_n' provided by Theorem 4.3 to other bases worked out much earlier by Young and Frame. The paper [12] does this for $W = S_n$, where the left cells are already irreducible. We also remark that Lusztig has attached a different basis to every simple left H_F-module M (for arbitrary finite or affine W) which shows that M admits a W-graph but which does *not* decompose left cells into their constituents [29].

5. Applications to the asymptotic Hecke algebra

In this section we will be working not with H but with a remarkable $\mathbb{Z}$-form of (a completion of) it, which is also a $\mathbb{Z}$-form of the group algebra $\mathbb{Q}W$. This was discovered by Lusztig; following him, we call it the *asymptotic Hecke algebra* and denote it by J. Like H, this algebra has a canonical basis $\{t_w\}$ (this time over $\mathbb{Z}$) indexed by W, but the behavior of the t_w under multiplication is much simpler than that of the C_w. Indeed, if $\mathcal{C}$ is a left (resp. right, two-sided) cell of W, then the span $J_{\mathcal{C}}$ of the t_w for $w \in \mathcal{C}$ is a left (resp. right, two-sided) ideal of J, not just a quotient of ideals as for H. Moreover, if we write $t_x t_y = \sum c_{x,y,z} t_{z^{-1}}$ for $x, y \in W$, then the structure constants $c_{x,y,z}$ lie in $\mathbb{N}$. As in the Benson-Curtis-Lusztig theorem, the isomorphism between $J \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q}W$ is complicated to write down; it does not just send t_w to C_w. For all of these facts and the precise definition of multiplication in J, see Lusztig's papers [25, 27, 28, 29], which also treat the case of affine Weyl groups W. For our purposes the main facts about J are the following ones. Given left cells $\mathcal{C}, \mathcal{C}'$ that are isomorphic as W-modules and an element x of $\mathcal{C}^{-1} \cap \mathcal{C}'$, any product $t_y t_x$ or $t_x t_y$ is either zero or t_z for some z. We have $t_y t_x \neq 0$ if and only if $y \in \mathcal{C}$, in which case z lies in the same left cell as x and the same right cell as y. A similar result holds of course for $t_x t_y$. The main result of this section shows how to compute z in terms of x and y. To state it, we need to extend the definition of extended open cycles of one tableau relative to another in [8] slightly. If the tableaux T_1 and T_2 do not necessarily have the same shape, but are equivalent under the relation $\approx$ of Section 3 to tableaux T_1', T_2' which do have the same shape, then the extended open cycles of T_1 relative to T_2 are defined to be those of T_1' relative to T_2'.

THEOREM 5.1. *Retain the above notation and suppose that $y \in \mathcal{C}$, so that $t_y t_x = t_z$. Then one can compute the left and right tableaux $T_L(z), T_R(z)$ of z as follows. Let d be the Duflo involution in $\mathcal{C}$ and let $T_L(y)$ (resp. $T_R(x)$) be obtained from $T_L(d)$ (resp. $T_R(d) = T_L(d)$) by moving through the open cycles $c_1, \ldots, c_k$ (resp. $c_1', \ldots, c_m'$). Let $e_1, \ldots, e_k$ (resp. $e_1', \ldots, e_m'$) be the extended open cycles containing $c_1, \ldots, c_k$ (resp. $c_1', \ldots, c_m'$) relative to $T_L(x)$ (resp. to $T_R(y)$). Denote by U (resp. U') the union of the extended open cycles appearing an odd number of times in the list $e_1, \ldots, e_k$ (resp. $e_1', \ldots, e_k'$). Then $T_L(z)$ (resp. $T_R(z)$) is the right tableau of $\mathbf{E}((T_L(x), T_L(y)); U, L)$ (resp. of $\mathbf{E}((T_R(y), T_R(x)); U', L)$), in the notation of [8].*

Proof. If $x = d$, then we know from [27] and [28] that $z = y$. In other words, t_d is the unit element of the subring $J_{\mathcal{C}^{-1}\cap\mathcal{C}} := J_{\mathcal{C}^{-1}} \cap J_{\mathcal{C}}$ of J and the right $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ module $J_{\mathcal{C}}$ is unital. In general, we know from Theorems 4.1 and 4.2 that we can get from d to x via a sequence of maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ and that this sequence of maps induces a left J-equivariant map from $J_{\mathcal{C}}$ to $J_{\mathcal{C}'}$ (by the definition of multiplication in J and the equivariance of the induced map on $[\mathcal{C}]$). So it suffices to compute the effect of the maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ on the level of domino tableaux. Garfinkle has done this in [8] and [10]. Her recipes reduce in this situation to the ones in the theorem. $\square$

There is a similar formula for $t_x t_y$ whenever x satisfies the hypothesis of Theorem 5.1. It can be proved in the same way, using the *right* H-equivariant analogues of the maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$. We will use these analogues below. Unfortunately Theorem 5.1 falls far short of determining the multiplication table of J completely. In a subsequent paper, we will adapt the ideas of this section to compute all the structure constants $c_{x,y,z}$.

COROLLARY 5.2. *Fix a Cartan subalgebra $\mathfrak{h}$ and a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing $\mathfrak{h}$. Let $\lambda \in \mathfrak{h}^*$ be a dominant regular integral weight. For $w \in W$, denote by $L(w \cdot \lambda)$ the simple module of highest weight $w(\lambda + \rho) - \rho$, where ρ as usual is the half sum of the positive roots. Let $x \in W$ satisfy the hypothesis of Theorem 5.1 and y be any other element of W. Then one can compute the socle of the bimodule $L(L(x \cdot \lambda), L(y \cdot \lambda))$ of Ad $\mathfrak{g}$-finite maps from $L(x \cdot \lambda)$ to $L(y \cdot \lambda)$, given a knowledge of the Duflo involution in $\mathcal{C}$. This socle is simple or zero and is nonzero if and only if $y \in \mathcal{C}^{-1}$.*

Proof. Begin by recalling the map $w \mapsto w_*$ introduced by Joseph in [18, Appendix]; this map takes left cells to left cells, right cells to right cells, and Duflo involutions to Duflo involutions. If we set $c^*_{x,y,z} := c_{x_*,y_*,z_*}$, then Joseph has shown in [18, 4.8] that the multiplicity of the simple Harish-Chandra bimodule with infinitesimal character (λ, λ) and Langlands parameter z in the given socle is $c^*_{x^{-1},y,z^{-1}}$; here we warn the reader that the integers $c_{x,y,z}$ in [18] have the same absolute value as the $c_{x,y,z}$ here, but can differ from the latter by a sign. (A formula for this sign is given in [18, Appendix].) For the purposes of this proof only, enlarge the domain of the map S_D above by decreeing that it send $w_0 w$ to $w_0 u$ whenever it sends w to u, where w_0 is the longest element of W (the other two maps $T_{\alpha\beta}$ and $S_{\alpha\beta}$ already have this property). The definition in [18, Appendix] then shows that the map $w \mapsto w_*$ commutes with the maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ (since these maps commute with left multiplication by w_0). Hence the same sequence of maps $T_{\alpha\beta}, S_{\alpha\beta}, S_D$ taking the Duflo involution d of $\mathcal{C}$ to x^{-1} also takes d_* to x_*^{-1}, and similarly for y and z^{-1}. Now the recipe for computing $c_{x,y,z}$ in Theorem 5.1 shows that $c^*_{x^{-1},y,z^{-1}} = c_{x^{-1},y,z^{-1}}$ and computes $c_{x^{-1},y,z^{-1}}$ in this situation. (In fact, Joseph has shown that $c^*_{x,y,z} = c_{x,y,z}$ for any $x, y, z \in W$.) The first assertion follows; the second is an easy consequence of the first and the basic facts about the $c_{x,y,z}$ given above. $\square$

As with Theorem 5.1 there is of course a parallel formula for the socle of $B := L(L(y \cdot \lambda), L(x \cdot \lambda))$. The most important special case of Theorem 5.1 occurs when x and y both lie in $\mathcal{C}^{-1} \cap \mathcal{C}$ for some left cell $\mathcal{C}$; then z also lies in $\mathcal{C}^{-1} \cap \mathcal{C}$. More precisely, Lusztig has shown in [28] that the t_w for $w \in \mathcal{C}^{-1} \cap \mathcal{C}$ form an elementary abelian 2-group under multiplication, but he has not shown how to compute this group structure explicitly. We can now do this, using standard tableaux. We already know that t_d is the identity element of this group, where d is the Duflo involution in $\mathcal{C}$. Now let x, y be any elements in $\mathcal{C}^{-1} \cap \mathcal{C}$ and $T_L(x), T_L(y)$ their left tableaux, which coincide with their right ones. Suppose that $T_L(x), T_L(y)$ are obtained from the common left and right tableau $T_L(d)$ of d by moving through the open cycles $c_1, \ldots, c_k$ and $c'_1, \ldots, c'_\ell$, respectively. Then it follows from Theorem 5.1 that the left tableau $T_L(z)$ is obtained from $T_L(d)$ by moving through those open cycles appearing exactly once in the list $c_1, \ldots, c_k, c'_1 \ldots, c'_\ell$. Recall now that we remarked after Theorem 2.12 that the set of representations in $\langle \mathcal{C} \rangle$, which of course coincides with the corresponding set for $\langle \mathcal{C}^{-1} \rangle$, also has the natural structure of an elementary abelian 2-group. In types B and C, each such representation corresponds to a unique partition via the map π of Section 2; the resulting set of partitions consists exactly of the tableau shapes of elements in $\mathcal{C}^{-1} \cap \mathcal{C}$. Now it is not difficult to produce a recipe for the group structure on this set of representations in terms of tableaux. Indeed, one just follows the above recipe for the group structure on the set $\mathcal{C}^{-1} \cap \mathcal{C}$, with one crucial exception: the identity element ι in the group of representations is the one whose tableau shape(= partition) is special, not the one whose tableau shape coincides with that of the Duflo involution. (Already in type C_2, one sees that these two elements can differ.) Thus the tableau (shape) corresponding to ι plays the role of $T_L(d)$ above. If we regard the two group structures as living on the same set (of tableaux, or tableau shapes), then they are conjugate to each other, but not the same in general. In type D, as noted above, the map from $\mathcal{C} \cap \mathcal{C}^{-1}$ to $\hat{W}$ can fail to be injective, so that the two group structures need not even live on the same set.

The above special case of Theorem 5.1 can be further specialized, namely to left cells $\mathcal{C}$ containing long elements w_S of parabolic subgroups W_S of W. Any such cell has w_S as its Duflo involution [17, 4.2] and is often Lusztig in the sense of Section 2 [3]. Thus Corollary 5.2 yields an explicit formula for the socle of $L(L(w \cdot \lambda), L(y \cdot \lambda))$ for any $w, y \in \mathcal{C} \cap \mathcal{C}^{-1}$. Translating this formula to a dominant infinitesimal character singular on exactly the simple roots corresponding to S, we obtain a formula for Soc $L(L(w' \cdot \lambda'), L(y' \cdot \lambda'))$ valid for any $w', y' \in W$ such that Ann $L(w' \cdot \lambda') =$ Ann $L(y' \cdot \lambda')$ is a maximal ideal. Moreover, it turns out that the bimodule $B := L(L(w' \cdot \lambda'), L(y' \cdot \lambda'))$ coincides with its socle [32, 4.1] and can be interpreted as a tensor product over $U(\mathfrak{g})/I$ of two simple Harish-Chandra bimodules with the same maximal left and right annihilator I. We thus obtain

THEOREM 5.3. *For any infinitesimal character μ, the set of simple Harish-Chandra bimodules with maximal left and right annihilator I_μ of infinitesimal character μ form an elementary abelian 2-group under tensor product over $U(\mathfrak{g})/I_\mu$.*

The group structure is explicitly computable on the level of domino tableaux of Langlands parameters.

The first assertion of this result was proved for special unipotent infinitesimal characters μ by Barbasch and Vogan [4] and later generalized by Barbasch to arbitary μ [1], making heavy use of the techniques of [4]. Later Joseph [19] showed how to obtain it more elegantly using the calculations in [23] and [28]. Barbasch's proof however does have one advantage over Joseph's in that it yields an explicit (but rather complicated) inductive recipe for the group structure on the level of Langlands parameters. The contribution of the present paper is to simplify this recipe considerably by using domino tableaux. We mention that the techniques of [4] and [19], unlike the ones in this paper, extend beyond the case of classical $\mathfrak{g}$. They show that for any infinitesimal character μ in any semisimple Lie algebra $\mathfrak{g}$, with two families of exceptions [2], the simple Harish-Chandra bimodules of maximal left and right annihilator I_μ tensor over $U(\mathfrak{g})/I_\mu$ like the irreducible characters (not elements) of a finite group A, which is a direct product of elementary abelian 2-groups (and nothing else in the classical case) and copies of the symmetric groups S_3, S_4, S_5. Again the techniques of [4], unlike those of [19], yield explicit formulas on the level of Langlands parameters; we do not know how to simplify them. In [28], Lusztig has generalized this result by determining the ring structure of $J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ for any left cell $\mathcal{C}$ (not necessarily containing the long element of a parabolic subgroup). His methods do not give explicit formulas for multiplying the t_w.

Of course a major drawback of Theorem 5.1 is that it requires a knowledge of the Duflo involution in the left cell $\mathcal{C}$ before it can be applied. Although there is a simple criterion for deciding when a tableau has special shape [7], there is no analogous rule for determining the tableau shape of the Duflo involution in a left cell. We therefore conclude the paper with the following useful result.

THEOREM 5.4. *As d runs over the Duflo involutions in W, the shape of $T_L(d)$ depends only on the module structure of the left or right cell to which d belongs, not on the cell itself.*

Proof. Any map $X := T_{\alpha\beta}, S_{\alpha\beta}$, or S_D has a 'right analogue' $T^R_{\alpha\beta}, S^R_{\alpha\beta}$, or S^R_D which sends w^{-1} to u^{-1} whenever X sends w to u. The maps X^R cannot be implemented on simple highest weight modules by translation functors, but they *can* be implemented on Harish-Chandra bimodules for the complex group by *right* translation functors. Now apply a typical composition $X \circ X^R$ to the bimodule $L := L(L(d\cdot\lambda), L(d\cdot\lambda))$, where as above λ is a dominant regular integral infinitesimal character and d is a Duflo involution. One obtains a bimodule of the form $L' := L(L(d'\cdot\lambda'), L(d'\cdot\lambda'))$, where λ' is a different integral infinitesimal character (no longer regular) and $d' = X \circ X^R(d)$ is an involution not yet known to be Duflo. But now the exactness of $X \circ X^R$ forces it to send the unique simple subbimodule of L to that of L'. These subbimodules have the Langlands parameters d, d', so d' is indeed a Duflo involution. Since we know from Theorem 4.2 that we can get from any left cell to any other with the same module structure via sequence of maps X, it follows that the same sequence of maps $X \circ X^R$ takes the Duflo involution of the first cell to that of the second. Now Garfinkle has shown how to compute any map X on the level of domino tableaux [8, 10]. It follows from her recipes and the fact that $T_L(w), T_R(w)$ have the same shape for any $w \in W$ that any map $X \circ X^R$ preserves tableau shapes. The result follows. □

One also has a weaker result for left cells $\mathcal{C}_1, \mathcal{C}_2$ with $\mathcal{C}_2$ obtained from $\mathcal{C}_1$ by a map $T'_{\alpha\beta}$ or T_D as in Section 4; then knowledge of the Duflo involution of $\mathcal{C}_1$ determines that of $\mathcal{C}_2$ up to a list of two candidates ([17, 5.7], [35]).

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